Totally geodesic submanifolds, superrigidity and arithmeticity

David Fisher (joint with Bader, Miller, Stover)

Indiana University, Bloomington

May 16, 2021

Motivations

Let X be a compact manifold of negative curvature. There are infinitely many closed geodesics, i.e. geodesic flow on $T^1(X)$ has infinitely many closed orbits.

Question

What about higher dimensional closed totally geodesic manifolds?

Theorem (Folklore)

A generic X has no totally geodesic submanifolds of dimension bigger than 1. (Local, no curvature constraint.)

Observation (Reid, 1991)

There are compact and finite volume hyperbolic manifolds with infinitely many closed totally geodesic submanifolds of dimension > 1.

Geodesic submanifolds and arithmeticity

Let $\tilde{X}$ be an irreducible symmetric space of non-compact type with no Euclidean factors.

Let $X = \tilde{X}/\Gamma$ have finite volume and *irreducible*.

Theorem (Bader, F, Miller, Stover 2020)

If X contains infinitely many maximal closed totally geodesic submanifolds of dimension at least two then Γ is arithmetic.

Maximal is not contained in another proper closed TG submanifold.

Can think $\tilde{X} = \mathbb{H}^n$, the hyperbolic space if you prefer, result is new there. Equivalently, Γ not arithmetic implies finiteness of maximal closed totally geodesic submanifolds.

Note that $G = \text{Isom}(\tilde{X})$ is a semisimple Lie group and $\Gamma < G$ is a lattice.

Arithmetic means there is a number field k such that Γ is commensurable to $G(\mathcal{O}_k)$.

Totally geodesic submanifolds

A geodesic is a length minimizing path.

Submanifold $\tilde{M} \subset \tilde{X}$ is totally geodesic if for every $x, y \in M$ the geodesic joining x to y is in M.

Submanifold $M \subset X$ is totally geodesic if it is covered by $\tilde{M} \subset \tilde{X}$ which is totally geodesic.

Even geodesics in X are not globally length minimizing.

If you describe geodesics via ODE, then the geodesics in X are exactly the images of geodesics in $\tilde{X}$.

In $\mathbb{H}^n$ all totally geodesic submanifolds are isometric to $\mathbb{H}^k$ for $k < n$.

History and reductions

If X has higher rank, i.e. if $\tilde{X}$ contains totally geodesic copies of $\mathbb{R}^2$, then Margulis proved Γ is always arithmetic. (1974)

If X is quaternionic hyperbolic space or the Cayley hyperbolic plane, then Corlete, Gromov-Schoen proved X is always arithmetic. (1992)

So the theorem is about real and complex hyperbolic manifolds of finite volume.

Asked by Reid and McMullen in the mid-2000's.

Motivation: Margulis' commensurator superrigidity and arithmeticity theorems.(1974)

Observation (Reid): If commensurator is dense, 1 TG submanifold $\Rightarrow$ infinitely many. (1991)

Commensurators and TG manifolds

Let $G = \mathrm{Isom}(\tilde{X})$ as above. $\Gamma < G$ lattice. Let

$$\mathrm{Comm}_G(\Gamma) = \{g \in G | [g\Gamma g^{-1} \cap \Gamma : \Gamma], [g\Gamma g^{-1} \cap \Gamma : g\Gamma g^{-1}] < \infty\}$$

Theorem (Margulis, 1974)

If $\mathrm{Comm}_G(\Gamma)$ *contains* Γ *at infinite index then* Γ *is arithmetic.*

Converse earlier due to Borel. E.g. $\mathrm{SL}(n, \mathbb{Q}) < \mathrm{Comm}_{\mathrm{SL}(n,\mathbb{R})}(\mathrm{SL}(n, \mathbb{Z}))$.

If g commensurates Γ then g is an isometry of a finite cover of X.

Reid's idea: let $M \subset X$ is a TG submanifold, lift to a cover X'

lift M to a cover X' where g is an isometry

push M by g, push back down to X. Check new TG submanifold.

