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Enhancing the sensitivity of mesoscopic light reflection statistics in weakly disordered media by interface reflections

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Reflection statistics have not been well studied for optical random media whose mean refractive indices do not match with the refractive indices of their surrounding media. Here, we theoretically study how this refractive index mismatch between a one-dimensional (1D) optical sample and its surrounding medium affects the reflection statistics in the weak disorder limit, when the fluctuation part of the refractive index (Δn) is much smaller than the mismatch as well as the mean refractive index of the sample ($\Delta n \ll \langle n \rangle$). In the theoretical derivation, we perform a detailed calculation that results in the analytical forms of the mean and standard deviation (STD) of the reflection coefficient in terms of disorder parameters ($\langle (\Delta n)^2 \rangle^{\frac{1}{2}}$ and its correlation length l_c) in an index mismatched backscattering system. Particularly, the orders of disorder parameters in STD of the reflection coefficient for index mismatched systems are shown to be lower ($\sim (\langle \Delta n^2 \rangle l_c)^{1/2}$) than that of the matched systems ($\sim \langle \Delta n^2 \rangle l_c$). By comparing STDs of the reflection coefficient values of index matched and mismatched systems, we show that reflection coefficient at the sample boundaries in index mismatched systems can enhance the signal of the STD to the "disorder parameters" of the reflection coefficient. In terms of biophotonics applications, this result can lead to potential techniques that effectively extract the sample disorder parameters by manipulating the index mismatched conditions. Potential applications of the technique for enhancement in sensitivity of cancer detection at the single cell level are also discussed.

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1. Introduction

The statistical properties of transport in one-dimensional (1D) mesoscopic optical and electronic disordered media have been well studied.[1–5] The quantum mechanical and optical systems are analyzed within the same formalisms because the Schrödinger equation and Maxwell's equations are reduced to the same Helmholtz equation.[6–9] After the Landauer formalism showed that the reflection coefficient is related to the resistance or conductance of a sample, the outward scattering information such as the reflection and transmission coefficients became important for the studies of localization and conductance fluctuations in electronic systems.[6,7] In optical research such as light scattering and localization, the disorder properties of the refractive index have been analyzed mainly under the assumption that the mean of a sample refractive index matches with the refractive index of its surrounding medium.[6,8–10] The results based on this assumption show that the mean ($\langle r \rangle$) and fluctuation [standard deviation (STD)] of the reflection coefficient (r), $\sigma(r)$, have the same analytical form in the weak disorder limit ($\Delta n \ll \langle n \rangle$). However, reflection statistics have not been thoroughly studied for systems where the mean of the sample refractive index does not match with the refractive index of the surrounding medium.

The STD of the reflectance $\sigma(r)$ exhibits significantly different behaviors in index mismatched systems in the weak disorder limit. The orders of disorder parameters ($\langle (\Delta n)^2 \rangle^{\frac{1}{2}}$ and l_c) in the mean of the reflectance are of the same order in the matched and mismatched systems ($\langle r \rangle \sim \langle \Delta n^2 \rangle l_c$). But the STD of the reflectance in mismatched systems deviates from that of matched systems, having different orders of disorder parameters in $\sigma(r)$ ($\sigma \sim \langle \Delta n^2 \rangle^{1/2} l_c^{1/2}$).

In this paper, we present a detailed theoretical derivation and a physical interpretation of the reflection statistics in index mismatched systems. In Sec. 2, by applying the index mismatched condition, we re-express the Langevin equation with two separate equations: a deterministic equation and a stochastic equation. Then, a short-range correlated random noise is introduced to the refractive index of an optical sample in the weak disorder limit. Finally, the mean and STD of the reflectance are calculated by the perturbative expansions in terms of disorder parameters and boundary index mismatch terms. In Sec. 3, based on the results in Sec. 2, the STDs of the reflection coefficient in matched and mismatched systems are compared, and the physical origin of their difference is explained. A potential application of the technique in mismatched systems is also discussed in the context of extracting the structural disorder parameters in a more efficient way in biological cell systems from the back scattering signal.

2. Theoretical Calculation

2.1. *Langevin equation with boundary refractive indices mismatched condition*

In this subsection, we construct the framework of theoretical derivation by reformulating the stochastic equation for the reflection coefficient with the refractive index mismatched condition. For this, let us first consider a 1D optical dielectric random medium that exists in $0 \leq x \leq L$ and its surrounding medium ($x > L$ or $x < 0$). In the stationary regime, a wavefunction $\psi(x)$ of an electromagnetic field is described by the Maxwell's equation:

$$\frac{d^2\psi}{dx^2} + k^2 n^2(x)\psi = 0, \tag{1}$$

where k is a wave vector in a vacuum and $n(x)$ is the medium refractive index in $0 < x < L$.

When a plane wave with a wave vector k impinges on the random medium from the right as shown in Fig. 1, the wavefunction has a form: $\psi(x) = e^{-ik(x-L)} + R(L)e^{ik(x-L)}$ ($x > L$) and $\psi(x) = T(L)e^{-ikx}$ ($x < 0$), where $R(L)$ and $T(L)$ are the reflection and transmission coefficients of the sample. Based on the invariant imbedding approach,[6] $R(L)$ is described by a stochastic differential equation (i.e., Langevin equation) with an initial condition:

$$\frac{dR(L)}{dL} = 2ikR(L) + \frac{ik}{2}\eta(L)[1 + R(L)]^2, \quad R(L = 0) = 0, \tag{2a}$$

where $\eta(x)$ is a randomly fluctuating function in $0 < x < L$ defined by $\eta(x) = n^2(x) - 1$.

