Final Exam
AST6112: Fall 2003
December 9, 2013

Full Marks 120

Weight towards grade: 30%

Full Name:

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1. Short answer type questions based on student presentations. (Total 15)

   (a) Choose one. Typical TTV magnitudes for the KPCs are of $\sim$ 1

   i. seconds.
   ii. minutes.
   iii. hours.
   iv. days.
   v. months.

   (b) Choose one. According to the analysis of Adams 2010, the birth cluster of the Sun likely had about N stars, where N is between $\sim$ 1

   i. 10 and 10^2.
   ii. 10^2 and 10^3.
   iii. 10^3 and 10^4.
   iv. 10^4 and 10^5.
   v. 10^5 and 10^6.

   (c) Name two moons of Saturn. 2

   Enceladous, Titan (For Example)

(d) Give an example of a known exo-planetary system with planets in the habitable zone. 1

Gliese 581

(e) Circle all that are correct. Eccentric giant planets are likely created via 3

i. planet-planet scattering.
ii. Kozai oscillations due to a distant highly inclined companion.
iii. Secular chaos.
iv. Type I migration.
v. Type II migration.
vi. planet-planetesimal scattering.

(f) Choose one that is the most applicable. A clear observational signature of a debris ring around a star is 1

i. an UV excess.
ii. an IR excess.
iii. a bimodal spectral energy distribution.

(g) Choose all that are correct. 3

i. The size distribution in the TNOs can be well approximated by a power law.
ii. The size distribution in the Asteroids can be well approximated by a power law.
iii. The typical sizes of the TNOs are expected to be larger than the asteroids.
iv. The TNO binaries have larger semimajor axes compared with asteroid binaries.
v. Size distributions for both asteroids and TNOs are likely results of collisional modifications.

(h) Choose one. Obliquity of Venus is about 1

i. 7.31°.
ii. 54.5°.
iii. 177.4°.
iv. 97.86°.

(i) In a standard, steadyly accreting, Shakura-Sunyaev disk around a star of mass $M_\star$, at a heliocentric distance r the sound speed $c_s \propto M_\star^{3/20} r^{-9/20}$, and the surface density $\Sigma \propto M_\star^{1/5} r^{-3/5}$. If two stars have a mass ratio $M_1/M_2 = 2$ and all else is kept fixed, find the ratio of the heliocentric distances r_1/r_2 where the disks can become Toomre unstable. 2

$$Q = \frac{c_s \Omega}{\pi G \Sigma}$$

$$c_s \sim m^{\frac{3}{20}} \cdot r^{-\frac{9}{20}}$$

$$\Sigma \sim m^{1/5} r^{-3/5}$$

$$\Omega \sim m^{1/2} r^{-3/2}$$

$$\rightarrow \quad Q \sim m^{\frac{9}{20}} r^{-\frac{27}{20}}$$

For GI in star 1 & star 2,

$$Q_1 = Q_2 \equiv 1$$

$$\Rightarrow \quad m_1^{\frac{9}{20}} r_1^{-\frac{27}{20}} = m_2^{\frac{9}{20}} r_2^{-\frac{27}{20}}$$

$$\Rightarrow \quad \left(\frac{m_1}{m_2}\right) = \left(\frac{r_1}{r_2}\right)^3$$

$$\Rightarrow \quad \frac{r_1}{r_2} = \left(\frac{m_1}{m_2}\right)^{1/3} = 2^{1/3}.$$

2. In this problem we will use simple assumptions to infer expected terrain features on self gravitating bodies. (Total 20)

(a) Assuming hydrostatic equilibrium find an expression for the expected ratio of heights, $H_{\text{Mars}}/H_{\oplus}$, between the highest stable mountains on Mars (H_{Mars}) and Earth ($H_{\oplus}$). For simplicity, assume that the average densities (ρ) of the rocks in the Earth and Mars mountains are equal. Further assume that the tensile strengths of the Earth and Mars surfaces are equal. Also assume that ρ is not z-dependent. 12