More history

Brief history of results:

2018 First cases proven by F, Lafont, Miller, Stover. See also Benoist-Oh.

2019 Real hyperbolic case by Bader, F, Miller, Stover. Partial results by Margulis-Mohammadi.

2020 Complex hyperbolic case by Bader, F, Miller, Stover. Partial results by Baldi-Ullmo.

Totally geodesic manifolds play a key role in hyperbolic geometry.

Totally geodesic manifolds in hyperbolic geometry

Totally geodesic manifolds play a key role in hyperbolic geometry, e.g. first examples of infinitely many compact real hyperbolic manifolds of dimension n with $vb_1 > 0$ (Millson, 1976)

finitely many earlier examples in dimension $2 < n < 10$ reflection groups (Vinberg)

construction of infinitely many non-commensurable non-arithmetic hyperbolic manifold in all dimension (Gromov and Piatetski-Shapiro 1988)

variants of GPS by many authors, Agol, 7 Samurai

Gelander-Levit: most compact or finite volume real hyperbolic manifolds of dimension > 3 are non-arithmetic.

Hyperbolic geometry in high dimensions

Question (GPS 1988)

Are all real hyperbolic manifolds of "high enough dimension" "built from subarithmetic pieces".

Question

Does every non-arithmetic hyperbolic manifold of dimension $n > 3$ contain a totally geodesic submanifold of codimension one?

All examples above dimension 3 are either variants of GPS or reflection groups.

no reflection groups constructed above dimension 21

no reflection groups possible above dimension 30 (compact) 997 (finite volume)

Complex hyperbolic manifolds: land of mysteries

Only 22 known non-arithmetic complex hyperbolic manifolds of complex dimension 2

only 2 known of complex dimension 3

early work of Mostow (reflection groups), Mostow-Deligne (monodromy groups), Thurston (moduli spaces)

found 17 and 1 in the $80's$

lots of effort since, few new manifolds. all examples can be built by all three methods.

Question

Are there non-arithmetic complex hyperbolic manifolds in dimension greater than 3?

All complex hyperbolic manifolds are *integral* while all real hyperbolic manifolds are not, Esnault-Greschonig 2018.

More Complex hyperbolic geometry:

Have $\mathbb{CH}^k \subset \mathbb{CH}^n$ for $1 \leq k < n$

but also $\mathbb{H}^k \subset \mathbb{CH}^n$ for $2 \leq k \leq n$

$\mathbb{CH}^1$ is the hyperbolic plane,but in $\mathbb{CH}^n$ it has curvature -4

while $\mathbb{H}^2$ has curvature -1

$\mathrm{Isom}(\mathbb{CH}^n) = \mathrm{SU}(n,1)$

$\mathbb{H}^2$ are orbits of $\mathrm{SO}(2,1)$ while $\mathbb{CH}^1$ are orbits of $SU(1,1)$.

The bridge to dynamics

The frame bundle of $\mathbb{H}^n$ is $\mathrm{SO}(n, 1)$.

The frame bundle of X is G/Γ.

In X, closed totally geodesic submanifolds of dimension k come from orbit closures of $W = SO(k, 1) < SO(n, 1)$ acting on G/Γ.

Note $\mathrm{SO}(1, 1) = a(t)$ is diagonal matrices and gives the geodesic flow.

For $\mathbb{CH}^n$, life is a bit more complicated: $\mathrm{SU}(n, 1)$ is the bundle of complex frames.

W can be $\mathrm{SU}(k, 1)$ (complex TG surfaces) or $\mathrm{SO}(k, 1)$ (real TG surfaces).

But still studying orbit closures of W on G/Γ.

Dynamical reformulations

Let W act on G/Γ as above. Call a W orbit closure maximal if it is not contained in a larger proper W orbit closure.

Theorem (BFMS)

If there are infinitely many maximal W orbit closures in G/Γ then Γ is arithmetic.

And the question motivated by GPS becomes:

Question

For $n > 3$, does minimality of the action of $SO(n-1,1)$ on $SO(n,1)/\Gamma$ imply Γ is arithmetic?

Lessons from the master

Margulis: to prove arithmeticity, prove superrigidity.

From now on, G simple group, $\Gamma < G$ lattice.

