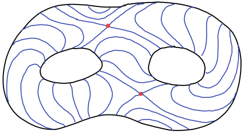
Midwest Dynamics and Group Actions

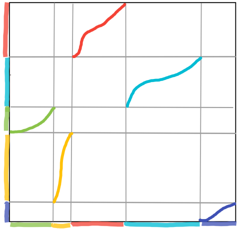
Indiana U. /U. of Chicago/U. of Illinois at Chicago/Northwestern U. /U. of Michigan

7 June, 2021

Rigidity of foliations on surfaces

and renormalization





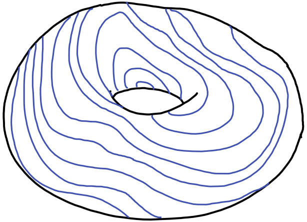
Corinna Ulcigrai

University of Zürich

University of Zurich[UZH]

(joint work with Selim Ghazouani)

Rigidity of foliations in genus one.



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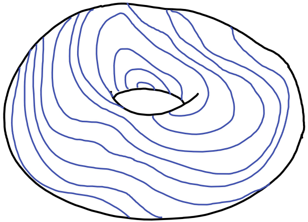

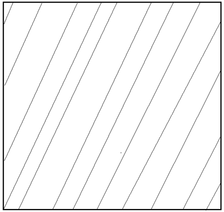

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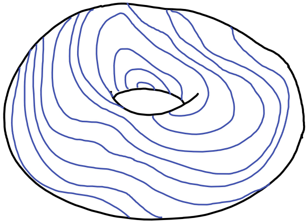

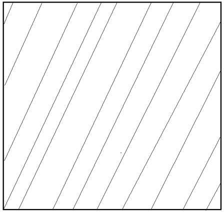

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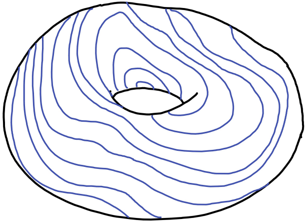

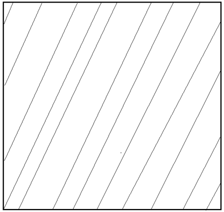

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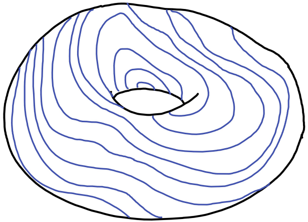

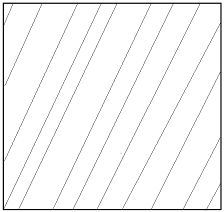

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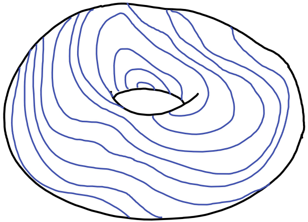

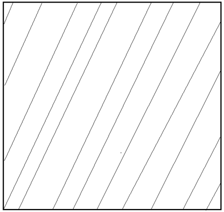

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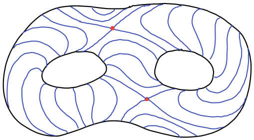
- For a *full measure set of rotation numbers* (i.e. for a.e. α) foliations in $g = 1$ are geometrically rigid [follows from M. Herman *global theorem* on *circle diffeos*.]

Rigidity of foliations in genus two.



- $\mathcal{F}$ orientable smooth foliation of compact S with $g = 2$ with only Morse type saddles;

[Morse type (simple) saddles:
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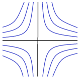

Theorem (Ghazouani-U', 2021)

Under a full measure arithmetic condition, if $\mathcal{F}$ is topologically conjugate to $\mathcal{F}_0$, then it is differentiably conjugate to it, i.e. $\mathcal{F}$ is geometrically rigid.

[*Full measure arithmetic condition:* for almost measured foliation (a.e. IET)]

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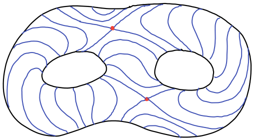

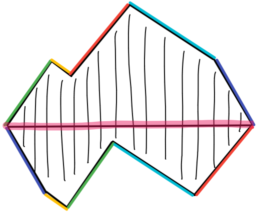

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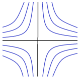

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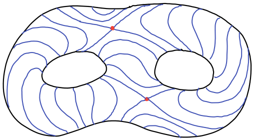

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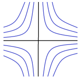

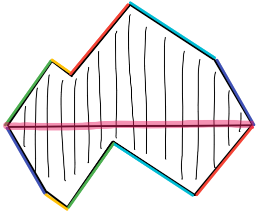

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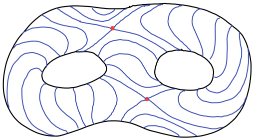

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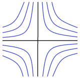

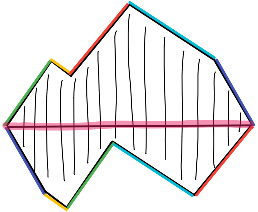

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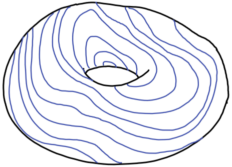
Poincaré maps in genus one: circle diffeomorphisms



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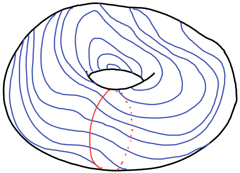

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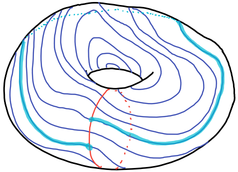

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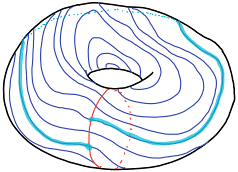

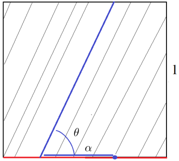

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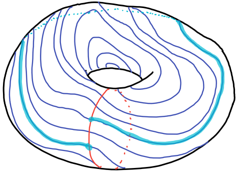

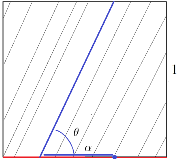

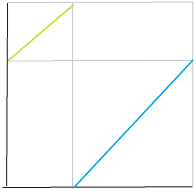

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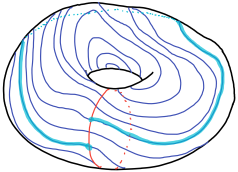
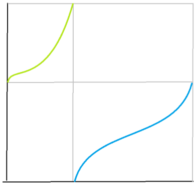
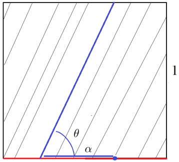

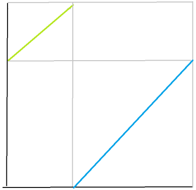

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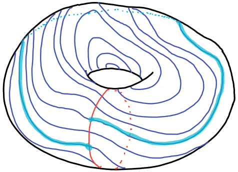

