Projective cones and hyperbolic Billiards

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Statistical properties

Let (X, f) or (X, ϕ_t) be a measurable dynamical system. Let μ be a (not necessarily invariant) probability measure.
We want to understand, for large n or t, the behaviour of

$$f_*^n\mu(g) := \int_X g \circ f^n d\mu$$
$$(\phi_t)_*\mu(g) := \int_X g \circ \phi_t d\mu$$

I'll consider only:
X is a Riemannian manifold, with volume form ω.
Let m be the corresponding measure, then $d\mu = hdm$.

Holes

I am interested also in the case in which there exists a forbidden set $H \subset X$.

Thus the dynamics can be iterated n times only in a subset $X_n = f^{-n+1}(H^c) \cap \cdots \cap f^{-1}(H^c) \cap H^c$ of X.

In the last decades several techniques to investigate these problems have been developed.

However all of them make use of some sort of *transfer operator*.

Transfer operator

Note that f may be non invertible.
We restrict to the case in which $f_* m$ is absolutely continuous with respect to m.
Let $d\mu = hdm$, then $df_*\mu = \mathcal{L}h\, dm$ for some linear operator $\mathcal{L}$ [the Ruelle transfer operator].

Transfer operator

Let $\{p_i\}$ be the invertibility partition of f, and set $\phi_i = f|_{p_i}^{-1}$.
Call $m_i(\varphi) = m(\mathbb{1}_{p_i}\varphi)$ and set $\rho_i = \frac{df_* m_i}{dm}$

$$
\begin{aligned}
m(h\mathbb{1}_{H^c}\varphi \circ f) &= \sum_i m(h\mathbb{1}_{H^c}\mathbb{1}_{p_i}\varphi \circ f) = \sum_i m_i(\{(\mathbb{1}_{H^c}h)\circ\phi_i\varphi\}\circ f) \\
&= \sum_i m(\rho_i(\mathbb{1}_{H^c}h)\circ\phi_i\varphi) = m(\varphi\mathcal{L}(\mathbb{1}_{H^c}h))
\end{aligned}
$$

where

$$
\mathcal{L}_f h(x) = \sum_{y\in f^{-1}(x)} \rho_i(y)h(y)
$$

which gives an explicit formula for the Ruelle transfer operator.

Transfer operator

Hence, setting $\mathcal{L}_{f,H}(h) := \mathcal{L}_f(\mathbb{1}_{H^c}h)$,

$$m(h\mathbb{1}_{X_n}\varphi\circ f^n) = m(h\mathbb{1}_{H^c}[\mathbb{1}_{X_{n-1}}\varphi\circ f^{n-1}]\circ f) = m(\varphi\mathcal{L}_{f,H}^n h).$$

The problem is thus reduced to studying the operator $\mathcal{L}_{f,H}$.
One way is to study the spectrum of $\mathcal{L}_{f,H}$.

Time dependency

There are important case where the dynamics changes with time. In this case we have at each time a different dynamics f_n and, possibly, different holes H_n. It follows that we are interested in the survival set $X_n = f_{n-2}(H_{n-1}^c) \cap \cdots \cap f_0^{-1}(H_1^c) \cap H_0^c$

$$m(h\mathbb{1}_{X_n}\varphi \circ f_{n-1} \circ \cdots \circ f_0) = m(\varphi \mathcal{L}_{f_{n-1},H_{n-1}} \cdots \mathcal{L}_{f_0,H_0} h)$$

We have to study the composition of operators, as n increases,

$$\mathcal{L}_{f_{n-1},H_{n-1}} \cdots \mathcal{L}_{f_0,H_0}$$

Spectral theory does not apply, we need an alternative.

Projective cone

A nice idea is to study the action of $\mathcal{L}$ on vector lattices $\mathbb{V}$.

Given a closed convex cone $\mathcal{C} \subset \mathbb{V}$, enjoying the property $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we can define an order relation by

$$f \preceq g \iff g - f \in \mathcal{C} \cup \{0\}.$$

Hilbert metric

Given $\mathcal{C}$ we can define a projective metric Θ (Hilbert metric):

$$\begin{aligned}
\alpha(f,g) &= \sup\{\lambda \in \mathbb{R}^+ \mid \lambda f \preceq g\} \\
\beta(f,g) &= \inf\{\mu \in \mathbb{R}^+ \mid g \preceq \mu f\} \\
\Theta(f,g) &= \log\left[\frac{\beta(f,g)}{\alpha(f,g)}\right]
\end{aligned}$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