Let $L = \mathbb{L}(k)$ where $\mathbb{L}$ is an algebraic group and k is a local field.

Superrigidity provides criteria for when $\rho : \Gamma \to L$ Zariski dense extends to G.

Superrigidity for certain $\mathbb{L}, k$ and ρ implies arithmeticity of Γ.

Only need $\mathbb{L}$ with the same complexification as G, so assume this. (Absolutely isogenous.)

Superrigidity theorem

From now on $G = \mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$.

Basically $L = \mathrm{SO}(p,q)$ with $p+q = n+1$ or $L = \mathrm{SU}(p,q)$ with $p+q = n+1$.

Totally geodesic submanifolds given by orbit closures of W on G/Γ where $W < G$ is simple non-compact.

Theorem

Assume L as above. Then any $\rho : \Gamma \to L$ with non-compact, Zariski dense image extends to G provided there is W as above and $J < L$ algebraic and a Γ equivariant measurable map $G/W \to L/J$.

Indication of proof:

"Two" steps in the proof:

Step 1: build Γ equivariant measurable map $G/W \to L/J$ with $J < L$ proper, algebraic.

Step 2: study properties of Γ equivariant measurable mappings to extend ρ.

Step 1 uses homogeneous dynamics, study of measures on projective bundles over G/Γ.

Step 2 uses ergodic theory of actions on algebraic varieties.

Margulis' proof of superrigidity for G higher rank:

Step 1: build Γ equivariant measurable map $G/P \to L/J$ with P minimal parabolic, $J < L$ proper, algebraic.

Step 1 uses that P is amenable and works for rank 1 groups too.

Step 2 used higher rank, centralizers, ergodic theory of algebraic actions.

We use normalizers. If $T < G$ is a subgroup, $N(T)$ is the normalizer in G of T.

Obstruction in the complex hyperbolic case:

In the complex hyperbolic case, we encounter an obstruction.

from which we build a *geometry preserving* map from $\partial\mathbb{CH}^n \to \partial\mathbb{CH}^n$.

History of geometry preserving maps in rigidity theorems:

Mostow's use of Tits' theorem in the proof of (higher rank) Mostow rigidity (1972)
Kleiner-Leeb, Eskin-Farb, Eskin: QI rigidity for higher rank lattices (1997-8)
Fisher, Nguyen, Whyte: QI superrigidity for higher rank lattices (2018-20)

Burger-Iozzi (2009), Pozzetti (2015), Duchesne-Lecureux-Pozzetti (2018): superrigidity for maximal representations of lattices in $\mathrm{SU}(n, 1)$

Margulis-Mohammadi 2019

Proof of Step 1:

First find a "good" representation of L on a vector space V.

Goal: find a W invariant measure on a $\mathbb{P}(V)$ bundle over G/Γ that projects to Haar measure.

From there one can use ideas of Furtenberg-Margulis-Zimmer to obtain the equivariant map $G/W \to L/J$.

Fact: A TG submanifold M_i in X correspond to W invariant ergodic measure μ_i on G/Γ.

The data define a vector bundle E over G/Γ with fiber V

and an associted $\mathbb{P}(V)$ bundle F over G/Γ.

Step I continued

Observation: if V is chosen correctly, there is an W invariant line bundle in E over $\mathrm{supp}(\mu_i)$

i.e. a section s_i of F over $\mathrm{supp}\,\mu_i$

get $s_{i*}\mu_i = \nu_i$ on the bundle. Take ν any weak-* limit of the ν_i.

Need to check that ν projects to Haar measure on G/Γ.

Key tools: Ratner's theorem classifying W invariant measures on G/Γ.

Theorem of Mozes-Shah on limits of W invariant measures proven using Ratner.

Step 2 of the proof:

We fix $\rho : \Gamma \to L$ Zariski dense, unbounded, where L is algebraic.

Definition (Algebraic representation)

Let $T < G$ be a closed subgroup then a T-algebraic representation is given by:

1. *an algebraic subgroup $J < L$*
2. *a homomorphism $\tau : T \to N(J)/J$ defining a right T action on L/J*
3. *a map measurable map $\phi : G \to L/J$ which is equivariant for Γ and T*

Algebraic representations of two distinct subgroups S, T *have the same map* if J and ϕ can be chosen to be the same.