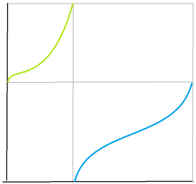

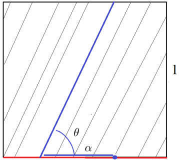

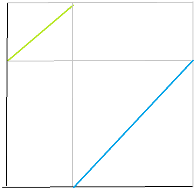

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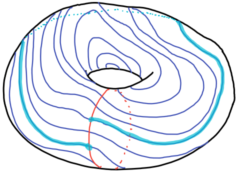
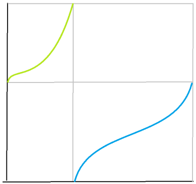
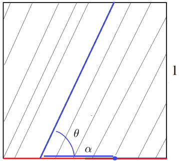

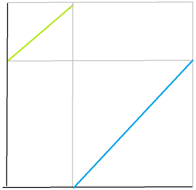

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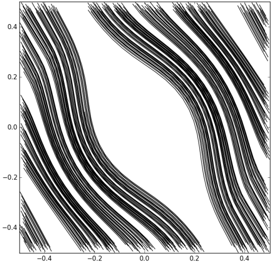
- The rotation number α of $f : S^1 \to S^1$ can be defined *dynamically* ($\alpha = \lim_{n\to\infty} \frac{f^n(x)-x}{n}$) or *combinatorially* (via *continued fractions* and the Euclidean algorithm).

Circle diffeomorphisms: a mini-survey of the classical theory.

Let $f : S^1 \to S^1$ be a *circle diffeomorphism* with *rotation number* α.



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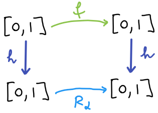
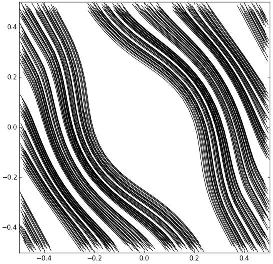


- if the rotation number $\alpha \notin \mathbb{Q}$, $\exists$ [Poincaré Thm] a topological semi-conjugacy, i.e. a continuous, surjective $h : S^1 \to S^1$ such that $h \circ f = R_\alpha \circ h$.



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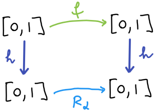

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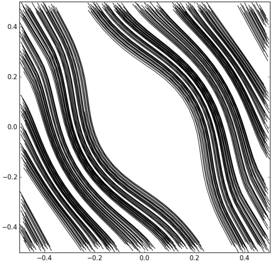
- h could fail to be a conjugacy, if there are wandering intervals, i.e. $J \subset S^1$ s.t. $f^n(J)$, $n \in \mathbb{Z}$ are all disjoint (Denjoy counterexamples).

  [*Idea:* $(f^n(J))_{n\in\mathbb{Z}}$ are obtained by *blow up* of an orbit.]



a Denjoy flow (*courtesy of J.Carrard*)

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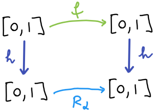
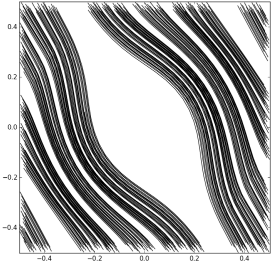


- if the rotation number $\alpha \notin \mathbb{Q}$, $\exists$ [Poincaré Thm] a topological semi-conjugacy, i.e. a continuous, surjective $h : S^1 \to S^1$ such that $h \circ f = R_\alpha \circ h$.

- h could fail to be a conjugacy, if there are wandering intervals, i.e. $J \subset S^1$ s.t. $f^n(J)$, $n \in \mathbb{Z}$ are all disjoint (Denjoy counterexamples).

  [*Idea:* $(f^n(J))_{n\in\mathbb{Z}}$ are obtained by *blow up* of an orbit.]



a Denjoy flow (*courtesy of J.Carrard*)

(2) Topology: if f is differentiable, e.g. $f \in \mathcal{C}^2$ ($\mathcal{C}^1$+ $f' \in BV$), then f is a conjugacy $\Leftrightarrow$ f is minimal [Denjoy theorem] (*Combinatorics (+smoothness) determines topology*).

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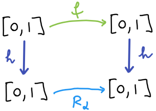

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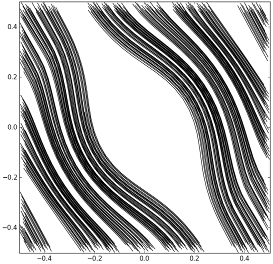

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(3) Geometry: What is the regularity of h? Is $h \in \mathcal{C}^1$? Is $h \in \mathcal{C}^\infty$?

(*Rigidity*: *topology* determines *geometry*)

Circle diffeomorphisms: local and global results.

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Remark: To get rigidity, one needs to impose (full measure) arithmetic conditions on α.

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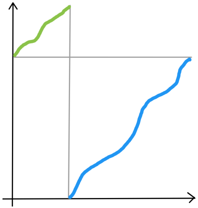

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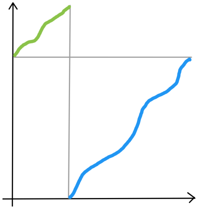

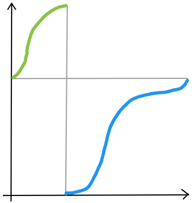

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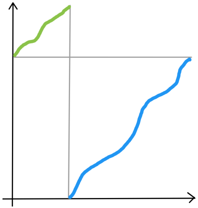

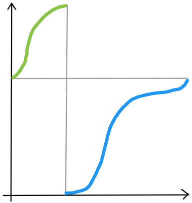

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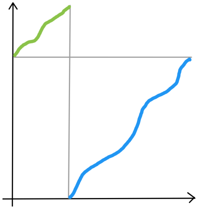

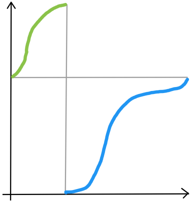

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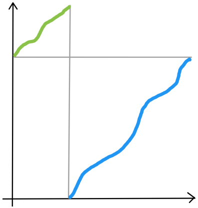

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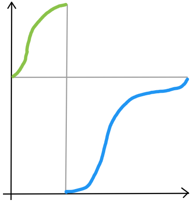

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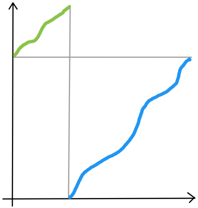

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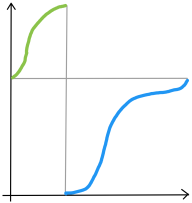

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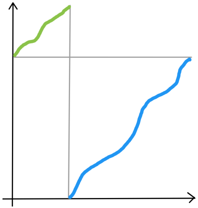

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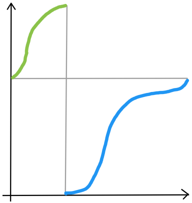

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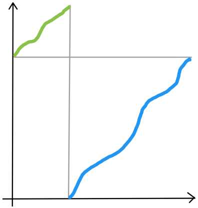

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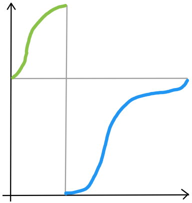