(b) Show that if you assume that gravitational acceleration g is *not* z-dependent, then your estimated ratio will have an error $\sim \left(\frac{H_{\oplus}}{R_{\oplus}} - \frac{H_{\text{Mars}}}{R_{\text{Mars}}}\right)$. Here, $R_{\oplus}$ and R_{Mars} are the average radii of Earth and Mars, respectively. 5

(c) Based on your answers in the previous parts, which term in the error derived above will dominate? 3

Hint: You may want to use the following identity based on binomial expansion.
$(1+x)^n = 1 + nx + \mathcal{O}(x^{\geq 2})$ for all rational n.
You may also need to assume that $H/R \ll 1$.

2. (a) $$\frac{dP}{dz} = -g\rho$$

$$\Rightarrow dP = -\frac{GM\rho}{r^2}\frac{1}{\left(1+\frac{z}{r}\right)^2}dz$$

r is the radius at surface (base).

Assuming, $\frac{z}{r} \ll 1$ we can expand up to the leading order in $\left(\frac{z}{r}\right)$.

$$dP = -\frac{GM\rho}{r^2}\cdot\left(1-\frac{2z}{r}\right)dz$$

$$\Rightarrow dP = -\frac{GM\rho}{r^2}dz + \frac{2GM\rho}{r^3}z\,dz \qquad (1)$$

Integrating both sides of (1) with appropriate limits

$$\int_{P_b}^{0} dP = -\int_0^H \frac{GM\rho}{r^2}\,dz + \int_0^H \frac{2GM\rho}{r^3}\,z\,dz$$

where, P_b is the pressure at the base of the mountain.

$$\Rightarrow \quad P_b = \frac{GM\rho H}{r^2} - \frac{GM\rho H}{r^2}\left(\frac{H}{r}\right).$$

$$= \frac{GM\rho H}{r^2}\left[1 - \frac{H}{r}\right].$$

Neglecting the z-dependance in g ~~we get~~ and using the tensile strength of surface the same,

$$\cancel{\frac{H_M}{H_\oplus}} \neq$$

$$\frac{\cancel{G}M_M \cancel{\rho} H_M}{R_M^2}\left[1 - \frac{H_M}{R_M}\right] = \frac{\cancel{G}M_\oplus \cancel{\rho} H_\oplus}{R_\oplus^2}\left[1 - \frac{H_\oplus}{R_\oplus}\right]$$

$$\frac{H_M}{H_\oplus} = \frac{M_\oplus R_M^2}{M_M R_\oplus^2}\left[\frac{1 - \frac{H_\oplus}{R_\oplus}}{1 - \frac{H_M}{R_M}}\right]$$

$$\Rightarrow \quad \frac{H_M}{H_\oplus} \approx \frac{M_\oplus R_M^2}{M_M R_\oplus^2}\left[\left(1 - \frac{H_\oplus}{R_\oplus}\right)\left(1 + \frac{H_M}{R_M}\right)\right] \qquad \text{For } \frac{H_M}{R_M} \ll 1$$

$$\Rightarrow \quad \frac{H_M}{H_\oplus} \approx \frac{M_\oplus R_M^2}{M_M R_\oplus^2}\left[1 - \frac{H_\oplus}{R_\oplus} + \frac{H_M}{R_M} - \text{Higher order terms in } \frac{H}{R}\right]$$

$$\Rightarrow \quad \frac{H_M}{H_\oplus} \simeq \frac{M_\oplus R_M^2}{M_M R_\oplus^2}\left[1 - \left(\frac{H_\oplus}{R_\oplus} - \frac{H_M}{R_M}\right)\right]. \qquad (2)$$

Putting values we get

$$\frac{H_M}{H_\oplus} \simeq 2.6$$

(neglecting the small terms after 1)
i.e. assuming g is not z-dependent.

b) From equation (2) derived in (a).