Typically in studies of quantum mechanical or optical systems,[1,6] the potential or refractive index within a random medium is assumed to fluctuate around the potential or mean refractive index level of its surrounding medium. When the surrounding medium is a vacuum, this means $\eta(x)$ has zero mean, i.e., $\langle\eta(x)\rangle = 0$.
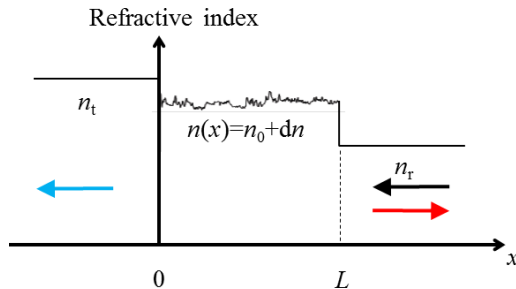

Fig. 1. (Color online) The 1D light scattering by a random optical medium is described. Light is incident on the medium from the right (black arrow), reflected (red arrow) and transmitted (blue arrow). Due to index mismatch at $x = 0$ and L, the light experiences an interference caused by boundary reflections. n_r and n_t are the refractive indices of the surrounding medium in $x > L$ and $x < 0$.

However, in this paper, we consider an optical random sample whose mean refractive index, $n_0 (= \langle n(x) \rangle)$, does not match with the index of its surrounding medium. (See Fig. 1.) Therefore, the effect of this refractive index mismatching at $x = 0$ and L needs to be taken into account in the analysis of the reflection statistics.

To include the effects of an index mismatch, we view the random medium of Fig. 1 as an optical medium with a constant index that is perturbed by weak spatial disorder. Then, the sample medium index can be written as $n(x) = n_0 + \Delta n(x)$, where $n_0 = \langle n(x) \rangle$ and Δn is a deviation around n_0. Accordingly, we can rewrite the random function $\eta(x)$ in Eq. (2a) as a sum of its mean and fluctuation parts:

$$\frac{dR(L)}{dL} = 2ikR(L) + \frac{ik}{2}[(n_0^2 - 1) + n_0\eta_d][1 + R(L)]^2, \tag{2b}$$

where $\eta_d = 2\Delta n + O(\Delta n^2)$. From here, $O(\Delta n^2)$ in η_d is ignored since the weak disorder limit ($\Delta n \ll n_0$) is applied. Now, we consider $R(L)$ in Eq. (2b) as a sum of two separate contributions: reflection due to the boundary index mismatch ($n_0 \neq n_{\text{out}}$) and a perturbation due to weak disorder Δn. Accordingly, we rewrite $R(L)$: $R(L) = R_s(L) + \Delta R(L)$, where $R_s(L)$ is a deterministic reflection coefficient due to the interference between two boundaries ($x = 0$ and L) and $\Delta R(L)$ is the remaining contribution due to Δn. The differential equation of $R_s(L)$ is given in Eq. (2c) by $\Delta n \rightarrow 0$ in Eq. (2b). Then, the stochastic equation for $\Delta R(L)$ can be derived as in Eq. (2d) by subtracting Eq. (2c) from Eq. (2b) (Ref. 6):

$$\frac{dR_s(L)}{dL} = 2ikR_s + \frac{ik}{2}(n_0^2 - 1)[1 + R_s]^2, \tag{2c}$$

$$\frac{d\Delta R(L)}{dL} = 2ik\Delta R + \frac{ik}{2}n_0\eta_d[1 + R_s + \Delta R]^2 + \frac{ik}{2}(n_0^2 - 1)[2\Delta R(1 + R_s) + \Delta R^2], \tag{2d}$$

where $R_s(L = 0) = 0$ and $\Delta R(L = 0) = 0$. Equation (2c) is a simple interference problem with a thin slab geometry of two boundaries ($x = 0$ and L). R_s is well described both as a perturbative form and as a closed form.[11] In Sec. 2.3, the perturbative form is used to calculate the mean and STD of the reflectance.

2.2. *Weak disorder with short-range spatial correlation: Different averages*

Before obtaining the mean and STD of reflectance, it is necessary to consider the stochastic behavior of a sample refractive index. This gives us the ability to calculate the moments of the complex reflection amplitude difference ΔR. To describe the stochastic properties of the refractive index, we introduce a spatially correlated disorder, that is Gaussian color noise with exponential decay, into the optical medium in Fig. 1. Specifically, we adopt an Ornestein–Uhlenbeck stochastic process and establish statistical properties of $\eta_d (= 2\Delta n)$ in Eq. (2b) (Ref. 6):

$$\langle \eta_d \rangle = 0, \quad \langle \eta_d(x)\eta_d(x') \rangle = \Delta^2 \exp(-|x - x'|/l_c), \tag{3a}$$

where l_c is a correlation length of Δn and $\Delta^2 = 4\langle \Delta n^2 \rangle$. In this paper, we assume a short-range correlation, $kl_c < 1$. When η_d satisfies Eq. (3a), its differentiation formula is derived for a functional of η_d (Ref. 6):

$$\frac{\partial}{\partial x}\langle \eta_d(x)\Phi[\eta_d(x)]\rangle = \left\langle \eta_d(x)\frac{\partial}{\partial x}\Phi[\eta_d(x)]\right\rangle - \frac{1}{l_c}\langle \eta_d(x)\Phi[\eta_d(x)]\rangle, \qquad (3b)$$

where Φ is a functional of η_d.