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  [*Renormalization approach*: Khanin-Sinai, Khanin-Teplisnky]

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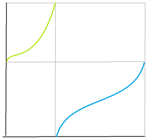

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  - discovered by Feigenbaum, Coullet-Tresser in the ′70s;
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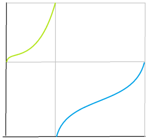

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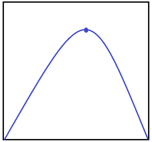

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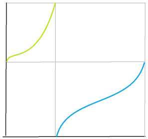

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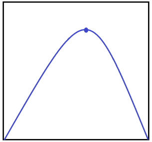

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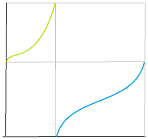

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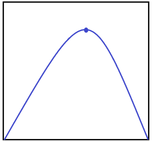

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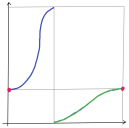

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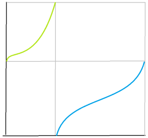

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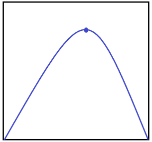

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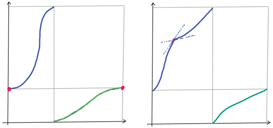

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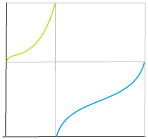

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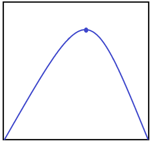

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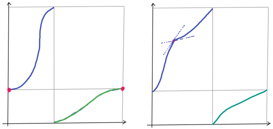

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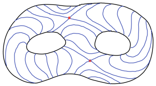
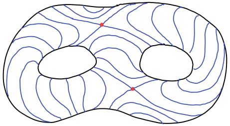


Poincaré maps in higher genus: generalized interval exchange maps



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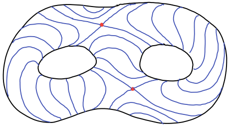

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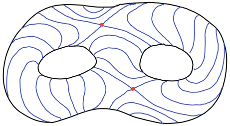

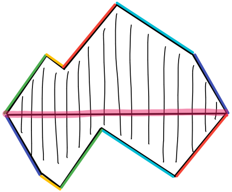

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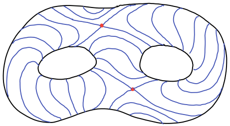

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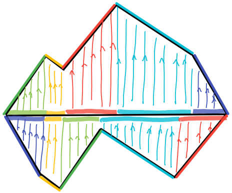

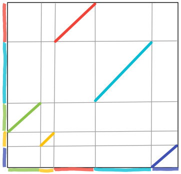

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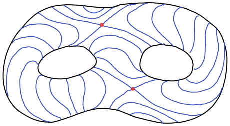

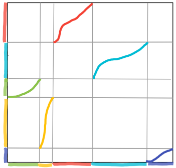

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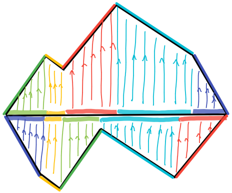

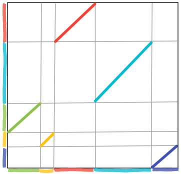

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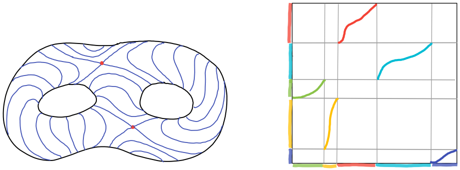

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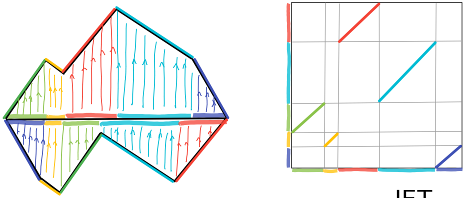

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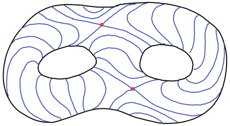

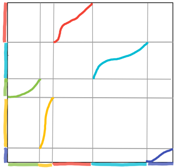

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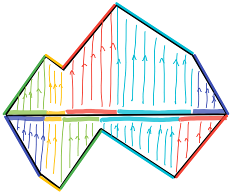

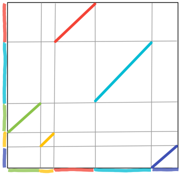

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Let $T : I \to I$ be a *Keane* GIET
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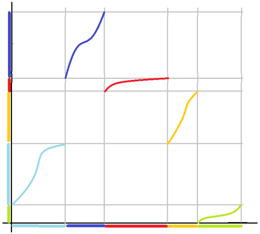

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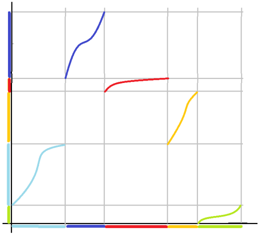

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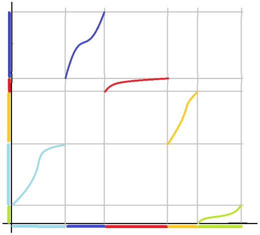

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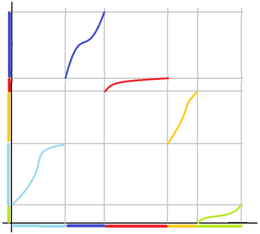

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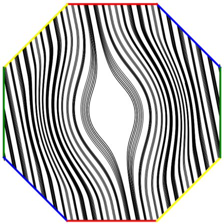

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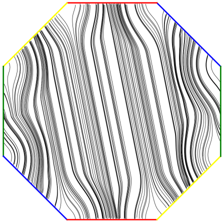

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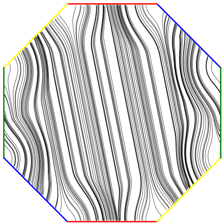

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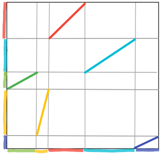

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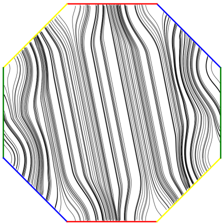

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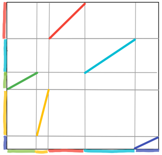

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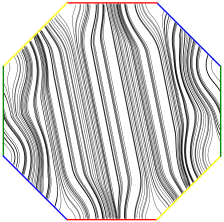

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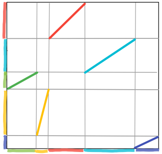

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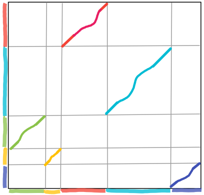

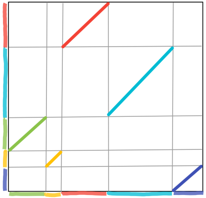

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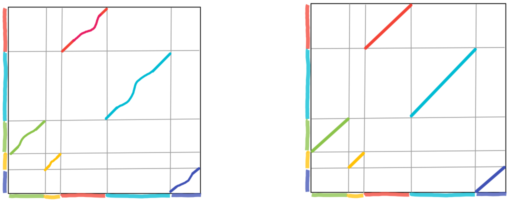