Error is $\sim \left(\frac{H_\oplus}{R_\oplus} - \frac{H_M}{R_M} \right)$.

c) $\frac{R_M}{R_\oplus} \approx 0.5$ $\qquad$ $\frac{H_M}{H_\oplus} \approx 2$

$\therefore \quad \frac{H_M}{R_M} \approx 4 \frac{H_\oplus}{R_\oplus}$. So the Mars term should dominate.

$$\frac{dp}{dz} = -g\rho \qquad \qquad d(R+z) = dz.$$

$$\rightarrow dp = -\frac{GM}{(R+z)^2}\rho\, dz$$

$$\rightarrow dp = -\frac{GM\rho\, d(R+z)}{(R+z)^2}$$

$$\rightarrow \int_{P_b}^{0} dP = -\int_{R}^{R+H} \frac{GM\rho\, d(R+z)}{(R+z)^2}$$

$$\rightarrow -P_b = -\frac{GM\rho}{-1} \left. \frac{1}{(R+z)} \right|_{R}^{R+H}$$

$$\rightarrow P_b = GM\rho \left[\frac{1}{R+H} - \frac{1}{R} \right].$$

3. Please note that this question should be taken with humor and no religious message is intended. This calculation is well known and first appeared in *Applied Optics* in the 70's. We base our problem on two passages from the Bible. (Total 20)

"Moreover, the light of the moon shall be as the light of the sun and the light of the sun shall be sevenfold as the light of seven days." - Isaiah 30:26, description of the Heaven.

"But the fearful and unbelieving... shall have their part in the lake which burneth with fire and brimstone." - Revelations 21:8, description of Hell.

(a) Derive an expression for the equilibrium temperature of the Earth as a function of albedo A_b, emissivity ϵ, radius $R_\oplus$, and distance from the Sun r. 10

(b) The first statement quoted above essentially means that Heaven receives $7 \times 7 = 49$ times more radiation flux compared to the Earth. What is the equilibrium temperature of Heaven assuming $A_b = 0.4$, $\epsilon = 1$, and that Heaven's radius is $= R_\oplus$? 8

(c) According to the quoted information, which is hotter, Heaven or Hell? 2

Hint: Boiling point of brimstone (Sulfur) is 444.6° C. Hell has a lake of that.

3. (a) Incident flux

$$P_{in} = (1 - A_b)\,\frac{49 L_\odot}{4\pi r^2}\,\pi R_\oplus^2$$

where r is 1AU, the Earth-Sun distance

$$P_{out} = 4\pi R_\oplus^2\,\epsilon\sigma T^4$$

at equilibrium $P_{in} = P_{out}$ & $T = T_{eq}$

$$\therefore\ T_{eq} = \left(\frac{49(1-A_b)L_\odot}{16\pi\epsilon\sigma r^2}\right)^{1/4}$$

8

(b) Putting values we get, $T_{eq} = 651\ K = 378\,^\circ c$

c) Hell must be at or below 445°c.

So Hell is hotter.

Note: If you assume a lower A_b value, you can easily change this answer. eg. with $A_b = 0$, Heaven T_{eq} becomes $\approx 467°C$. Similarly, reducing ϵ, you can make Heaven hotter than Hell. Of course arguments can be made that may be in Hell pressure is so high that S can still be liquid at a much higher temperature. In any case, neither Hell nor Heaven seem to be very comfortable. You would hope Heaven has very powerful AC.

This calculation can be made more rigorous using dP&L (2nd Ed.) 3.3.2 concepts. Here one must realize that likely the lake is on the surface of Hell. So the equilibrium temperature could be quite lower.