Now differential equations for any moments of $\Delta R(L)$ can be derived based on Eq. (3b). Therefore, the mean and STD of the reflectance can also be calculated (see the next subsection). To assist the calculation, we first calculate $\langle \eta_d \Delta R \rangle$, which is a stochastic contribution of the mean and STD in the leading order of Δ^2. Using Eqs. (2d) and (3b) with $\Phi = \Delta R$ and keeping ΔR up to $O(\eta_d^1)$, a formal expression of $\langle \eta_d \Delta R \rangle$ up to $O(\Delta^2)$ can be written (see Appendix A):

$$\langle \eta_d \Delta R \rangle = \frac{i}{2}n_0 kl_c\Delta^2 \left(\frac{1}{1 - 2ikl_c\beta}\right)\left[1 - l_c\frac{\partial}{\partial L}\right](1 + R_s)^2, \qquad (4)$$

where $\beta = 1 + (n_0^2 - 1)(1 + R_s)/2$. Here, we introduce a phase transformation which simplifies Eq. (2d) by removing $O(\eta_d^0)$ and makes the perturbative calculation straightforward in the next subsection:

$$\Delta R(L) = \Delta Q(L) \cdot e^{2ik\alpha(L)}, \qquad (5)$$

where $\alpha = \int_0^L \beta(L')dL'$. The same relationship exists between R and Q, and between R_s and Q_s. The phase term, $e^{2ik\alpha}$, is used to separate the rapid phase oscillation caused by $2ikR$ in Eqs. (2a)–(2d). When the constant part of the random potential η, $(n_0^2 - 1)$, vanishes, the phase factor $e^{2ik\alpha}$ in Eq. (5) becomes e^{2ikL}, which is a mere phase delay due to a round trip within $[0, L]$ in a vacuum. In this Q representation, Eq. (2d) becomes:

$$\frac{d\Delta Q}{dL} = \frac{i}{2}n_0 k\eta_d e^{-2ik\alpha}[1 + R_s + \Delta Q e^{2ik\alpha}]^2 + \frac{i}{2}k(n_0^2 - 1)(\Delta Q)^2 e^{2ik\alpha}$$

$$= \frac{i}{2}n_0 k\eta_d e^{-2ik\alpha}[1 + R_s]^2 + O(\eta_d^2). \qquad (6)$$

The leading order of η_d in Eq. (6) is $(\eta_d)^1$, whereas that of Eq. (2d) is $(\eta_d)^0$. By using Eq. (5), the expression corresponding to Eq. (4) can be written again up to $O(\Delta^2)$:

$$\langle \eta_d \Delta Q \rangle = \frac{i}{2}n_0 kl_c\Delta^2 \left(\frac{e^{-2ik\alpha}}{1 - 2ikl_c\beta}\right)\left[1 - l_c\frac{\partial}{\partial L}\right](1 + R_s)^2. \qquad (7)$$

Now, the mean and STD of reflectance can be readily derived based on Eq. (7) in the Q representation.

2.3. *Mean and STD of the reflectance*

In this subsection, we calculate the mean and STD of the reflectance perturbatively up to $O(\Delta^2)$. For a perturbative expansion, the mean and variance can be decomposed into deterministic terms and disorder terms by using $R(L) = R_s(L) + \Delta R(L)$.

$$\langle r \rangle = \langle (R_s + \Delta R)(R_s + \Delta R)^* \rangle = r_s + R_s^* \langle \Delta R \rangle + \text{c.c.} + \langle |\Delta R|^2 \rangle, \qquad (8a)$$

$$\sigma^2(r) = \langle r^2 \rangle - (\langle r \rangle)^2 = 2r_s[\langle |\Delta R|^2 \rangle - \langle \Delta R^* \rangle \langle \Delta R \rangle] + R_s^2[\langle \Delta R^{*2} \rangle - \langle \Delta R^* \rangle^2] + \text{c.c.}$$

$$+ 2R_s[\langle \Delta R^* |\Delta R|^2 \rangle - \langle \Delta R^* \rangle \langle |\Delta R|^2 \rangle] + \text{c.c.}$$

$$+ \langle |\Delta R|^4 \rangle - \langle |\Delta R|^2 \rangle^2, \qquad (8b)$$

where $r = |R|^2 = |Q|^2$ and $r_s = |R_s|^2 = |Q_s|^2$. The moments of ΔR, $f_{m,n} = \langle \Delta R^m \Delta R^{*n} \rangle$, satisfy $f_{m,n} \sim \delta_{m,n}$ in the weak disorder limit as discussed elsewhere.[6] Then, Eqs. (8a) and (8b) can be simplified in the R and Q representations:

$$\langle r \rangle \approx r_s + \langle |\Delta R|^2 \rangle = r_s + \langle |\Delta Q|^2 \rangle, \qquad (9a)$$

$$\sigma^2(r) \approx 2r_s \langle |\Delta R|^2 \rangle + \langle |\Delta R|^4 \rangle - \langle |\Delta R|^2 \rangle^2 = 2r_s \langle |\Delta Q|^2 \rangle + \langle |\Delta Q|^4 \rangle - \langle |\Delta Q|^2 \rangle^2. \qquad (9b)$$

From here, we use the Q representation that establishes simpler perturbative expansions. Among the three terms in Eq. (9b), $r_s \langle |\Delta Q|^2 \rangle$ is $O(\Delta^2)$ and the rest of the terms have higher orders in Δ^2. Therefore, $\sigma^2(r) = 2r_s \langle |\Delta Q|^2 \rangle$ in $O(\Delta^2)$. Now Eqs. (9a) and (9b) can be calculated by integrating the differential form of $\langle |\Delta Q|^2 \rangle$:

$$\frac{d\langle |\Delta Q|^2 \rangle}{dL} = \left\langle \left(\frac{d\Delta Q}{dL} \right) \cdot \Delta Q^* \right\rangle + \left\langle \Delta Q \cdot \left(\frac{d\Delta Q^*}{dL} \right) \right\rangle. \qquad (10)$$