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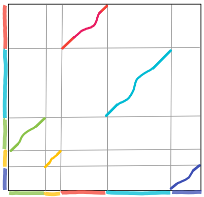

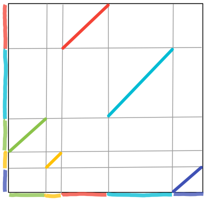

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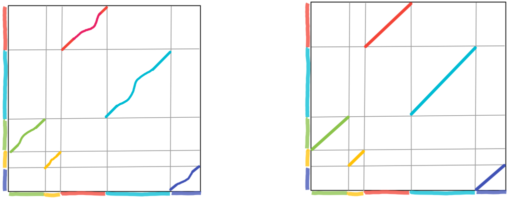

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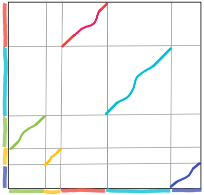

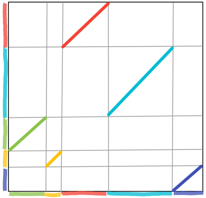

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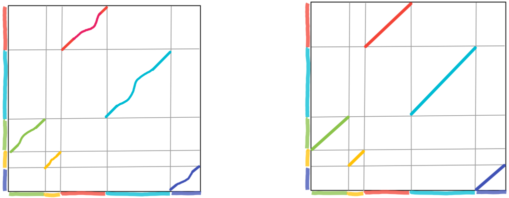

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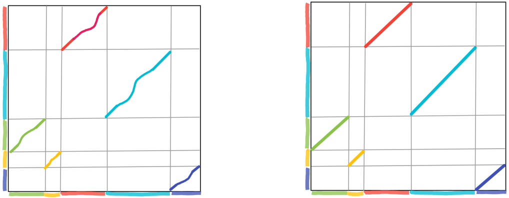

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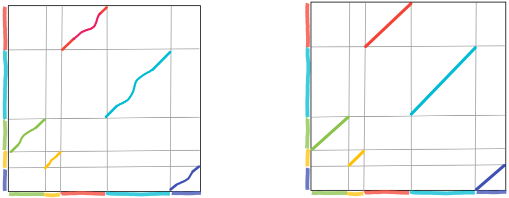

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  - Ghazouani, 2020: for T_0 hyperbolic periodic-type, the GIETs $\mathcal{C}^3$-*close* to T_0 (+simple def.) $\mathcal{C}^1$ conjugate to it have codim $(d-1)+(g-1)$;

Global theory: rigidity conjecture.

Geometric rigidity: if T and T_0 are topologically conjugate ($h \in \mathcal{C}^0$), are they differentiably conjugate ($h \in \mathcal{C}^1$)?

- *Def*: the boundary $B(T)$ is a $\mathcal{C}^1$-conjugacy invariant s.t:
  - $B(T) = (B(T)_i)_{1 \leq i \leq \kappa} \in \mathbb{R}^\kappa$, κ number of saddles;
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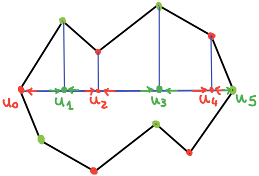

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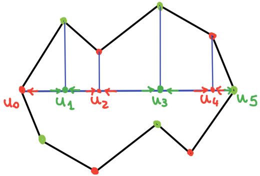

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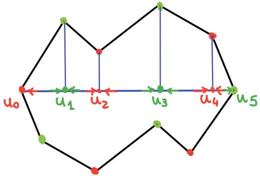

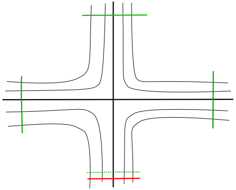

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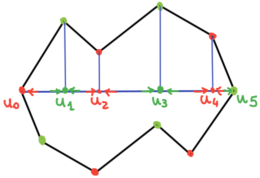

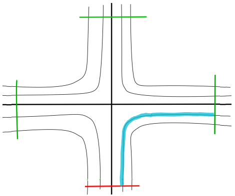

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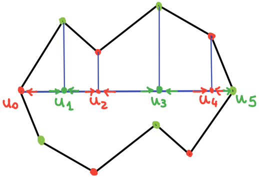

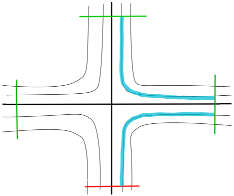

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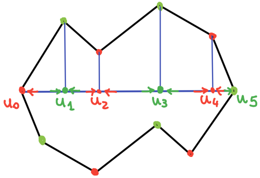

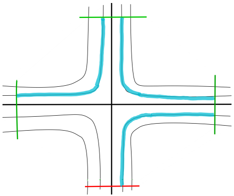

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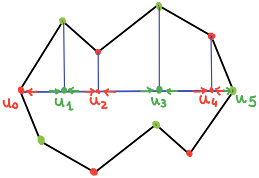

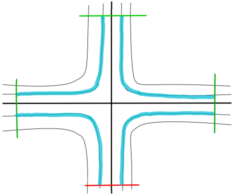

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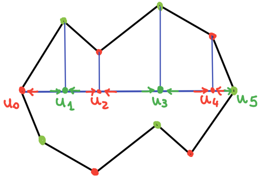

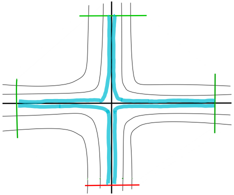

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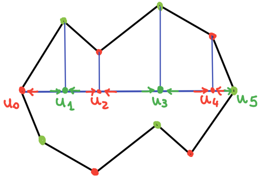

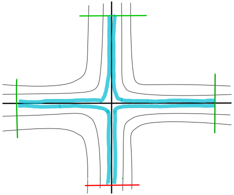

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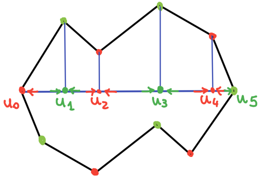

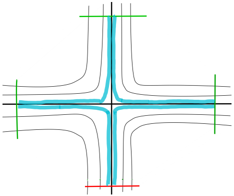

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Theorem (Ghazouani-U', 2021)

For a full measure set of IETs T_0 with $d = 4, 5$ intervals (Poincaré sections of $g = 2$, π irreducible),

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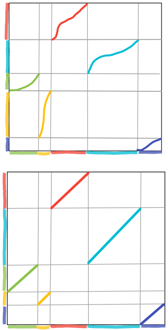

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- proves Marmi-Moussa-Yoccoz conjecture in $g = 2$;
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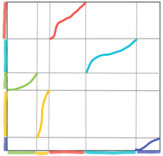

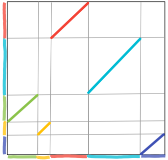

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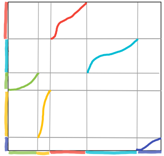