Projective cone

In this setting, the basic theorem, due to Garret Birkhoff, is

Theorem

Let $\mathbb{V}_1$*, and* $\mathbb{V}_2$ *be two vector spaces;* $\mathcal{L} : \mathbb{V}_1 \to \mathbb{V}_2$ *a linear map such that* $\mathcal{L}(\mathcal{C}_1) \subset \mathcal{C}_2$*, for two closed convex cones* $\mathcal{C}_1 \subset \mathbb{V}_1$ *and* $\mathcal{C}_2 \subset \mathbb{V}_2$ *with* $\mathcal{C}_i \cap -\mathcal{C}_i = \emptyset$*. Let* Θ_i *be the Hilbert metric of the cone* $\mathcal{C}_i$*. Setting* $\Delta = \sup_{f,g \in T(\mathcal{C}_1)} \Theta_2(f,g)$ *we have*

$$\Theta_2(\mathcal{L}f, \mathcal{L}g) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta_1(f,g) \qquad \forall f,g \in \mathcal{C}_1$$

($\tanh(\infty) \equiv 1$*).*

A simple example

Let us consider the map $f \in \mathcal{C}^2(\mathbb{T},\mathbb{T})$, $f' \geq \lambda > 1$.
We want to study the transfer operator

$$\mathcal{L}h(x) = \sum_{y \in f^{-1}(x)} \frac{h(y)}{f'(y)}.$$

Consider the cone

$$\mathcal{C}_a = \left\{ h \in \mathcal{C}^1 \;:\; h \geq 0\,;\, \frac{|h'(x)|}{h(x)} \leq a \right\}$$

A simple example: cone contraction

Then, setting $D = \|\frac{f''}{(f')^2}\|_{\mathcal{C}^0}$ (*distortion*),

$$\begin{aligned}\left|\frac{d}{dx}\mathcal{L}h(x)\right| &= \left|\sum_{y\in f^{-1}(x)}\frac{h'(y)}{f'(y)^2} - \frac{f''(y)h(y)}{f'(y)^3}\right| \\ &\leq \sum_{y\in f^{-1}(x)}\frac{|h'(y)|}{h(y)}\frac{h(y)}{f'(y)^2} + D\frac{h(y)}{f'(y)} \\ &\leq (a\lambda^{-1}+D)\mathcal{L}h(x).\end{aligned}$$

Thus, for $\sigma \in (\lambda^{-1}, 1)$,

$$\mathcal{L}\mathcal{C}_a \subset \mathcal{C}_{a\lambda^{-1}+D} \subset \mathcal{C}_{\sigma a}$$

provided $a \geq D(\sigma - \lambda^{-1})^{-1}$.

A simple example: Diameter

Note that, if $h \in \mathcal{C}_a$, then $h(x) \geq e^{-a} \int_{\mathbb{T}} h$. Then, using the definition one can compute for each $h, g \in \mathcal{C}_{\sigma a}$

$$\Theta(h,g) \leq \ln \frac{(1+\sigma)^2}{1-\sigma)^2} e^{4a} = \Delta$$

Hence, by Birkhoff theorem,

$$\Theta(\mathcal{L}h, \mathcal{L}g) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta(h,g) =: \nu\Theta(h,g) < \Theta(h,g).$$

So what?

A simple example: Contraction

Let $\|\cdot\|$ be a norm on $\mathbb{V}$, and suppose that it is *order preserving*. That is, for each $h, g \in \mathbb{V}$,

$$-h \preceq g \preceq h \Longrightarrow \|h\| \geq \|g\|.$$

In addition, let $\mathfrak{n} : \mathcal{C} \to \mathbb{R}_+$ be a homogeneous of degree one and order preserving function, i.e.

$$\forall h \in \mathcal{C},\ \lambda \in \mathbb{R}_+ \quad \mathfrak{n}(\lambda h) = \lambda \mathfrak{n}(h)$$
$$\forall h, g, \in \mathcal{C} \quad h \preceq g \Longrightarrow \mathfrak{n}(h) \leq \mathfrak{n}(g),$$

then, given $h, g \in \mathcal{C} \subset \mathbb{V}$ for which $\mathfrak{n}(h) = \mathfrak{n}(g) > 0$, then

$$\|h - g\| \leq \left(e^{\Theta(h,g)} - 1\right) \min\{\|h\|, \|g\|\}.$$

A simple example: Contraction

The above applies to $\|\cdot\|_{\mathcal{C}^0}$ and $\mathfrak{n}(h) = \int_{\mathbb{T}} h$, yielding, for all $h, g \in \mathcal{C}_a$,

$$\begin{aligned}
\|\mathcal{L}^n h - \mathcal{L}^n g\|_{\mathcal{C}^0} &\leq \left(e^{\Theta(\mathcal{L}^n h, \mathcal{L}^n g)} - 1\right) \min\{\|h\|_{\mathcal{C}^0}, \|g\|_{\mathcal{C}^0}\} \\
&\leq \left(e^{\nu^n \Theta(h,g)} - 1\right) \min\{\|h\|_{\mathcal{C}^0}, \|g\|_{\mathcal{C}^0}\} \\
&\leq c\nu^{n-1} \Delta \min\{\|h\|_{\mathcal{C}^0}, \|g\|_{\mathcal{C}^0}\}.
\end{aligned}$$

