Efficiency in Bargaining with Externalities

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Abstract

Consider a principal contracting with multiple agents. The principal engages in simultaneous bilateral negotiations with each agent over an allocation (referred to as the "trade") and payment. There are externalities to trade; each agent's payoffs depends on their own trade and that of the other agents. I identify general conditions on the bargaining game under which the payoff of the principal is increasing in the aggregate surplus generated by trade. In particular, this implies that the principal-optimal trade profile is efficient.

The question of when efficient trade arises in multi-agent contracting problems is addressed by Segal (1999) for the special case in which the principal makes take-it-or-leave-it offers. My setting is the same as in Segal (1999), but I allow for agents with bargaining power.

Payoffs

The setting is as follows. There are N agents. The principal's trade with each agent is denoted by $z_i \in \mathcal{Z}_i$, and the trade profile by $z = (z_1, \ldots, z_N) \in \mathcal{Z} \equiv \mathcal{Z}_i \times \cdots \times \mathcal{Z}_N$. Agent i's utility from trade profile z is given by $u_i(z) - p_i$, where p_i is the payment made to the principal. The principal's payoff is given by $g(z) + \sum_{i=1}^{N} p_i$.

Bargaining

The principal negotiates simultaneously with each agent. Assume that the the bargaining outcome between the principal and agent i is given by that which would be attained if it was common knowledge that all other agents received their equilibrium trades. Formally,

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let z^*, p^* be the equilibrium trade and payment profiles. Then agent i negotiates with the principal assuming that the profile of trades and payments agreed to with other agents is z^*_{-i}, p^*_{-i}. This assumption is made in the Nash-in-Nash bargaining solution. It also holds, for example, in dynamic bargaining games in which agents have "passive beliefs". The outcome of negotiations between the principal and agent i is given by $D_i(U, d) \in U$, where U is the set of feasible payoff pairs for the principal and the agent, and d is the pair of disagreement payoffs, i.e. the payoffs obtained if either party walks away from the negotiation. I assume, as is typical, that $D_i(U, d) \geq d$ for all i. A bargaining outcome $D_i(U, d)$ is *efficient* if there is no $u \in U$ such that $u \geq D_i(U, d)$. I will assume that this holds for all agents.

A1. Efficiency. D_j is efficient for all i.

Since the principal and all agents have quasi-linear payoffs, the set U of feasible utilities can always be described by a scalar X. Moreover, given the efficiency assumption, I will abuse notation and denote by $D_i(X, d)$ the payoff of the principal in the bargaining problem defined by X, d. The payoff of agent i is then given by $X - D_i(X, d)$. I assume that the principal always benefits by expanding the set of feasible utilities.

A2. Monotonicity. $D_i(\cdot, d)$ strictly increasing for all i, d.

Monotonicity means that the principal always has some bargaining power with every agent. Efficiency and Monotonicity are relatively innocuous assumptions, and are made in most models of multi-agent contracting. For some results, I will also make use of the assumption that D_i is continuous for all i. This assumption guarantees equilibrium existence. However it will also play a substantive role in the proof of some results.

A3. Continuity. D_i is continuous for all i.

I now turn to three more substantive assumptions. It will be useful to restrict attention to environments in which the principal commits to the trade profile, and negotiations with each agent are only over the payments.

A4. Trade commitment. The principal can commit to the trade profile.

Given that the agents' payoffs are quasi-linear, it seems reasonable to assume that the principal can get them to agree to the proposed trade by adjusting the payments. Moreover, since there are externalities from trades, the principal can claim in their negotiation with each agent that modifying the trade would constitute a violation of the terms agreed to with other agents.

In some settings, it is natural to think that the outside option of an agent is independent of the trades of other agents. For example, suppose the principal is a retail monopolist in a given market, and the agents are sellers of different goods. Should an agent enter the market, they may well care about the retail space dedicated to their competitors goods. However if they do not enter then in may be natural to assume that their payoffs do not depend on the trades of other agents.

A5. Agent outside option independence. The default payoff to an agent does not depend on the trades of other agents.

Similarly, we can consider settings in which the default payoff of the principal, should the negotiation with firm i break down, does not depend on the trades agreed to with other agents.

A6. Principal outside option independence. The payoff of the principal if negotiations with firm i break down does not depend on the trades of other firms.

When assuming principal and agent outside option independence, I will abuse notation and write $D_i(X)$ instead of $D_i(x, d)$. Principal outside option independence is perhaps a less natural condition than agent outside option independence. It will not be satisfied if the trades agreed to with other agents remain in place after agent i walks away. However will be satisfied, for example, if the contracts with other agents are re-negotiated following a breakdown with agent i. If principal outside option independence is satisfied then it is natural to assume that agent outside option independence is satisfied as well, since the former implies that what happens after an agent walks away does not depend on the equilibrium trade profile.