$\Omega = 2.816377 \times 10^{-4} s^{-1}$

4. For this test, assume that ISON has an angular velocity $\cancel{\Omega = 2.8 \times 10^{-4}\,\mathrm{s}^{-1}}$ at its perihelion $q = 0.01\,\mathrm{AU}$. Also assume that its orbit is not perturbed by any of the solar system planets. Find the following properties of ISON. (Total 30)

   (a) What is the orbital eccentricity? 10

   (b) What is the aphelion distance? 5

   (c) What is the angular speed at aphelion? 5

   (d) What is the period? 5

   (e) If right after the perihelion passage the apparent brightness of ISON is $\sim 3 \times 10^{-12}$ erg s^{-1} cm^{-2}, find an upper limit of its brightness at aphelion assuming no mass is lost. 5

Reminders:
(i) Recall Kepler's laws.
(ii) The speed at any r along an ellipse is given by $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$, where, a is the semimajor axis.
(iii) The apparent brightness of a comet depends both on the distance from the star and the distance from the Earth.
(iv) Typically, the brightness decreases with distance from the star faster than $r_{\odot}^{-2}$.
Hint: If, for some reason, you could not find the value of eccentricity in problem (a), assume $e = 0.999995$ for the later parts. The values you obtain should be very similar to the real ISON properties with the exception that the comet is probably on an ejection trajectory after perihelion passage.

a) $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$

at perihelion,

$$v_q^2 = GM_\odot\left(\frac{2}{q} - \frac{1}{a}\right)$$

$$\Rightarrow \Omega^2 q^2 = GM_\odot\left(\frac{2}{q} - \frac{1}{a}\right)$$

$$\Rightarrow \frac{1}{a} = \frac{2}{q} - \frac{\Omega^2 q^2}{GM_\odot}$$

$$\Rightarrow a = \left(\frac{2}{q} - \frac{\Omega^2 q^2}{GM_\odot}\right)^{-1}$$ Putting in values we get

$$= 1882.04 \text{ AU}$$

$e \simeq 0.999995$

b) $r_{ap} = a(1+e) = 3764.1$ AU

c) $r^2 \dot{\theta}$ = constant

$$\frac{\dot{\theta}_{peri}}{\dot{\theta}_{ap}} = \frac{r_{ap}^2}{q^2} \Rightarrow \dot{\theta}_{ap} = \dot{\theta}_{peri}\frac{q^2}{r_{ap}^2} = \cancel{1.987781}\ s^{-1}$$

$$7.951 \times 10^{-11} s^{-1}$$

d) $P^2 \propto a^3 \rightarrow \frac{P^2}{P_\oplus^2} = \frac{1882.04^3}{1^3}$

$$\rightarrow P = a^{3/2} \text{ yr} = 81647.76 \text{ yr.}$$

e) $B \propto \frac{1}{r_\odot^{2+\delta} r_\Delta^2}$

$$\frac{B_q}{B_{ap}} \simeq \frac{r_{ap}^{2+\delta} \cdot (r_{ap}-1)^2}{q^{2+\delta} \, 1\,AU^2}$$

The upper limit is if $\delta = 0$.

$\therefore B_{ap} \leq B_q \cdots$ —

Putting values,

$B_{ap} \leq 1.5 \times 10^{-30}$ erg s^{-1} cm^{-2}.

5. Through the following questions we will derive the atmospheric structure of Earth, and figure out what happens to a meteorite during its fall. (Total 35)

   (a) What are the two major types of meteorites? 2

   (b) What are the origins of the top three most numerous meteorites on Earth? 3

   (c) What is the dominant heat transfer mechanism in Earth's troposphere? 2

   (d) Find an expression for an altitude (z)-dependent temperature $T(z)$. For simplicity, assume troposphere has a constant lapse rate $\frac{dT}{dz} = -\mathcal{L}$. Assume ground temperature is T_0. 2

   (e) Assuming hydrostatic equilibrium, show that the expression for a z-dependent pressure can be written as follows:
   $P(z) = P_0 \left(1 - \mathcal{L}\frac{z}{T_0}\right)^{g\mathcal{M}/R\mathcal{L}}$,
   where, $\mathcal{M}$ is the mean molecular weight of dry air, and g is the gravitational acceleration. You can assume that air behaves as an ideal gas. 10
   Hint: You may need to use the ideal gas law $PV = nRT$, where, n is the number of moles. You will also need to use the expression you just derived for $T(z)$. You may also need to use the following identity:
   $\int \frac{dx}{1-Kx} = -\frac{1}{K}\ln(1 - Kx)$.