A simple example: loss of memory

Consider $\{f_i\}$ with $f_i \geq \lambda > 1$ and $\|\frac{f_i''}{(f_i')^2}\|_{\mathcal{C}^0} \leq C$, then

$$|\mathcal{L}_n \cdots \mathcal{L}_{f_1} h - \mathcal{L}_{f_n} \cdots \mathcal{L}_{f_1} g\|_{\mathcal{C}^0} \leq c\nu^{n-1} \Delta \min\{\|h\|_{\mathcal{C}^0}, \|g\|_{\mathcal{C}^0}\}$$

Hence, for each $m < n$,

$$\begin{aligned} \left|\mathcal{L}_{f_n} \cdots \mathcal{L}_{f_1} h - \mathcal{L}_{f_n} \cdots \mathcal{L}_{f_{n-m}} 1\right\|_{\mathcal{C}^0} &\leq c\nu^m \Delta \min\{\|\mathcal{L}_{f_{n-m-1}} \cdots \mathcal{L}_{f_1} h\|_{\mathcal{C}^0}, 1\} \\ &\leq c\nu^m \Delta \min\{\|h\|_{\mathcal{C}^0}, 1\} \end{aligned}$$

A simple example: Holes

If we want to study the operator $\mathcal{L}_{f,H}$, then the previous cone will not work because $\mathcal{L}_{f,H}$ does not leave invariant $\mathcal{C}_0$.
A good substitute is

$$\mathcal{C}_a := \left\{ h \in \mathrm{BV} \mid h \not\equiv 0;\ h \geq 0;\ \bigvee h \leq a \int h \right\}.$$

A less simple example: hyperbolic billiards

The simplest hyperbolic Billiard consists of a particle in a bounded regions which moves in straight lines and collides elastically against finitely many obstacles.
Such Billiards are hyperbolic, and have stable and unstable invariant cones.
Let $\mathcal{W}^s$ the collection of curves with tangent in the stable cone.

A less simple example: hyperbolic billiards

The basic idea, going back to Liverani 1995, is to consider not the pointwise value of a density, but only its value when integrated along a stable curve.

To this end we have to be a bit more specific about the curves

By $\mathcal{W}^s(\delta)$ we mean the curves of length between δ and 2δ.

By $\mathcal{W}^s_-(\delta)$ stands for all the curves of length less than δ.

A less simple example: hyperbolic billiards

If we want to consider h as a function from the space of stable curves to $\mathbb{R}$, and we want to talk about the regularity of such a function, then we need to introduce a "distance" among curves.

A less simple example: hyperbolic billiards

$W \in \mathcal{W}^s$ be the graphs of C^2 functions over an interval I_W:

$$W = \{G_W(r) = (r, \varphi_W(r)) : r \in I_W\}.$$

Then

$$d_{\mathcal{W}^s}(W^1, W^2) = |\varphi_{W^1} - \varphi_{W^2}|_{C^1(I_{W^1} \cap I_{W^2})} + |I_{W^1} \triangle I_{W^2}|,$$

if W^1 and W^2 lie in the same homogeneity strip and $|I_{W^1} \cap I_{W^2}| > 0$; otherwise, we set $d_{\mathcal{W}^s}(W^1, W^2) = \infty$.

A less simple example: hyperbolic billiards

One last thing, we want to integrate h along stable curves, but against which density? (henceforth called "test function"?)
For $W \in \mathcal{W}^s$, $\alpha \in (0,1]$ and $a \in \mathbb{R}^+$, define a cone of test functions by

$$\mathcal{D}_{a,\alpha}(W) = \left\{ \psi \in C^0(W) : \psi > 0, \frac{\psi(x)}{\psi(y)} \leq e^{a d(x,y)^\alpha} \right\},$$

where $d(\cdot,\cdot)$ is the arclength distance along W.

A less simple example: hyperbolic billiards

$$|||f|||_+ = \sup_{\substack{W\in\mathcal{W}^s(\delta)\\ \psi\in\mathcal{D}_{a,\beta}(W)}} \frac{\int_W f\psi\, dm_W}{\int_W \psi\, dm_W}, \qquad |||f|||_- = \inf_{\substack{W\in\mathcal{W}^s(\delta)\\ \psi\in\mathcal{D}_{a,\beta}(W)}} \frac{\int_W f\psi\, dm_W}{\int_W \psi\, dm_W},$$

Denote the average value of ψ on W by
$⨍_W \psi\, dm_W = \frac{1}{|W|}\int_W \psi\, dm_W$.

