The strong unstable foliation of an Anosov diffeomorphism

Partial hyperbolicity: M = closed Riem. mfld

$f : M \circlearrowleft$ diffeo. is partially hyperbolic if $\exists$

$$TM = E^{uu} \oplus E^{c} \oplus E^{ss} \circlearrowleft Df$$

- E^{uu}: unstable, vectors uniformly expanded.
- E^{c}: center, dominated by other two
- E^{ss}: stable, uniform contraction

E^c can be chosen to be $\{0\}$ — f is <u>Anosov</u>.

E^{ss}, E^{uu} always integrable, tangent to f-invariant folns $\mathcal{W}^{ss}$, $\mathcal{W}^{uu}$.

<u>Question</u> (D. Fisher) How do the leaves of $\mathcal{W}^{uu}$ distribute in M?

<u>Examples</u>:

1) $M = \mathbb{T}^3$ $\quad f(x) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \mathrm{Id}_{S^1}$

$x = (x_1, x_2, x_3)$

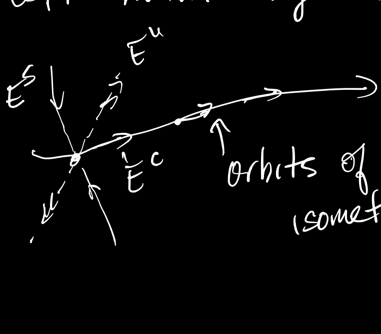
$\mathcal{W}^{ss}_f(x) = \mathcal{W}^s_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)}(x_1, x_2) \times \{x_3\}$

$\mathcal{W}^{ss}, \mathcal{W}^{uu}$ linear

$\forall x \in M$ $\quad \overline{\mathcal{W}^{uu}(x)} = \mathbb{T}^2 \times \{x_3\}$.

2) $M = SL(2,\mathbb{R})/\Gamma$ $\quad \Gamma$ = cocompact

f = left mult. by $\underline{\underline{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}}}$ $\quad |a| > 1$



orbits of $\left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right\}$

isometry.

$\mathcal{W}^{uu}$ = foliation by orbits of $u^+_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$\mathcal{W}^{ss} = \ldots \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$

$\forall x \in M \quad \overline{\mathcal{U}_{\mathbb{R}}^{+}} = M \Rightarrow \overline{\mathcal{W}^{uu}(x)} = M$

3) $M = \mathbb{T}^3$ f given by $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ eigenvals $\approx .2$ $\approx \underline{1.55}$ $\approx \underline{3.25}$

$TM = \underbrace{E^{uu} \oplus E^{c}}_{E^u} \oplus E^s$

$E^c = \{0\}$ Anosov

E^c tangent to 1.55... eigendir. Partially hyp.

$\mathcal{W}^{uu}$ linear (affine) foliation.

$\forall x \in \mathbb{T}^3 \quad \overline{\mathcal{W}^{uu}(x)} = \mathbb{T}^3$

All 3 examples, $\mathcal{W}^{uu}$ are orbits of a unipotent flow. Ratner's theorem $\Rightarrow$ orbit closures closed submflds.

<u>Question</u>: Is there a Ratner-type orbit closure thm for general $\mathcal{W}^{uu}$ of transitive partially hyp?

- In particular, under what hypotheses is $\mathcal{W}^{uu}$ <u>minimal</u> meaning every leaf is dense?

Partial hyperbolicity is C^1-open. Consider perturbations of 1-3.

Examples 1 & 2 Bonatti-Díaz-Ures '02: C^1 densely in a nbd of linear examples like these, either $\mathcal{W}^{uu}$ or $\mathcal{W}^{ss}$ is <u>robustly</u> minimal.

Example 3: Golgolev-Maimon-Kolmogorov ('19) Numeric study. Conjecture that generically $\mathcal{W}^{uu}$ is minimal

Today: example 3 & relatives.