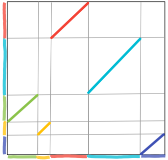

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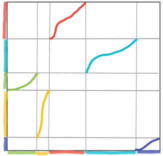

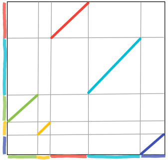

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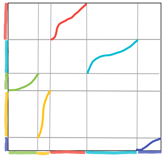

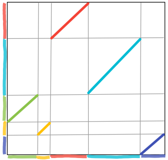

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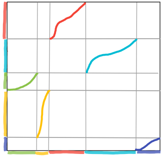

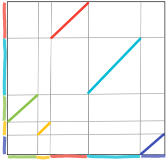

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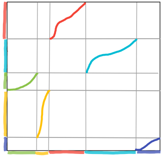

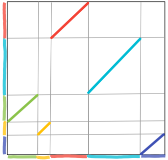

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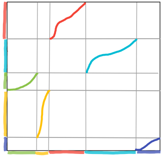

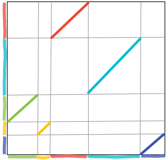

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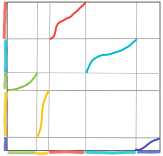

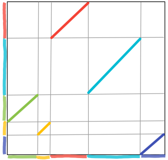

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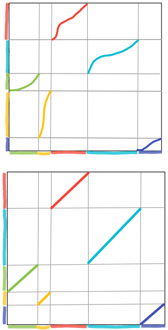

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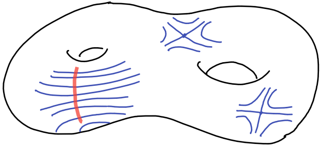

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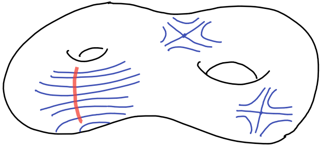

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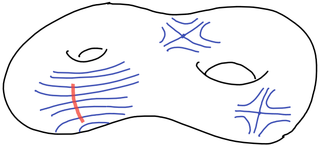

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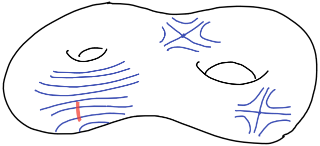

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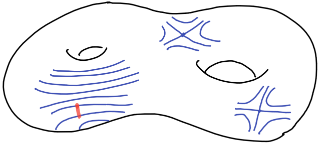

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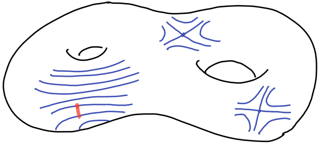

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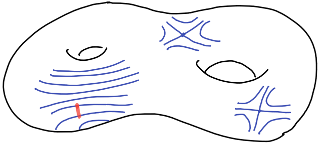

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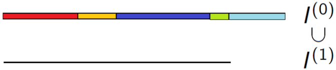

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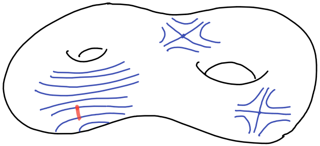

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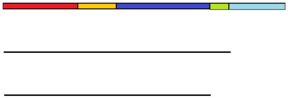

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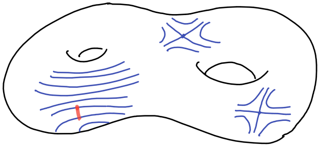

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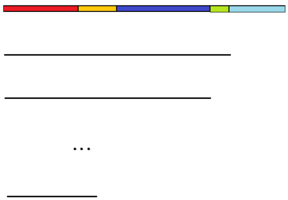

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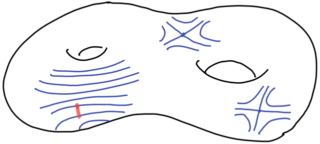

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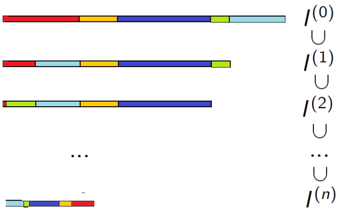

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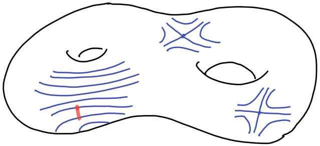

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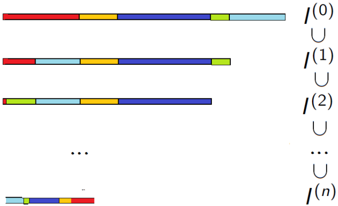

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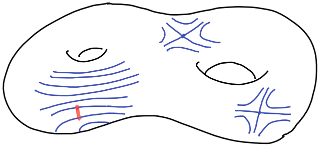

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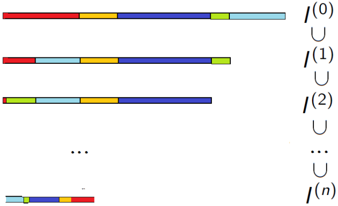

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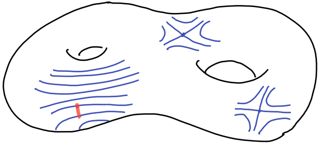

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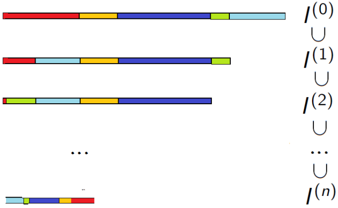

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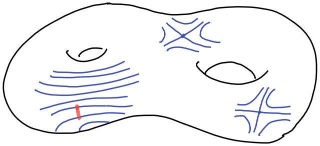

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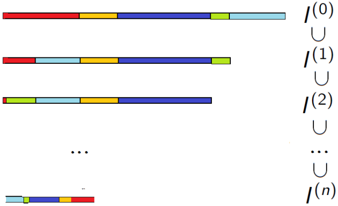

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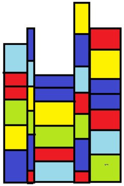

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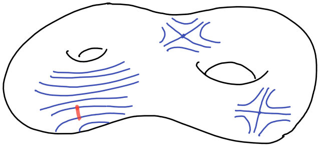

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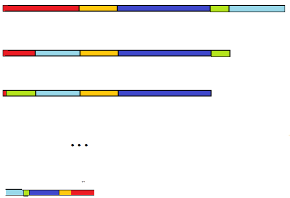

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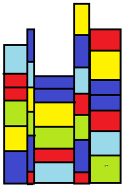

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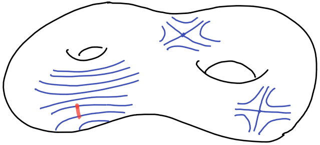

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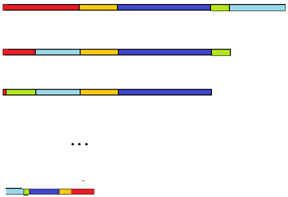

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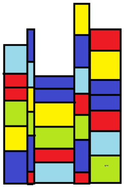