Using Eq. (6) and keeping the order of disorder only up to Δ^2, $\langle |\Delta Q|^2 \rangle$ is expressed as follows:

$$\langle |\Delta Q|^2 \rangle = -\frac{i}{2} n_0 k \int_0^L dL' [e^{2ik\alpha^*}(1 + R_s^*)^2 \langle \eta_d \Delta Q \rangle] + \text{c.c.}$$

$$= \frac{1}{4}(n_0 k)^2 l_c \Delta^2 \int_0^L dL' e^{2ik(\alpha^* - \alpha)} \left(\frac{1}{1 - 2ikl_c\beta} \right) (1 + R_s^*)^2 \left[1 - l_c \frac{\partial}{\partial L} \right]$$

$$\times (1 + R_s)^2 + \text{c.c.} \qquad (11)$$

Here, it is important to note that $\sigma^2(r) = 2r_s \langle |\Delta Q|^2 \rangle$ in Eq. (9b) is valid only when R_s is significantly greater than ΔR. For example, $\sigma^2(r) = 2r_s \langle |\Delta Q|^2 \rangle$ is invalid under a complete destructive interference condition which reduces R_s to zero for some periodic ks with a given L. At these periodic ks, $\sigma^2(r)$ tends to the expression in the index matched system[6]: $\sigma^2(r) = \langle |\Delta Q|^4 \rangle - \langle |\Delta Q|^2 \rangle^2 \sim \langle \Delta n^2 \rangle^4 l_c^2$.

To perform the integration in Eq. (11), we consider R_s, which is simply a solution to Eq. (2c). A closed form of R_s is not suitable for performing the integration due to its fractional form. [See (4.27) and (4.30) in Ref. 11]. Therefore, we expand R_s

as a sum of all successive rays that are reflected and transmitted at $x = 0$ and L and approximate R_s by keeping the two lowest order terms[11]:

$$R_s \approx R_s^{(2)} = A_0 + A_1 e^{2in_0 kL}, \tag{12}$$

where $A_0 = (n_r - n_0)/(n_r + n_0)$ and $A_1 = 4 n_r n_0/(n_r + n_0)^2 \cdot (n_0 - n_t)/(n_0 + n_t)$ ($|A_0|$ and $|A_1| < 1$). We also define B_0 as $B_0 = A_0 + 1$ for the calculations below. The approximation in Eq. (12) is accurate when reflections at the boundaries are not very strong ($|A_0|$ and $|A_1| \ll 1$) as in biological samples. In Appendix A, the integration of Eq. (11) is performed expanding the integrand in terms of A_1 with a periodic condition of k: $2n_0 kL = \pi N$ (N = even integers). Then, $\langle |\Delta Q|^2 \rangle$ is approximated in $O(\Delta^2)$:

$$\langle |\Delta Q|^2 \rangle = \frac{1}{2}(n_0 k_N)^2 l_c \Delta^2 \frac{L}{1 + 4(\rho k_N l_c)^2} F(n_0, n_r, n_t), \tag{13}$$

where $2n_0 k_N L = \pi N$ ($N = 2, 4, 6, \ldots$), $\rho = 1 + (n_0^2 - 1)B_0/2$, $F(n_0, n_r, n_t) = e^{\frac{(n_0^2-1)A_1}{n_0}} I_0(|\frac{(n_0^2-1)A_1}{n_0}|) \times B_0^4$ and I_0 is a modified Bessel function of the first kind. Accordingly, the mean and STD in Eqs. (9a) and (9b) are expressed by Eq. (13):

$$\langle r \rangle \approx r_s + \langle |\Delta Q|^2 \rangle$$

$$= (A_0^2 + A_1^2 + 2A_0 A_1 \cos x_N) + \frac{1}{2}(n_0 k_N)^2 l_c \Delta^2 \frac{L}{1 + 4(\rho k_N l_c)^2} F(n_0, n_r, n_t)$$

$$= (A_0 + A_1)^2 + \frac{1}{2}(n_0 k_N)^2 l_c \Delta^2 \frac{L}{1 + 4(\rho k_N l_c)^2} F(n_0, n_r, n_t), \tag{14}$$

$$\sigma^2(r) \approx 2r_s \langle |\Delta Q|^2 \rangle$$

$$= 2(A_0^2 + A_1^2 + 2A_0 A_1 \cos x_N) \times \frac{1}{2}(n_0 k_N)^2 l_c \Delta^2 \frac{L}{1 + 4(\rho k_N l_c)^2} F(n_0, n_r, n_t)$$

$$= (A_0 + A_1)^2 \cdot (n_0 k_N)^2 l_c \Delta^2 \frac{L}{1 + 4(\rho k_N l_c)^2} F(n_0, n_r, n_t), \tag{15}$$

where $x_N = 2n_0 k_N L = \pi N$ ($N = 2, 4, 6, \ldots$).