Anosov diffeos with expanding center

$f: M \circlearrowleft$ Anosov has expanding center if

$$TM = E^u \oplus E^s \circlearrowleft Df \text{ (Anosov)}$$

$$E^u = E^{uu} \oplus E^c$$

Remark: If $M = \mathbb{T}^k$ then f is conjugate via h to linear Anosov A & $h(\mathcal{W}^u_f) = \mathcal{W}^u_A$ $h(\mathcal{W}^s_f) = \mathcal{W}^s_A$

But do not expect $h(\mathcal{W}^{uu}_f) = \underline{\mathcal{W}^{uu}_A}$ ← even if this exists

Theorem 1 (Avila-Crovisier-W, in progress) C^1 generically among Anosovs with expanding center, the fol'n $\mathcal{W}^{uu}$ is minimal.

Theorem 2 (ACW) Fix $r > 1$. $\exists$ C^1-open C^r-dense set among Anosovs with 1-dim'l expanding center s.t. $\mathcal{W}^{uu}$ is minimal.

Theorem 3 (ACW + Eskin-Potrie-Zhang) If $f: \mathbb{T}^3 \circlearrowleft$ Anosov with 1-d expanding center, $\mathcal{W}^{uu}_f$ is minimal. (Answers Q. of G-M-K.)
Measure-theoretic version, uses work A. Katz.

Proofs

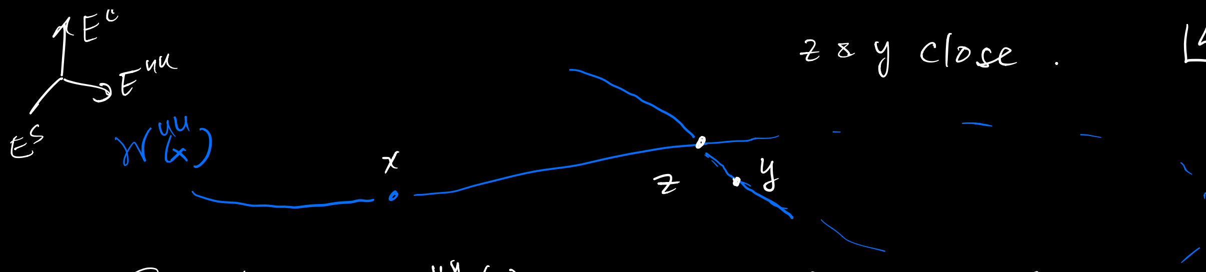
Thm 1 & 2: construct horseshoe-like blenders $\rightsquigarrow$ C^r generic $\mathcal{W}^{uu}$ minimal.

Thm 2 & 3: $\dim E^c = 1$ openness in thm 2 ...

Key property: s-transversality of $\mathcal{W}^{uu}$

Let f be An. w/ 1-d exp. center.
A compact $\mathcal{W}^{uu}$-saturated invariant set Λ is s-transverse ($s\pitchfork$) if $\forall x \in \Lambda$, $\exists z, y, \in \mathcal{W}^{uu}(x)$

z & y close.



s.t. Projection of $W^{uu}_{loc}(y)$ onto $W^{u}_{loc}(z)$ along W^{s}_{loc} leaves crosses topologically $\pitchfork$ to $W^{uu}_{loc}(z)$

If M is s$\pitchfork$ for f, then it is for any f' C^1 close to f.

Thm 4: If Λ is s$\pitchfork$, then Λ is an attractor.
In particular if f is transitive, then W^{uu} is minimal.

Thm 5 (ACW) $\exists$ C^r generic set of AEC w dim 1 s.t. M is s$\pitchfork$

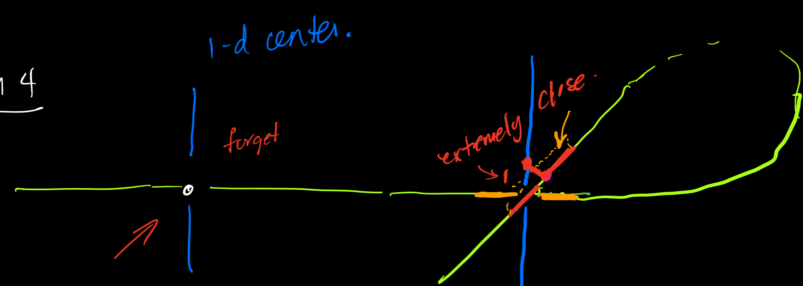
Thm 6 ("Rafael's Argument") On $\mathbb{T}^3$ & f C^1 AEC.
then either

- $E^{uu} \oplus E^s$ is integrable & $h(W^{uu}_f) = W^{uu}_A$
  $\Rightarrow W^{uu}$ minimal.