A less simple example: hyperbolic billiards

For $a, c, A, L > 1$, and δ small enough, define the cone

$$\mathcal{C}_{c,A,L}(\delta) = \Big\{ f : \qquad |||f|||_+ \le L |||f|||_-;$$

$$\sup_{W \in \mathcal{W}^s_-(\delta)} \sup_{\psi \in \mathcal{D}_{a,\beta}(W)} |W|^{-q} \frac{|\int_W f\psi|}{\fint_W \psi} \le A\delta^{1-q} |||f|||_-;$$

$$\forall\, W^1, W^2 \in \mathcal{W}^s_-(\delta) : d_{\mathcal{W}^s}(W^1, W^2) \le \delta, \forall \psi_i \in \mathcal{D}_{a,\alpha}(W_i) : d_*(\psi_1, \psi_2) = 0,$$

$$\left| \frac{\int_{W^1} f\psi_1}{\fint_{W^1} \psi_1} - \frac{\int_{W^2} f\psi_2}{\fint_{W^2} \psi_2} \right| \le d_{\mathcal{W}^s}(W^1, W^2)^\gamma \delta^{1-\gamma} cA |||f|||_- \Big\}.$$

A less simple example: hyperbolic billiards

Let f be the Poincarè associated to the billiard flow.
Then there exists $n_0 > 0$ and $\nu \in (0,1)$ such that, for all $n \geq n_0$,

$$\mathcal{L}^n \mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\nu c,\nu A,3L}(\delta).$$

To have a contraction of L we need mixing of f.

A less simple example: hyperbolic billiards

If f is mixing, then there exists $n_1 \geq n_0$, such that, for all $n \geq n_1$,

$$\mathcal{L}^n \mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\nu c,\nu A,\nu L}(\delta).$$

And

$$\text{diam}_{\mathcal{C}_{c,A,L}(\delta)}(\mathcal{C}_{\nu c,\nu A,\nu L}(\delta)) < \infty$$

Applications: Time varing billiards

This applies when the billiard configuration changes in time.

Note that the change can be drastic, for example after every collision the obstacle configuration can change completely, not just a small perturbation.

Applications: Chaotic scattering

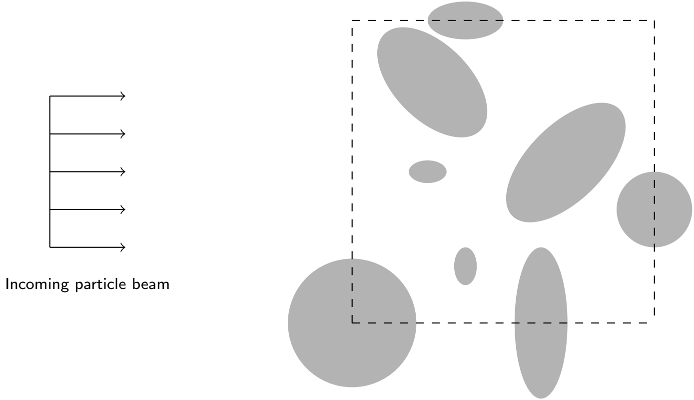
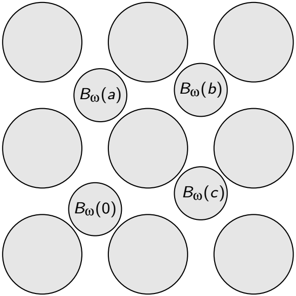


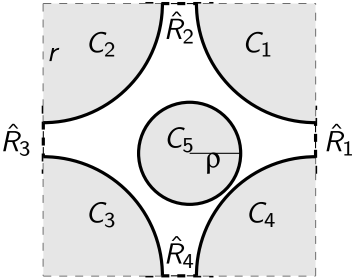
Figure: Obstacle configuration for which the non-eclipse condition fails and the box R (dashed line).

Applications: random Lorentz gas



$a = (1,0); b = (1,1); c = (1,0)$

Random obstacles $B_\omega(z)$



Poincaré section C_i *and gates* $\hat{R}_i$

Applications: random Lorentz gas

There exist $C_* > 0$ and $\vartheta \in (0,1)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, if the particle belongs to the cell zero with initial condition distributed according to $f \in \mathcal{C}_{c,A,L}(\delta)$ with $\int_M f = 1$, then for all $n > m \geq 0$ and all paths $(w_1, \dots, w_n)$,

$$\left|\mathbb{P}_\omega(w_{k_n} \mid w_{k_0} \dots w_{k_{n-1}}) - \mathbb{P}_{\xi_{z_m}\omega}(w_{k_n} \mid w_{k_m} \dots w_{k_{n-1}})\right| \leq C_* \vartheta^{n-m}.$$