Finally, we compare the above theoretical result with direct numerical simulation data obtained from iteration equations (2a) and (2b) with the initial condition $R(L = 0) = 0$. For the simulation, 10,000 realizations of correlated random noise, η, were generated for each parameter set (Δ, l_c, n_0, k and L). Then, the corresponding reflectance is calculated with those realizations based on Eq. (2a). During the calculation, the Riccati form of Eq. (2a) was linearized for stable numerical integration. In Figs. 2 and 3, STDs based on Eq. (15) are plotted with the exact numerical simulation data in two disorder parameter spaces, $\langle \Delta n^2 \rangle^{1/2}$ and l_c, to verify the accuracy of the analytical calculations. In Fig. 2, we observe a clear difference between index mismatched systems and a matched system in terms of the order of $\langle \Delta n^2 \rangle^{1/2}$ in STD (see Sec. 3).
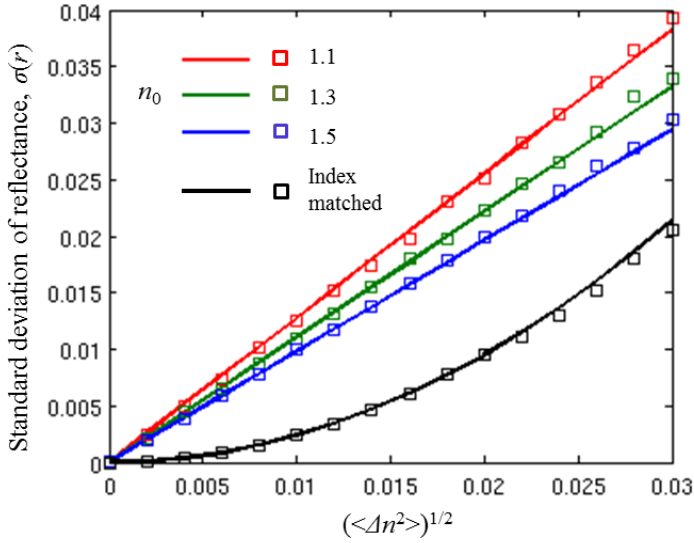
Fig. 2. (Color online) STD versus $\langle \Delta n^2 \rangle^{1/2}$ ($= \Delta/2$) of reflection statistics. A set of parameters is: $n_r = 1$, $n_t = 1.5$, $n_0 = 1.1$–1.5, $l_c = 20$ nm, $\lambda = 550$ nm and $2L = (1/2 \cdot \lambda/n_0) \cdot 20$. The colored solid lines are theoretical values based Eq. (15) and the colored markers are simulational data points. The black markers and black solid line are simulation data points and theoretical calculations based on Eq. (16b) with $n_r = 1$, $n_t = 1.0$ and $n_0 = 1.0$ (an index matched system) shown for comparison.
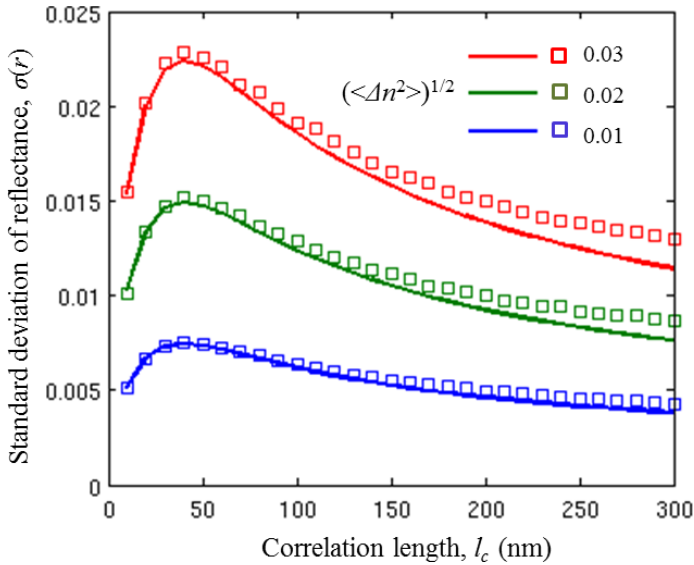

Fig. 3. (Color online) STD versus l_c of reflection statistics. A set of parameters is: $n_r = 1$, $n_t = 1.5$, $n_0 = 1.3$, $\langle \Delta n^2 \rangle^{1/2} = 0.01$–$0.03$, $\lambda = 650$ nm and $2L = (1/2 \cdot \lambda/n_0) \cdot 16$. The solid lines are theoretical values based Eq. (15) and the markers are simulational data points.

3. Discussion: Analysis of Mean and STD of Reflection Coefficient r and its Applications

The reflection statistics have been well established for index matched systems. The mean and STD of the reflection coefficient for index matched systems exhibit the same behavior in the weak disorder limit [Eq. (4.56) in Ref. 6]:

$$\langle r_{\text{mat}} \rangle = L/\xi = \frac{1}{2} k^2 l_c \Delta^2 \frac{L}{1 + 4(kl_c)^2}, \tag{16a}$$

$$\sigma(r_{\text{mat}}) \approx \langle r_{\text{mat}} \rangle, \tag{16b}$$

where the inverse of localization length is defined as: $\xi^{-1} = 2^{-1} k^2 l_c \Delta^2 / [1 + (2kl_c)^2]$ and $L \ll \xi$.

$\langle r \rangle$ of a mismatched system in Eq. (14) has essentially the same stochastic behavior as Eq. (16a). This is because the interference between boundary reflection and disorder scattering, $Q_s^* \Delta Q + $c.c., in Eq. (8a) vanishes upon ensemble averaging. Therefore, Eq. (14) is an additive combination of deterministic boundary interference r_s and stochastic contribution $\langle |\Delta Q| \rangle^2$. Note that $\langle |\Delta Q|^2 \rangle$ exhibits the same behavior as in Eq. (16a):

$$\langle |\Delta Q|^2 \rangle \sim L/\xi \sim \frac{1}{2} (n_0 k_N)^2 l_c \Delta^2 \frac{L}{1 + 4(\rho k_N l_c)^2}.$$

Here, the correction to the wave number ($k \to n_0 k$) is due to the mean refractive index, and the rest of the n_0-dependent terms, ρ and $F(n_0)$, in Eq. (14) reduce to one as $n_0 \to 1$.