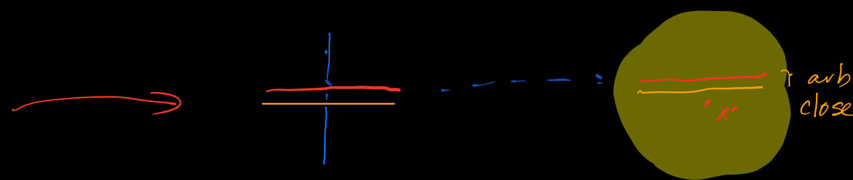
- $\mathbb{T}^3$ is s$\pitchfork$

thm 4 & 6 $\Rightarrow$ Thm 3
thm 4 & 5 $\Rightarrow$ Thm 2

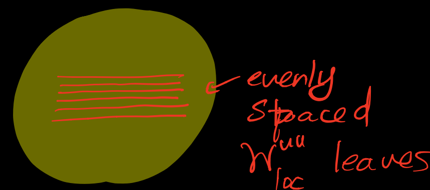
Thm 4



Iterate forward

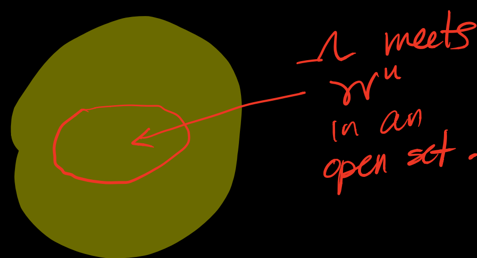


Inductively



In Λ in same $\mathcal{W}^u_{loc}$ leaf. arb. close together

Limit



Λ meets $\mathcal{W}^u$ in an open set.

$\Rightarrow \Lambda$ is $\mathcal{W}^u$ saturated

$\Rightarrow \Lambda$ is an attractor

Thm 6 ("Rafael's Argument") On $\mathbb{T}^3$ & f C^1 AEC.
then either

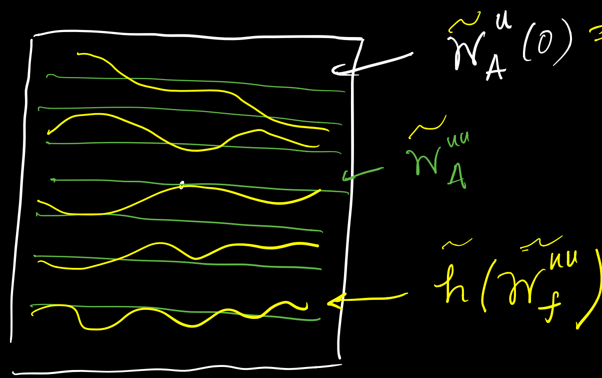
- $E^{uu} \oplus E^s$ is integrable & $h(\mathcal{W}^{uu}_f) = \mathcal{W}^{uu}_A$
  $\Rightarrow \mathcal{W}^{uu}$ minimal.

- $\mathbb{T}^3$ is s Λ

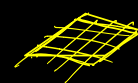
Sketch Proof: $f : \mathbb{T}^3$ AEC $\quad h : \mathbb{T}^3$ FM conjugacy to $A : \mathbb{T}^3 \circlearrowleft$

$h(\mathcal{W}^u_f) = \text{linear } \mathcal{W}^u_A$

$\mathcal{W}^s_f \quad \cdot \quad \cdot \quad \mathcal{W}^s_A$



WTS: $h(\mathcal{W}^{uu}_f) = \mathcal{W}^{uu}_A \Leftarrow$ $E^{uu}_f \oplus E^s_f$ (uniquely) integrable.



WTS: $h(\mathcal{W}^{uu}_f)$ is linear (done)

Not linear, then can do following.

Consider in $\mathbb{R}^3$ the $\pi_1(\mathbb{T}^3) = \mathbb{Z}^3$ action on P. Translation composed w/ projection along $\mathcal{W}^s_A$ leaves

Since $\mathcal{W}^s_A$ is minimal, this action is minimal