   (f) From the ideal gas law eliminate V and find an expression for ρ as a function of P and T. 4

   (g) Find the approximate air density at the tropopause, $z = 20$ km. Use $T_0 = 300\,\mathrm{K}$, $P_0 = 10^6\,\mathrm{dyne\ cm^{-2}}$, $\mathcal{L} = -6.5 \times 10^{-5}\,\mathrm{K\ cm^{-1}}$, $\mathcal{M} = 29\,\mathrm{g\ mole^{-1}}$, $R = 8.31 \times 10^7$ erg $\mathrm{K^{-1}\ mole^{-1}}$, and $g = 981$ cm $\mathrm{s^{-2}}$. 4

(h) Find an expression for the terminal velocity of a meteorite. 4
Hint: The drag force is given by $|F_{\text{drag}}| = \frac{C_D \rho A v^2}{2m}$. Terminal velocity is reached when the acceleration is zero. Drag force always acts in the opposite direction of the velocity.

(i) Assume that a spherical meteorite of radius $R = 10$ cm and density $\rho_m = 8$ g cm^{-3} reaches terminal velocity v_∞ at the tropopause. Find v_∞. Use $C_D \approx 0.1$. 4

a) Chondrites & Achondrites.

b) In order of decreasing frequency
Asteroids, Mars, Moon.

c) Convection

d) $\frac{dT}{dz} = -\alpha$

$\Rightarrow dT = -\alpha\, dz \Rightarrow \int_0^{T(z)} dT = -\alpha \int_0^z dz$

$\Rightarrow T(z) = (T_0 - \alpha z).$

e) $\frac{dP}{dz} = -g\rho$ HS eq. (1)

$PV = nRT$ (2) ideal gas.

$\Rightarrow P = \frac{m}{\mathcal{M}V} RT$

$= \frac{\rho}{\mathcal{M}} RT.$

$\therefore \rho = \frac{P\mathcal{M}}{RT}$ (3)

using (3)

$$\frac{dP}{dz} = -g\frac{P\mathcal{M}}{RT}$$

$$\int_{P_0}^{P(z)} \frac{dP}{P} = -\frac{g\mathcal{M}}{RT_0} \int_0^z \frac{dz}{(1 - \frac{\alpha z}{T_0})}$$

Integrating both sides,

$$\ln\left[\frac{P(z)}{P_0}\right] = \ln\left[\left(1-\frac{\alpha}{T_0}z\right)^{\frac{g\mathcal{M}}{R\alpha}}\right]$$

$$\therefore P(z) = P_0\left(1-\frac{\alpha}{T_0}z\right)^{\frac{g\mathcal{M}}{R\alpha}}$$

(f)

$$PV = nRT$$

n # of moles in V $= \frac{m}{\mathcal{M}}$

$$P = \frac{m}{\mathcal{M}V} RT$$

$$P = \frac{\rho RT}{\mathcal{M}}$$

$$\rho = \frac{P\mathcal{M}}{RT}.$$

$$= \frac{\mathcal{M}}{R}\cdot\frac{1}{T_0\left(1-\frac{\alpha}{T_0}z\right)}\cdot P_0\left(1-\frac{\alpha}{T_0}z\right)^{\frac{g\mathcal{M}}{R\alpha}}$$

(g) Putting in values,

$$\rho \approx 10^{-4}\ g\,cm^{-3}, \quad \text{at } z = 20\text{ km}.$$

(h)

$$\frac{d\vec{v}}{dt} = -\frac{C_D\rho A v^2}{2m}\hat{v} + g\hat{v}$$

vertical drop. $\hat{v} = \hat{z}$.