On the other hand, $\sigma(r)$ in Eq. (15) is described as a multiplicative combination of r_s and $\langle |\Delta Q|^2 \rangle$. This leads to a drastic deviation of $\sigma(r)$ from $\sigma(r_{\text{mat}})$ in terms of the order of disorder parameters as in Eqs. (17a) and (17b). The orders of Δ and l_c decrease half from $\sigma(r_{\text{mat}})$ to $\sigma(r)$, and $\sigma(r)$ has a deterministic term $(2r_s)^{1/2}$ as a prefactor that modulates the amplitude of $\sigma(r)$ as demonstrated in Fig. 2. The origin of this deviation is the interference between boundary reflection and disorder scattering, $Q_s^* \Delta Q + $c.c. in r of Eq. (8b):

$$\sigma(r_{\text{mat}}) \sim \Delta^2 \frac{k^2 l_c L}{1 + 4(kl_c)^2} \sim \Delta^2 (kl_c)^1 \cdot (kL)^1, \tag{17a}$$

$$\sigma(r) \approx \sqrt{2r_s} \sqrt{\langle |\Delta Q|^2 \rangle}$$

$$\sim \sqrt{2(A_0^2 + A_1^2 + 2A_0 A_1 \cos x_N)} \cdot \Delta \sqrt{\frac{k_N^2 l_c L}{1 + 4(n_0 k_N l_c)^2}}$$

$$\sim \sqrt{2r_s} \cdot \Delta^1 (k_N l_c)^{1/2} (k_N L)^{1/2}, \tag{17b}$$

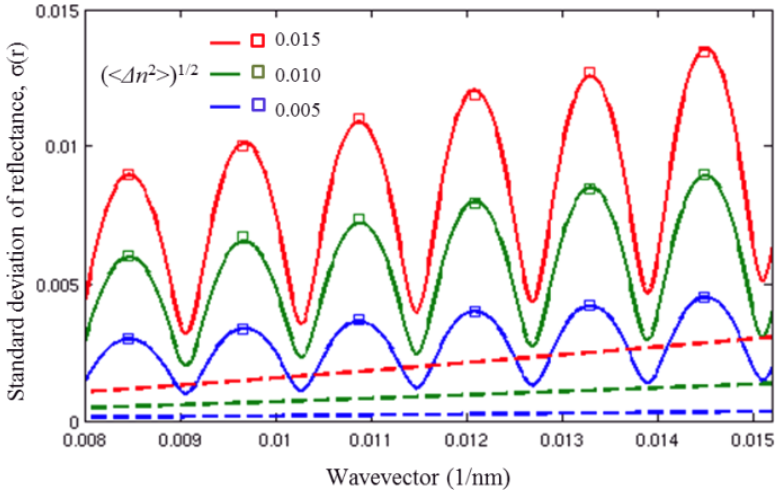where $x_N = 2n_0 k_N L = \pi N$ ($N = 2, 4, 6, \ldots$).

Fig. 4. (Color online) $\sigma(r)$ versus k of the reflection statistics. A set of parameters for refractive indices for outside medium and the sample: $n_r = 1$, $n_t = 1.5$, $n_0 = 1.3$, $l_c = 20$ nm, $\langle \Delta n^2 \rangle^{1/2} = 0.005$–$0.015$, $\lambda = 400$–800 nm and $L = 2\mu$. The solid lines are simulation data and the markers are theoretical calculation based Eq. (15). For the theoretical calculation, a condition $2L = (1/2 \cdot \lambda_N/n_0) \cdot N$ (N = even integers) was used, where $k_N = 2\pi/\lambda_N$. The dashed lines are theoretical values based on Eq. (16b) with $n_r = 1$, $n_t = 1.0$, $n_0 = 1.0$ (an index matched system) and the same $\langle \Delta n^2 \rangle^{1/2}$ values for each color. The number of generated realizations is 10,000 for each k.

3.1. Enhancements of STD of the reflectance ($\sigma(r)$)

To analyze the effect of $(2r_s)^{1/2}$ and decreased orders of Δ and l_c in Eq. (17b), STDs of several Δs are presented for both mismatched and matched systems in Fig. 4. First, $\sigma(r)$ values show a pseudo-periodic and 'amplified' behavior in the k domain (solid lines of simulation data), whereas $\sigma(r_{\text{mat}})$ values show a small monotonic increase as k increases. The periodic modulation is driven by the boundary interference $(2r_s)^{1/2}$ and, therefore, $\sigma(r)$ is maximally amplified around maxima of $(2r_s)^{1/2}$. Second, the decreased orders of disorder parameters ($\Delta^2(kl_c) \rightarrow \Delta(kl_c)^{1/2}$) improve the sensitivity of $\sigma(r)$ to disorder parameters (Δ and l_c) in the weak disorder and short-range correlation limits. In Fig. 4, we find that $\sigma(r)/\sigma(r_{\text{mat}}) > 10$ for $\langle \Delta n^2 \rangle^{1/2} = 0.005$ and $(\sigma(r)_{0.010} - \sigma(r)_{0.005})/(\sigma(r_{\text{mat}})_{0.010} - \sigma(r_{\text{mat}})_{0.005}) \sim 5$ around maxima of $(2r_s)^{1/2}$. This clearly is an example of the enhancement of the STD of reflection coefficient by the value of the refractive index mismatch.

