Pointwise normality and Fourier decay for self-conformal measures

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Midwest dynamics and group actions seminar, 24.5.2021

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Method Est. fast decay of $L^2(\mu)$ norms of trig. poly. as in Weyl's cri.

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Definition

We call ν a *Rajchman measure* if

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   Exp: Fix $r \in (0,1)$ consider $\{r \cdot x - 1, r \cdot x + 1\}$ with $\mathbf{p} = (\frac{1}{2}, \frac{1}{2}) \Rightarrow \nu_{\mathbf{p}}$ = Ber. Conv. with para. r.
2. If every f_i is diff., K = self conformal set, $\nu_{\mathbf{p}}$ = self conformal measure.

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Unified proof of Rajchman property for C^ω IFS's

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We call this the derivative cocycle.

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May assume For $k \in \mathbb{N}$ and ω, $\quad S_{\tau_k(\omega)}(\omega) \in [k, k+1]$

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Use: Hochman's Lemma: Let $\theta \in \mathcal{P}(\mathbb{R})$. Then for any $r > 0$

$$\int_0^1 |\mathcal{F}_q(M_{e^{-t}}\theta)|^2 dt \leq \frac{e^2}{r \cdot |q|} + \int \theta(B_r(y)) d\theta(y)$$