Solution concept

Fix a trade profile z. Let $d_i(z_{-i})$ be the pair of disagreement payoffs in the negotiation with individual i, given the trades of other agents (we are not assuming any form of outside-option independence). For each agent i, the set of feasible payoffs for the principal and agent is given by

$$X_i = \sum_{k \neq i} p_k + u_i(z) + g(z). \tag{1}$$

An equilibrium is defined as a pair (z, p) of trade and payment profiles such that $D_i(X_i, d_i(z_{-i})) = \sum_{k=1}^{N} p_k + g(z)$ for all i.

Results

Under the assumptions discussed above, a somewhat surprising result obtains: the trade profile chosen by the platform maximize the total surplus, independent of what the functions D^j are. Before stating the stating the main theorem, I will describe a characterization of equilibrium trade and payment profiles, which does not depend on some of the more restrictive assumptions.

Fix a trade profile z. It will be convenient to make a change of variables. Rather than looking for equilibrium payment profiles, we can instead look directly for a profile of equilibrium surplus levels. That is, given a profile $\{X_i\}_{i=1}^N$, define p_i by $p_i = D_i(X_i, d(z_{-i}) - X_i + u_i(z)$. Say that $\{X_i\}_{i=1}^N$ is an equilibrium iff $(\{p_i\}_{i=1}^N, z)$ constitutes an equilibrium. Equilibrium X_j's will be characterized by two simple conditions. This characterization does not require any additional assumptions, and may be of more general interest.

Lemma 1. $\{X_j\}_{j=1}^N$ *is an equilibrium iff*

1. *There exists a number D such that $D_i(X_i, d_i(z_{-i})) = D \ \forall \ i$.*

2. $\sum_{i=1}^N X_i = (N-1) \cdot D + \sum_i u_i(z) + g(z)$

Proof. Suppose the two conditions hold. Given X_i, define p_i by $p_i = D_i(X_i, d(z_{-i}) - X_i + u_i(z)$. I wish to show that the $p = \{p_i\}_{i=1}^N$ and z constitute an equilibrium. From the definition of equilibrium, this will hold if $D = \sum_i p_i + g(i)$. Using the definition of p_i, we can write

$$\sum_i p_i + g(z) = N \cdot D - \sum_i X_i + \sum_i u_i(z) + g(z).$$

Condition 2 implies that the right hand side is equal to D.

For the converse direction, start with an equilibrium payment profile p. Let $D = \sum_i p_i + g(z)$ and define $X_i = \sum_{k \neq i} p_k + u_i(z) + g(z)$. Since p is an equilibrium payment profile, $D_i(X_i, d_i(z_{-i})) = \sum_k p_k + g(z)$ for all i, and so condition 1 is satisfied given the definition of D. Summing the X_i's, we obtain

$$\sum_i X_i = (N-1) \sum_i p_i + \sum_i u_i(z) + N g(z)$$
$$= (N-1) \cdot D + \sum_i u_i(z) + g(z)$$

where the second inequality follows from the definition of D. This is exactly condition 2. $\square$

This alternative characterization of equilibrium will be helpful, as it will allow us to characterize the equilibrium directly in terms of surplus. Define aggregate surplus from trade

profile z as

$$S(z) = \sum_{i=1}^{N} u_i(z) + g(z).$$

Let $S^* = \sup_z S(z)$ and $S_* = \inf_z S(z)$. The set of efficient trade profiles are those that maximize $S(z)$ (which may be empty without further assumptions). For any given trade profile there may be multiple equilibrium payment profiles. The main theorem is a monotone comparative statics result. Let $E(z)$ be the set of equilibrium payoffs for the principal, given trade profile z.[1] The theorem states that the highest and lowest payoffs in $E(z)$ are increasing in $S(z)$.[2] This implies that the equilibrium that maximizes the principal's payoff is efficient, when an efficient trade profile exists.

Theorem 1. *Under A1-A6,* $\min E(z)$ *and* $\max E(z)$ *are increasing in* $S(z)$.

Proof. Given principal and agent outside option independence, we can normalize the default payoffs of all the principal and agents to 0 (by subtracting the default payoff of agent i from u_i, and similarly for the principal). Let $\bar{X} = \sum_i X_i$. Under Monotonicity, for each $\bar{X}$ there is a unique vector $\{X_i\}_{i=1}^{N}$ such that $\sum_i X_i = \bar{X}$ and $D_i(X_i) = D_k(X_k)$ for all k, i. Define $D(\bar{X})$ as the value achieved by this vector. Then, by Lemma 1, finding an equilibrium consists of finding $\bar{X}$ such that

$$\bar{X} = (N - 1) \cdot D(\bar{X}) + S(z). \tag{2}$$

I first want to show that $\max E(z)$ is increasing in $S(z)$.[3] Equivalently, I want to show that for any z, at the highest $\bar{X}$ at which the right hand side of (2), as a function of $\bar{X}$, crosses the 45 degree line it does so from above. This will imply that the highest such $\bar{X}$ is increasing in $S(z)$.