3.2. Applications

This periodic amplification and enhanced sensitivity of $\sigma(r)$ has strong potential applications in the context of disorder information that are extracted efficiently from weakly spatial disorder fluctuations embedded in a higher refractive index media, such as biological cells. In particular, it can be a physical basis for various techniques in biophotonics since the index mismatch condition and weak disorder limit are

readily applicable to biological cells, which have mean refractive index $n_0 \sim 1.3$–1.5 and spatial refractive index fluctuations $\langle \Delta n^2 \rangle^{1/2} \sim 0.01$–$0.1$. The light backscattering by weakly disordered samples such as biological cells can be approximated as a quasi-1D parallel multichannel problem.[12] It was recently demonstrated that quasi-1D multichannel backscattering statistics of biological cells are sensitive to changes in nanoscale disorder properties.[13–16] Based on the first-order Born approximation, the reflection statistics were also theoretically analyzed in three-dimensional (3D) backscattering, which can be applied to a 1D case as well.[17] It has been demonstrated that the applications of the light backscattering statistics can be applied for early precancer screening by detecting changes in the refractive index fluctuation in cells, corresponding to the progress of carcinogenesis in different types of cancer. By enhancing the STDs of reflected signals via controlling $(2r_s)^{1/2}$ factor, we expect to achieve a higher detection sensitivity to weak disorder parameters of the nanoscale refractive index fluctuations in biological cells.

4. Conclusion

In conclusion, we performed a theoretical analysis to understand the effects of a refractive index mismatching between a weakly disordered optical sample and its surrounding medium on the reflection statistics. By separating the interference induced by the sample boundaries and the disorder scattering, we show that the mean of the reflectance $\langle r \rangle$ in a mismatched system has the same stochastic behavior as that of a matched system. On the other hand, the STD of the reflection coefficient of a mismatched system $\sigma(r)$ has lower orders of disorder parameters compared to that of a matched system, and the boundary interference term $(2r_s)^{1/2}$ functions as a prefactor. In particular, the prefactor $(2r_s)^{1/2}$ causes a periodic amplification of the disorder signal in the spectral domain. The origin of this difference between $\sigma(r)$ and $\sigma(r_{\mathrm{mat}})$ is the interference between the boundary reflection and disorder scattering in $\sigma^2(r)$. From a technological point of view, the boundary index mismatch condition provides a handle for enhancing the signal from the disorder part of the refractive index fluctuations buried in a higher uniform refractive index media. The intracellular mass density fluctuations or refractive index fluctuations increase with the progress of carcinogenesis,[13–16] therefore, probing weakly structural disorder parameters in biological cells with high detection sensitivity by tuning the index mismatch condition has strong potential applications in early cancer screening, as well as for other biophotonics applications that probe mass density variations or refractive index fluctuations.

Appendix A. Derivations of Eqs. (4) and (13)

A.1. *Derivation of Eq. (4)*

First, to derive Eq. (4), we use Eq. (3b) under the assumption of an Ornestein–Uhlenbeck stochastic process. Then, a differentiation formula for ΔR can be established:

$$\left[1 + l_c \frac{\partial}{\partial L}\right] \langle \eta_d \Delta R \rangle = \left\langle \eta_d l_c \left(\frac{\partial}{\partial L} \Delta R\right) \right\rangle. \tag{A.1}$$

Using Eq. (2d) and maintaining the order of η_d only up to $O(\eta_d^2)$ on the right-hand side of Eq. (A.1), we obtain:

$$\left[1 + l_c \frac{\partial}{\partial L}\right] \langle \eta_d \Delta R \rangle$$

$$= \left\langle \eta_d l_c \left(2ik\Delta R + \frac{ik}{2} n_0 \eta_d [1 + R_s + \Delta R]^2 + \frac{ik}{2}(n_0^2 - 1)[2\Delta R(1 + R_s) + \Delta R^2]\right)\right\rangle$$

$$= 2ikl_c\beta\langle \eta_d \Delta R\rangle + \frac{i}{2}n_0 kl_c \Delta^2 (1 + R_s)^2, \tag{A.2}$$

where $\beta = 1 + (n_0^2 - 1)(1 + R_s)/2$. Multiplying Eq. (A.2) by $[1 - l_c \frac{\partial}{\partial L}]$, we obtain:

$$\langle \eta_d \Delta R \rangle - l_c^2 \frac{\partial^2}{\partial L^2} \langle \eta_d \Delta R \rangle$$

$$= \left[1 - l_c \frac{\partial}{\partial L}\right] \left(2ikl_c\beta\langle \eta_d \Delta R\rangle + \frac{i}{2}n_0 kl_c \Delta^2 (1 + R_s)^2\right)$$

$$= 2ikl_c\beta\langle \eta_d \Delta R\rangle - 2ikl_c^2 \frac{\partial}{\partial L}\beta\langle \eta_d \Delta R\rangle + \frac{i}{2}n_0 kl_c \Delta^2 \left[1 - l_c \frac{\partial}{\partial L}\right](1 + R_s)^2. \quad \text{(A.3)}$$

By rewriting Eq. (A.3) in orders of kl_c, Eq. (A.3) is expressed as follows:

$$(1 - 2ikl_c\beta)\langle \eta_d \Delta R\rangle + \hat{O}\langle \eta_d \Delta R\rangle = \frac{i}{2}n_0 kl_c \Delta^2 \left[1 - l_c \frac{\partial}{\partial L}\right](1 + R_s)^2, \tag{A.4}$$

where $\hat{O}$ is an operator whose lowest order is $(kl_c)^2$ and the lowest order of kl_c in $\langle \eta_d \Delta R\rangle$ is $(kl_c)^1$. Approximating Eq. (A.4) by ignoring $\hat{O}$ in the short spatial correlation limit, we obtain:

$$\langle \eta_d \Delta R\rangle = \frac{i}{2}n_0 k(\Delta^2 l_c)\left(\frac{1}{1 - 2ikl_c\beta}\right)\left[1 - l_c \frac{\partial}{\partial L}\right](1 + R_s)^2. \tag{A.5}$$