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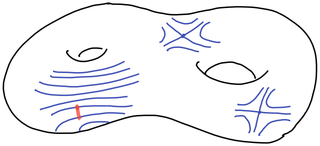

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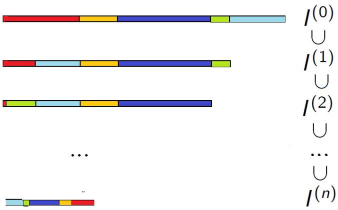

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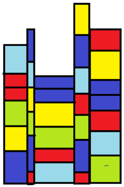

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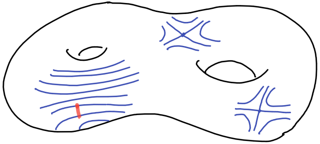

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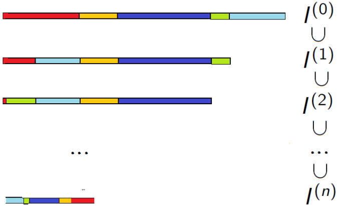

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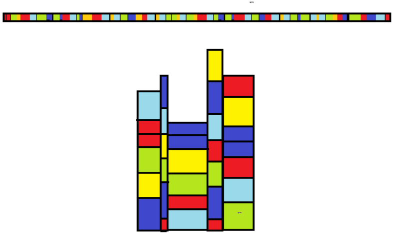

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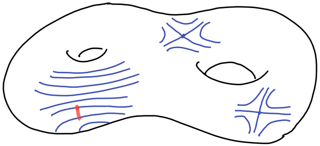

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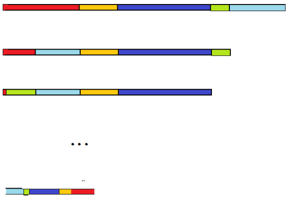

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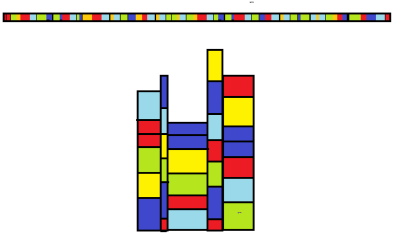

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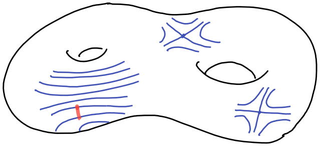

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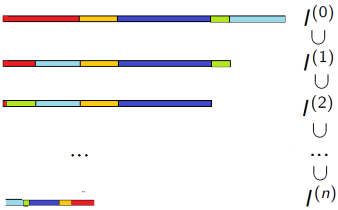

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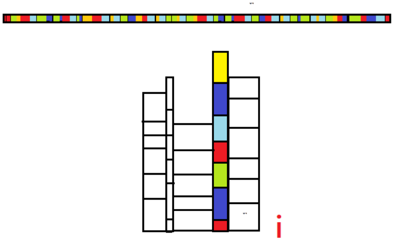

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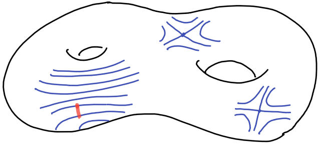

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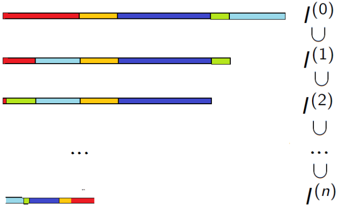

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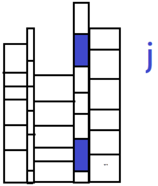

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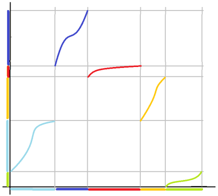

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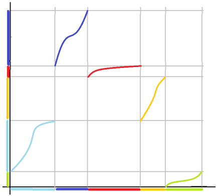

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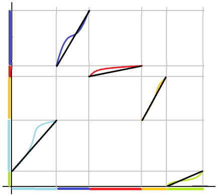

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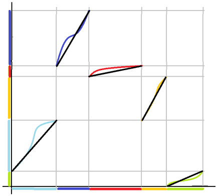

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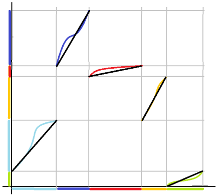

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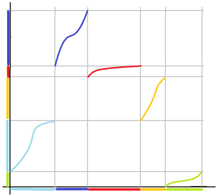

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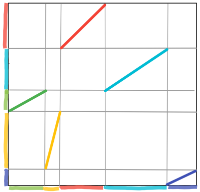

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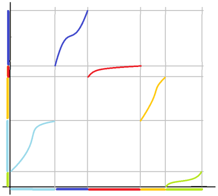

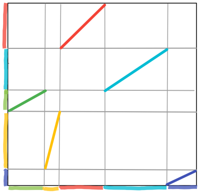

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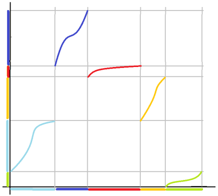

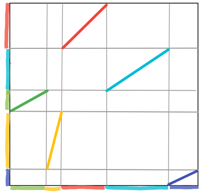

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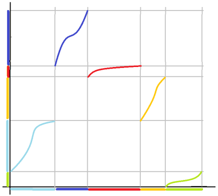

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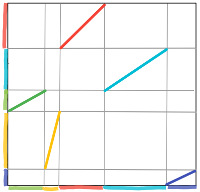

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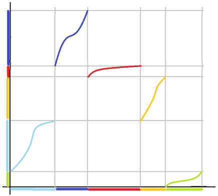

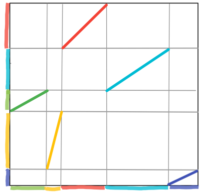

Theorem (Dynamical dichotomy, Ghazouani-U', 2021)

For any $d \geq 2$, for a full measure set of rotation numbers $\gamma(T)$, $\exists\ (n_k)_{k\in\mathbb{N}}$ s. t.