Proposition (Bader-Furman)

Let $T_1, \dots T_k$ which generate G topologically and assume all have algebraic representations with the same map and $J = 1$, then ρ extends.

Building algebraic representations

The existence of $\phi : G/W \to L/J$ is an obstruction to triviality of algebraic representations.

We say a T-algebraic representation is *initial* if J is minimal among possible J's (up to conjugacy).

All other algebraic representations then factor through $\phi : G \to L/J$.

I.e. given J' and ϕ' such that $\phi' : G \to L/J'$ is T-algebraic, we have $J < J'$ and $\phi' = \phi \circ \pi$ where $\pi : L/J \to L/J'$ is L equivariant projection (up to conjugacy).

Proposition (Bader-Furman)

If the T action on G/Γ is weak mixing, then there exists an initial T algebraic representation.

The initial T representation is also the initial $N(T)$ representation.

Basic strategy of proof

Start with $\phi : W\backslash G \to L/J$ from step 1.

View as $\phi : G \to L/J$ a W-algebraic representation.

Let P be a parabolic and $U < P$ the unipotent radical.

ϕ is a W algebraic representation so also $W \cap U$-algebraic representation.

Replace ϕ with the initial $W \cap U$-algebraic representation $\tilde{\phi}$.

Play with normalizers to show

1. $\tilde{\phi}$ is initial P representation
2. J for $\tilde{\phi}$ is trivial
3. $\tilde{\phi}$ is an initial $N(A)$ representation where A is Cartan.

Since $N(A)$ and P generate G, this suffices.

Works unless $G = L = SU(n, 1)$.

Breaks at (2).

Towards the complex hyperbolic case:

For (2) the unique obstruction is $\tilde{\phi} : G \to G/Z(U)$ (Recall $G = L$).

Chain = Totally geodesic $\mathbb{H}^2$ in $\mathbb{CH}^2$ that is complex submanifold (Cartan)

Orbits of $SU(1,1)$ in $SU(n,1)/K = \mathbb{CH}^n$.

Recall $\partial\mathbb{CH}^n = G/P$.

Theorem (Pozzetti)

Any measurable map from G/P to G/P with Zariski dense image that sends almost every chain to a chain agrees almost everywhere with a rational map.

Superrigidity follows.

Prior work by Cartan, Burger-Iozzi.

Pozzetti studies maps to higher rank targets as well.

How to get a geometry preserving map from $\tilde{\phi}$?

$\tilde{\phi} : G \to G/Z(U)$ is P equivariant. So covers a map $\bar{\phi} : G/P \to G/P$.

U is a Heisenberg group.

$P = MAU$ stabilizes a point at infinity.

$P \cap SU(1,1) = MAZ(U)$ stabilizes a chain and a point on that chain.

i.e. $G/MAZ(U)$ is the space of pointed chains.

$\tilde{\phi}$ also covers a map on pointed chains $G/MAZ(U) \to G/MAZ(U)$.

Use fiber product constructions to show that $\bar{\phi}$ sends triples of points on a chain to triples of points on a chain.

More general geometries and superrigidities?

Now let L be a general simple group and $\rho : \Gamma \to L$ with Zariski dense unbounded image.

Assume there exists Γ equivariant $\phi : W\backslash G \to L/J$ with $J < L$ proper algebraic.

Call ρ *compatible* if at step (2) above we win. I.e. $\tilde{\phi}$ has J trivial.

If ρ is not compatible, then there is a geometry and a geometry preserving map.

More general geometries and superrigidities?

Meaning?

There is a parabolic $Q < L$ and a map $\bar{\phi} : P\backslash G \to L/Q$

and proper subgroups $C < P$ and $D < Q$ such that $\bar{\phi}$ is dominated by a map

$\hat{\phi} : C\backslash G \to H/D$.

Question: can one prove superrigidity for general L using this?

The proof that Q, C, D exist puts some constraints on these groups.

One can really choose $C < U$.

Applications of more general superrigidity theorems?