A.2. *Derivation of Eq. (13)*

Secondly, to derive $\langle|\Delta Q|^2\rangle$ in Eq. (13), we first insert Eq. (12) into Eq. (11) and change the variable L by $x = 2n_0 kL$. This gives:

$$\langle |\Delta Q|^2 \rangle = -\frac{i}{2} n_0 k \int_0^L dL' [e^{2ik\alpha^*} (1 + R_s^*)^2 \langle \eta_d \Delta Q \rangle] + \text{c.c.}$$

$$= \frac{1}{8} (n_0 k l_c) \Delta^2 \int_0^x dx' e^{2ik(\alpha^* - \alpha)} \left(\frac{1}{1 - 2ikl_c\rho - ikl_c(n_0^2 - 1) A_1 e^{ix'}]} \right)$$

$$\times (B_0 + A_1 e^{-ix'})^2 \left[1 - 2n_0 k l_c \frac{\partial}{\partial L} \right] (B_0 + A_1 e^{ix'})^2 + \text{c.c.}, \qquad (A.6)$$

where $\rho = 1 + (n_0^2 - 1)B_0/2$. The exponent of $e^{2ik(\alpha*-\alpha)}$ in Eq. (A.6) can be integrated:

$$2ik(\alpha^* - \alpha) = -ik(n_0^2 - 1) \int_0^L dL' (R_s - R_s^*)$$

$$= -ik(n_0^2 - 1) \int_0^L dL' [(A_0 + A_1 e^{2in_0 kL}) - (A_0 + A_1 e^{-2in_0 kL})]$$

$$= \frac{(n_0^2 - 1) A_1}{n_0} (1 - \cos x). \qquad (A.7)$$

Equation (A.6) can, then, be expanded in terms of A_1 to perform the integral because $A_1 < 1$. By maintaining the order of A_1 only up to $(A_1)^1$, we obtain:

$$\langle |\Delta Q|^2 \rangle = \frac{1}{8} n_0 k l_c \Delta^2 e^{\frac{(n_0^2 - 1) A_1}{n_0}} \int_0^x dx' e^{\frac{-(n_0^2 - 1) A_1}{n_0} \cos x} \left(\frac{1 + \frac{i(kl_c)(n_0^2 - 1) A_1 e^{ix'}}{1 - 2i\rho kl_c}}{1 - 2i\rho kl_c} \right)$$

$$\times (B_0 + A_1 e^{-ix'})^2 \left[1 - l_c \frac{\partial}{\partial L} \right] (B_0 + A_1 e^{ix'})^2 + \text{c.c.}$$

$$= \frac{1}{8} n_0 k l_c \Delta^2 e^{\frac{(n_0^2 - 1) A_1}{n_0}} \int_0^x dx' e^{\frac{-(n_0^2 - 1) A_1}{n_0} \cos x} \frac{1}{1 + 4(\rho kl_c)^2}$$

$$\times \left[\begin{array}{l} 2B_0^4 + (B_0^3 A_1) \left(8(1 + 2n_0\rho(kl_c)^2) - \dfrac{8B_0(n_0^2 - 1)\rho(kl_c)^2}{(1 + 4(\rho kl_c)^2)} \right) \cos x' \\[3mm] + (B_0^3 A_1) \left(8n_0(kl_c) - \dfrac{2B_0(n_0^2 - 1)(kl_c)(1 - 4(\rho kl_c)^2)}{(1 + 4(\rho kl_c)^2)} \right) \sin x' \end{array} \right].$$

$$(A.8)$$

The integral in Eq. (A.8) can be performed by a set of known integral formulas in Eq. (A.9) for $x = 2n_0 k_N L = N\pi$ when N is an even integer:

$$\int_0^\pi dx \cdot e^{p \cos x} \cos mx = \pi \cdot I_m(|p|), \qquad \int_0^{N\pi} dx \cdot e^{p \cos x} \cos mx = N\pi \cdot I_m(|p|),$$

$$\int_0^{2\pi} dx \cdot e^{p \cos x} \sin mx = 0,$$

$$(A.9)$$

where I_m is a modified Bessel function of the first kind and m is an integer. Then, $\langle|\Delta Q|^2\rangle$ can be expressed up to $O(A_1^1)$ and $O(\Delta^2)$:

$$\langle|\Delta Q|^2\rangle$$

$$= \frac{1}{2}n_0^2 k_N^2 l_c \Delta^2 \frac{L}{1+4(\rho k_N l_c)^2} e^{\frac{(n_0^2-1)A_1}{n_0}}$$

$$\times \left[\begin{array}{l} B_0^4 I_0\left(\left|\frac{(n_0^2-1)A_1}{n_0}\right|\right) \\ +4B_0^3 A_1 \left\{1+2n_0\rho(k_N l_c)^2 - \frac{(n_0^2-1)B_0\rho(k_N l_c)^2}{1+4(\rho k_N l_c)^2}\right\} I_1\left(\left|\frac{(n_0^2-1)A_1}{n_0}\right|\right) \end{array} \right].$$

$$(A.10)$$

By ignoring the second term (leading order, A_1^1) in the bracket, Eq. (A.10) can be approximated:

$$\langle|\Delta Q|^2\rangle = \frac{1}{2}(n_0 k_N)^2 l_c \Delta^2 \frac{L}{1+4(\rho k_N l_c)^2} F(n_0, n_r, n_t), \qquad (A.11)$$

where $F(n_0, n_r, n_t) = e^{\frac{(n_0^2-1)A_1}{n_0}} I_0(|\frac{(n_0^2-1)A_1}{n_0}|) B_0^4$.

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