(1) *either we have recurrence, i.e. $\exists C > 0$ s.t. $||\omega^{(n_k)}|| \le C\ \forall k$ (and $(\mathcal{R}^n(T))_{n\in\mathbb{N}}$ is recurrent to a $\mathcal{C}^1$-bounded set $\mathcal{K}$);*

(2) *or $(\mathcal{R}^n(T))_{n\in\mathbb{N}}$ diverges and we have affine shadowing, i.e. there exists v (the shadow) s.t.*

$$||\omega^{(n)} - B(0,n)v|| \le C_\epsilon ||B(0,n)v||^\epsilon, \quad \forall \epsilon > 0.$$

Strategy to prove rigidity ($\mathcal{C}^0 \Rightarrow \mathcal{C}^1$)

Assume that T is such that the dynamical dichotomy holds. Consider two cases:

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- $\frac{1}{\nu} \leq \rho^{(n_k)} \leq \nu$ (a priori bounds);
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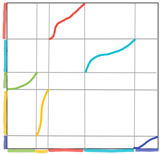
Convergence of renormalization in the recurrent case



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2. Convergence to Moebius IET: no B assumption!
   - *Tool*: Schwarzian derivative $\mathcal{S}(T) := \frac{D^3 T}{DT} - \frac{3}{2}\left(\frac{D^2 T}{DT}\right)^2$;
   - *Show*: mesh of dynamical partition goes to zero;
3. Convergence to AIET: requires $\sum_{s=1}^{\kappa} B(T)_s = 0 \Leftrightarrow \int \eta_T = 0$;
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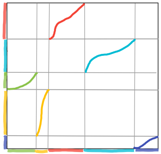

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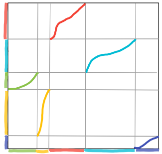

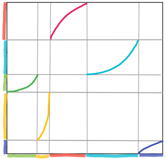

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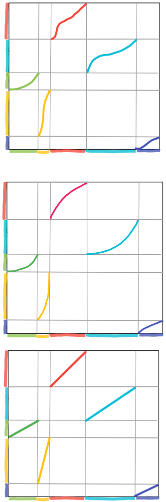

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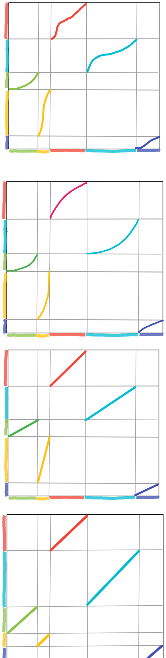

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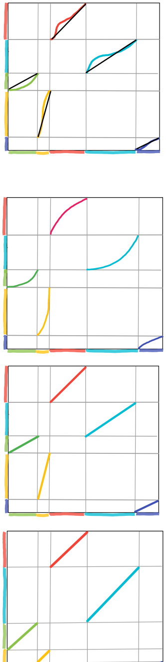

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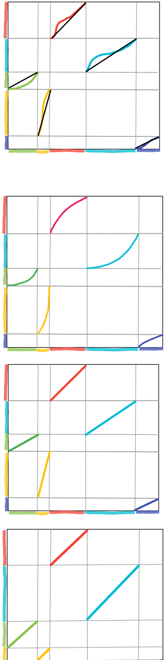

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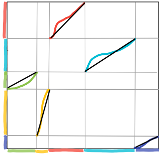

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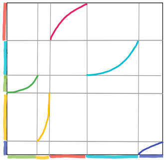

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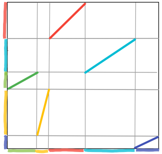

3. Convergence to AIET: requires $\sum_{s=1}^{\kappa} B(T)_s = 0 \Leftrightarrow \int \eta_T = 0$;
   - *Tool*: non-linearity $\eta_T(x) := D \log DT(x) = \frac{D^2 T}{DT}$;
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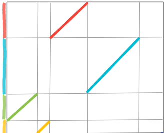

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Convergence of renormalization in the recurrent case

Assume to be in the recurrent case [at special times, $||\omega^{(n_k)}|| \leq C$]

1. Show a priori bounds at $(n_k)_k$ [i.e. $\frac{1}{C} \leq DT^{(n)} \leq C$];
   - Consider separately *shape* and *profile* coordinates:
     - the shape is the *affine* IET with log-slope ω_n;
     - profiles φ_i^n are $T_i^{(n)}$ *rescaled* to be in $Diff^+[0,1]$;
   - Classical *distortion bounds* control $|\varphi_i^n(x)/\varphi_i^n(y)|\ \forall x, y, \forall n$;
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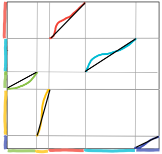
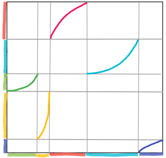
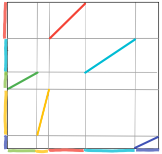
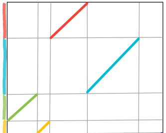

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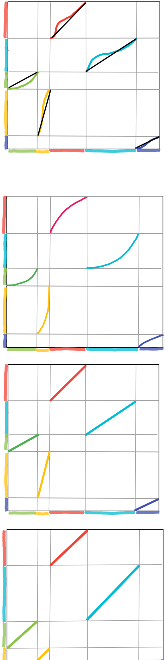

Divergent case: building the shadow

[Assume we are in Case 2. *Goal*: build the *shadow* v.]

- *Example*: periodic type case, i.e. $B(0, np) = A^n$, for any n, where $A > 0$;
  - Assume A has g exponents $\lambda_i > 1$;
  - Split $\mathbb{R}^d = E^s \oplus E^c \oplus E^u$ (positive/neutral/negative eigenvalues);
  - Denote by P_u the projection on E^u;

Definition (Shadow in periodic case)

$$v := \sum_{i=1}^{\infty} A^{-i} \left(P_u(\underbrace{\omega^{(i)} - A\omega^{(i-1)}}_{e_i})\right) + P_u(\omega^{(0)}).$$

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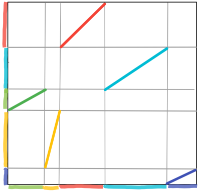

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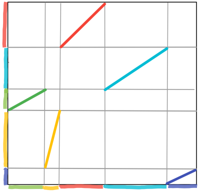

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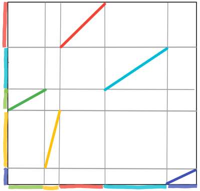

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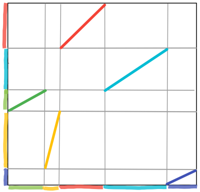

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Condition on the rotation number $\gamma(T) = \gamma(T_0)$ (valid for full measure set of IET T_0):

- Assume T is *Oseledets generic*; consider an *effective Oseledets* acceleration $\mathcal{R}$;
- Let $B(0, n)$ be the matrices of the acceleration.

Definition (Regular Diophantine condition, or RDC)

$\gamma(T_0)$ satisfy the (RDC) if there exists a *linearly growing* sequence $(n_k)_{k\in\mathbb{N}}$ of effective Oseledets times such that:

(i) at time n_k, one has a double occurrence $A\,A$ of $A > 0$;

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$$\sum_{n=1}^{n_k} ||B(n, n_k)_{|E_s^{(n)}}||\,||P_s^{(n)}||\,||B(n-1, n)|| \leq C, \quad \text{for all } k \in \mathbb{N}; \ (\text{Backward series})$$

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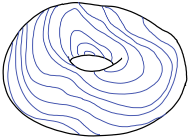
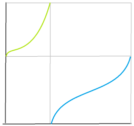
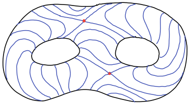
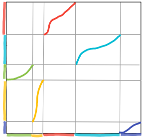
An overview to conclude