To reach terminal velocity v_∞, $\frac{d\vec{v}}{dt}$ must be zero.

$$\Rightarrow \frac{C_D\rho A v_\infty^2}{2m} = g$$

$$\Rightarrow v_\infty = \left(\frac{2gm}{C_D\rho A}\right)^{1/2}.$$

(i) Further simplifying using a spherical meteorite,

$$v_\infty = \left[\frac{2g \cdot \frac{4}{3}\cancel{\pi} R^{\cancel{3}} \rho_m}{C_D \rho \cancel{\pi} \cancel{R^2}}\right]^{1/2}$$

$$= \left(\frac{8g}{3C_D\rho}\right)^{1/2} \cdot \left(R\rho_m\right)^{1/2}$$

Putting in values we get.

$$v_\infty\,(\text{at } 20\,\text{km}) \approx 1.4\ \text{km/s}.$$

List of Constants **All values are in CGS unless otherwise noted**

#Astronomical constants
AU = 1.496e13
pc = 3.086e18
lightyear = 9.463e17
Msun = 1.99e33
Rsun = 6.96e10
Lsun = 3.9e33
Tsun = 5.780e3
Solar magnitude = -26.73
Solar absolute magnitude in B = 5.48
Solar absolute magnitude in V = 4.83
spinSun = 25. days

#Planets

#Masses of planets
MMercury = 0.3302e27
MVenus = 4.8685e27
MEarth = 5.9736e27
MMars = 0.64185e27
MJupiter = 1.8986e30
MSaturn = 5.6846e29
MUranus = 86.832e27
MNeptune = 102.43e27
MPluto = 1.27e25

#Radii of planets
RMercury = 2440e5
RVenus = 6051.8e5
REarth = 6371e5
RMars = 3389.9e5
RJupiter = 69.911e8
RSaturn = 57.316e8
RUranus = 25054.86e5
RNeptune = 2.4764e9

#Density of planets
rhoMercury = 5.427
rhoVenus = 5.204
rhoEarth = 5.515
rhoMars = 3.933
rhoJupiter = 1.326
rhoSaturn = 0.6873
rhoUranus = 1.318
rhoNeptune = 1.638

#Eccentricity of planets
eMercury = 0.205
eVenus = 0.007
eEarth = 0.0168
eMars = 0.0934
eJupiter = 0.0485
eSaturn = 0.055
eUranus = 0.046
eNeptune = 0.009
INeptune = 1.770

#Semimajor axes of planets in AU
aMercury = 0.387
aVenus = 0.72333201
aEarth = 1.
aMars = 1.52368946
aJupiter = 5.2027584
aSaturn = 9.5428244
aUranus = 19.19206
aNeptune = 30.06893

#Spin periods of planets (day)
spinEarth = 0.997
spinMars = 1.026
spinJupiter = 0.414
spinSaturn = 0.44
spinUranus = 0.718
spinNeptune = 0.671

#Inclination of planets degree w.r.t. Earth's orbit
IMercury = 7.005
IVenus = 3.39447
IEarth = 0.0
IMars = 1.84973
IJupiter = 1.3033
ISaturn = 2.4889
IUranus = 0.773

#Universal constants
Planck constant h = 6.6260755e-27
Speed of light c = 2.99792458e10
k = 1.380658e-16
hbar = 1.05457266e-27
G = 6.67259e-8
σ = 5.67051e-5
Rydberg constants = 2.1798741e-11
Hydrogenmass = 1.67e-24

#Conversion of units
km = 100000
year = 31556925.9936
day = 86400.0
Angstrom = 1e-8
hour = 3600.