For any z and any equilibrium payment profile p, define $\{X_i\}_{i=1}^{N}$ as in (1). Since the agents' outside options are normalized to zero, $X_i \leq S(z)$ for all i, with equality if and only if $p_i = u_i(z)$ for all i. Then $D_i(X_i) \leq S(z)$ (the principal can't appropriate more than the available surplus). Therefore $D(\bar{X}) \leq S(z)$, and so $(N - 1) \cdot D(NS(z)) + S(z) \leq N \cdot S(z)$. Suppose $(N - 1) \cdot D(NS(z)) + S(z) = N \cdot S(z)$. Consider some z' with $S(z') > S(z')$. Then

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[1] Assuming Continuity, for any trade profile an equilibrium exists by Brouwer's fixed point theorem.

[2] This is a *weak monotone comparative statics* result, in the sense of Che et al. (2019). Say that a set S'' dominates S' in the *weak set order* if, for each $x' \in S'$, one can find $x'' \in S''$ such that $x'' \geq x'$, and likewise for each $x'' \in S''$, one can find $x' \in S'$ such that $x'' \geq x'$. When the order on the space is complete, as is the case here, the weak set order reduces to a comparison of the smallest and largest elements of the sets, when these exist.

[3] In fact, the result is stronger. The best and worst equilibria improve, from the principal's perspective, as $\sum_j v_j \cdot g_A(|s_A(j)|s_B) + \int \varphi(v) \cdot g_B(|s_B(v)|)dF_B(v)$ increases. The worst equilibrium improves as long as $D^j(0) \geq 0$, since this implies that the right hand side of (2) crosses the 45 degree line first from above.

$(N-1) \cdot D(NS(z)) + S(z') > N \cdot S(z)$, which implies that there is an $\bar{X} > NS(z)$ such that $\bar{X} = (N-1) \cdot D(\bar{X}) + S(z')$, which is the desired comparative statics result. If the inequality is strict then for $\bar{X}$ close enough to $NS(z)$ the RHS of (2) is strictly below the 45 degree line (by Continuity). Thus the last time at which it crosses it must do so from above. This implies that the higher is $S(z)$, the higher is $D(\bar{X})$ at this highest crossing.

Similarly at the lowest $\bar{X}$ such that the RHS of 2 crosses the 45 degree line, it must do so from above. The argument for this is similar to that given above. Since the principal's outside option is normalized to zero $D_i(X_i) \geq 0$, and there cannot be an equilibrium with $X_i < 0$. This implies that there is no equilibrium if $S(z) < 0$. Then $(N-1)D(0) + S(z) \geq 0$. If this holds with equality then the comparative statics result is trivial. If it holds with a strict inequality then the first time the RHS of 2 crosses the 45 degree line it must do so from above, as desired. $\qquad\square$

Corollary 1. *Under A1-A6, the principal-optimal equilibrium is efficient.*

The most restrictive assumption of Theorem 1 is A6, principal outside option independence. In some settings we can obtain similar comparative statics results without making this assumption. On prominent case is the popular Nash-in-Nash bargaining solution. Continue to assume agent outside option independence, and normalize the outside option payoff of each agent to zero. Let $d_i(z)$ be the principal's outside option when bargaining with agent i, given trade profile z. Then we have

$$D_i(X_i, d_i(z)) = \beta_i \cdot (X_i - d_i) + d_i.$$

Let $\tilde{X}_i = X_i - d_i$. For any $\bar{\tilde{X}}$ and any trade profile z, there is a unique vector $\{\tilde{X}_i\}_{i=1}^N$ and number $\tilde{D}$ satisfying $\sum_{i=1}^N \tilde{X}_i = \bar{\tilde{X}}_i$ and $\beta\tilde{X}_i + d_i = \tilde{D}$ for all i. In particular, $\tilde{D}$ is given by

$$\tilde{D}(\bar{\tilde{X}}, z) = \frac{1}{\sum_i \frac{1}{\beta_i}}\bar{\tilde{X}} + \frac{1}{\sum_i \frac{1}{\beta_i}}\sum_i \frac{1}{\beta_i}d_i(z).$$

Let $B = \sum_i \frac{1}{\beta_i}$. Using Lemma 1, we can conclude that for any z, finding an equilibrium consists of finding $\bar{\tilde{X}}$ such that

$$\bar{\tilde{X}} = \frac{N-1}{B}\bar{\tilde{X}} + S(z) + \sum_i d_i(z)\left(\frac{1}{B\beta_i} - 1\right). \tag{3}$$

Since expression is affine in $\bar{\tilde{X}}$ the equilibrium is unique. The following comparative statics result follows along the same lines as Theorem 1.

Proposition 1. *Under A1-A5 and Nash-in-Nash bargaining, there is a unique equilibrium for any z, and the principal's payoff is increasing in*

$$S(z) + \sum_i d_i(z)\left(\frac{1}{B\beta_i} - 1\right).$$

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Proof. This is immediate from the characterization in (3) since $(N-1)/B < 1$. □

References

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