	$g = 1$	$g = 2$
Combinatorics	▶ rotation number α;	▶ rotation number $\gamma(T) = (\pi^{(n)})_{n\in\mathbb{N}}$
Topology	▶ α irrational + $\mathcal{C}^2 \Rightarrow \mathcal{C}^0$-conjugacy [Denjoy thm]	▶ *Obstructions* to topological conjugacy: for a.e. $\gamma(T_0)$, affine T with $\gamma(T) = \gamma(T_0)$ has *wandering intervals* [Marmi,Moussa,Yoccoz]
Geometry	▶ α Diophantine $\Rightarrow h \in \mathcal{C}^1$ (rigidity) [Herman,Yoccoz thm]	▶ *Obstructions* to differentiable conjugacy [Forni, Marmi-Moussa-Yoccoz, Ghazouani] ▶ Still geometric rigidity: for a.e. $\gamma(T_0)$, T, T_0 $\mathcal{C}^0$-conjugate, $B(T) = B(T_0) \Rightarrow \mathcal{C}^1$-conjugate[G'-U']

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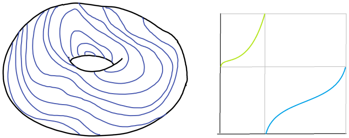

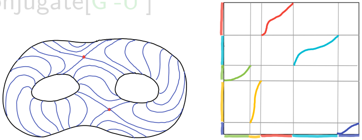

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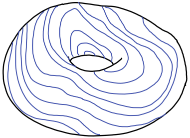
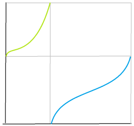
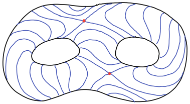
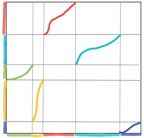

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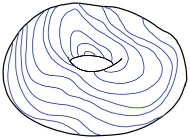
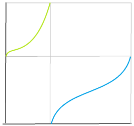
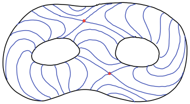
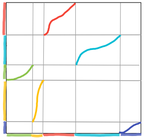

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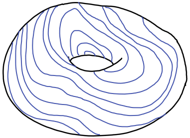

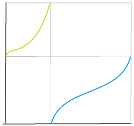

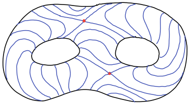

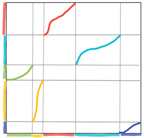

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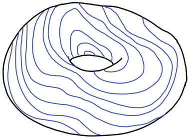

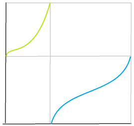

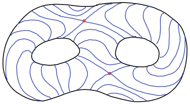

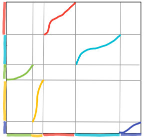

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For a.e. T, if T_0 is an affine IET such that:

- $\gamma(T) = \gamma(T_0)$ *(same rotation number);*
- $v := \log \rho(T)$ *belongs to* $E_2 \setminus E_1$ *i.e.* $\frac{\log ||B(0,n)v||}{n} = \theta_2 > 0$,

then T has wandering intervals.

To show: the result also holds for every v s.t. $\frac{\log ||B(0,n)v||}{n} = \theta_i > 0$.

To show this, [*MMY*] prove that for a sequence $(n_\ell)_\ell$, the partitions $\mathcal{P}_{n_\ell}$ are *exponentially distorted*, i.e. for every j there exists a floor of the j-tower s.t.

$$|T^i F_0| = |T^{k_0+i} I_j^{(n)}| \leq C \exp(-c|i|^\gamma) |F_0|.$$

In particular, for every $1 \leq j \leq d$

$$\mathrm{Leb}(\mathcal{P}_n^j) \leq C \max_{0 \leq k < q_j^{(n)}} \left| T^k(I_j^{(n)}) \right| = C \max \{ \mathrm{Leb}(T^k(I_j^{(n)}), \quad 0 \leq k < q_j^{(n)} \}.$$

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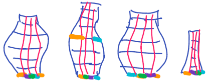

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E.g.: uses of the double occurrence AA of a positive matrix $A > 0$.

Proposition: $\text{mesh}(\mathcal{P}_{n_k}) \leq C\nu^k$ for $\nu < 1$ (i.e. the mesh decay exponentially), where:

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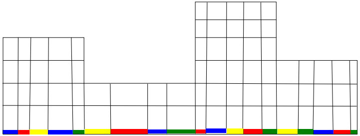

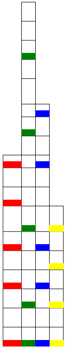

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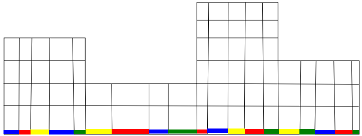

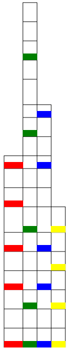

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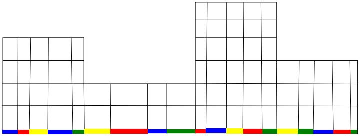

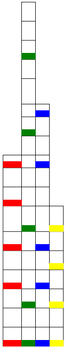

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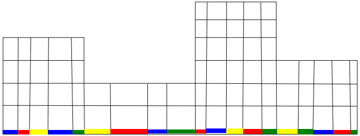

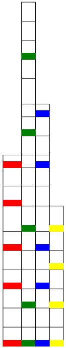

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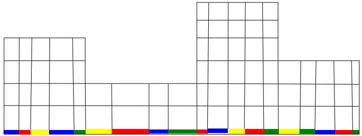

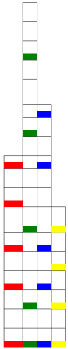

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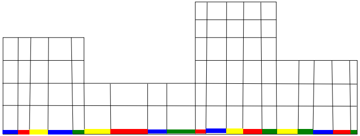

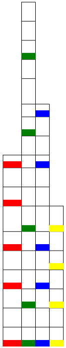

Extra: Effective Oseledets estimates

Given T, let $\hat{T}$ an Oseledets generic *extension*, so that we have *splittings*:

$$\mathbb{R}^d = E_s^{(n)} \oplus E_c^{(n)} \oplus E_u^{(n)}, \qquad \forall n \in \mathbb{N}.$$

Definition (Effective Oseledets sequence)

A sequence $(k_m)_{m\in\mathbb{N}}$ is an *effective Oseledets sequence* if for s $C_1 > 0, \theta > 0, \epsilon > 0, c_2(\epsilon) > 0$ we have:

$$||B(n_k, n)|_{E_s^{(n_k)}}||_\infty \leq C_1 e^{-\theta(n-n_k)} \qquad \text{for every } n \geq n_k, \tag{EO1}$$

$$||B(n, n_k)^{-1}|_{E_u^{(n_k)}}||_\infty \leq C_1 e^{-\theta(n_k-n)} \qquad \text{for every } n \leq n_k, \tag{EO2}$$

$$|\angle(E_x^{(n)}, E_y^{(n)})| \geq c_2 \; e^{-\epsilon|n-n_k|}, \qquad \text{for all } n \in \mathbb{Z}, \text{distinct } x, y \in \{s, c, u\}; \tag{EO3}$$

$$\lim_{k\to+\infty} \frac{\log ||B(n_k, n_{k+1})||}{k} = 0. \tag{EO4}$$