# Amenability of covers and critical exponents

**Seminar**

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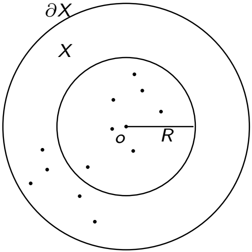
# Critical exponents

A discrete group $\Gamma$ acts on a hyperbolic space $X$ (for ex. $X = \mathbb{H}^n$).

The critical exponent of this action is

$$\delta_\Gamma = \limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma,\ d(o, \gamma o) \leq R\}.$$



## Critical exponents

Coincides with

$\rightarrow$ the dimension of the (radial) limit set $\Lambda_{rad}(\Gamma)$ inside $\partial X$

$\rightarrow$ the entropy of the geodesic flow on $T^1(X/\Gamma)$

Of course, if $\Gamma' < \Gamma$, $\delta_{\Gamma'} \leq \delta_\Gamma$.

**Question :** When do we have equality $\delta_{\Gamma'} = \delta_\Gamma$ ?

# Our result

**Theorem :** (Coulon-Dougall-Sch.-Tapie 2018) Let $\Gamma$ be a discrete group acting on a proper hyperbolic space $X$, with *entropy gap at infinity* $\delta_\Gamma^\infty < \delta_\Gamma$. Let $\Gamma' < \Gamma$ be a subgroup. Then

$$\delta_{\Gamma'} = \delta_\Gamma \quad \text{iff} \quad \Gamma/\Gamma' \quad \text{amenable}\,.$$

Result already known in particular cases :
$\to$ Brooks (81,85) : convex-cocompact actions on $\mathbb{H}^n$, with $\delta_\Gamma > n/2$
$\to$ Grigorchuk, Cohen (80, 82) : action of a free group on its Cayley graph
$\to$ Stadlbauer: convex-cocompact (and some geom. finite) actions on $\mathbb{H}^n$
$\to$ Dougall-Sharp : convex-cocompact actions in variable neg. curvature
$\to$ Coulon-Dal'bo-Sambusetti : cocompact actions on $CAT(-1)$-spaces
$\to$ Roblin (03) : amenability implies equality when $\Gamma' \triangleleft \Gamma$

# Amenability

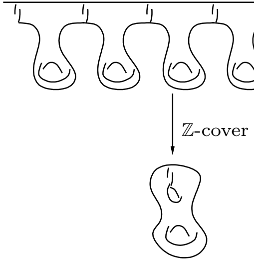
The group $\Gamma'$ is coamenable in $\Gamma$ if the regular representation

$$\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma/\Gamma'))$$

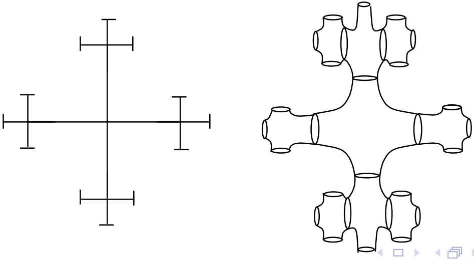
defined by $\rho(\gamma)(\varphi)(\cdot) = \varphi(\gamma\ \cdot)$ almost admits invariant vectors : $\|\rho(\gamma)\varphi - \varphi\| < \varepsilon\|\varphi\|$.

Typical amenable group : $\mathbb{Z}, \mathbb{Z}^d$
Typical nonamenable group : $\mathbb{F}^n$

# (Non)-Amenable covers




And an attempt to draw a $\mathbb{F}_2$-cover of a compact hyperbolic surface



# Entropy gap at infinity

The group $\Gamma$ acts on $X$ (for ex. $X = \mathbb{H}^n$)
Let $K \subset X$ be a compact set, $o \in K$. Define

$$\Gamma_K = \{\gamma \in \Gamma, [o, \gamma o] \cap \Gamma K \subset K \cup \gamma K\} \subset \Gamma\,.$$

The entropy at infinity is

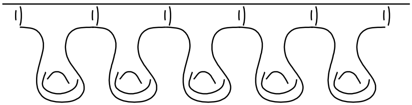
$$\delta_\Gamma^\infty = \inf_{K \subset X} \delta_{\Gamma_K} \leq \delta_\Gamma\,.$$

The action of $\Gamma$ on $X$ admits a entropy gap at infinity when $\delta_\Gamma^\infty < \delta_\Gamma$.

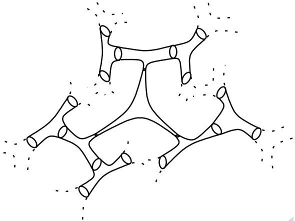
We call these actions strongly positively recurrent actions.

# Manifolds with(out) entropy gap at infinity

Typical examples without entropy gap : infinite covers.



Typical examples with entropy gap : compact or convex-cocompact manifolds, geometrically finite locally symmetric manifolds, Schottky products, Ancona surfaces.



# Optimality of the assumptions

Result false without hyperbolicity :
$\Gamma$ amenable group with exponential growth, $X = Cay(\Gamma)$, $\Gamma' = \{1\}$.
Then $\delta_\Gamma > 0$ whereas $\delta_{\Gamma'} = 0$, and $\Gamma/\Gamma' = \Gamma$ is amenable

On higher rank symmetric spaces, if $\Gamma/\Gamma'$ is amenable, then $\delta_\Gamma = \delta_{\Gamma'}$ (Glorieux-Tapie).

Result false without critical gap :
Let $S = X/\Gamma$ be a negatively curved surface without critical gap.
For example, $S$ is a $\mathbb{Z}$-cover of a compact hyperbolic surface. Build a $\mathbb{F}_2$-cover $S' = X/\Gamma'$ of $S$ by cutting $S$ along two disjoint nonseparating closed curves. Then there is no critical gap :

$$\delta_\Gamma \geq \delta_{\widehat{\Gamma}} \geq \delta_{\widehat{\Gamma}}^\infty = \delta_\Gamma^\infty = \delta_\Gamma\,.$$

# Recall the Patterson-Sullivan construction

The Poincaré series

$$P(s) = \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)}$$

has critical exponent $\delta_\Gamma$. For $s > \delta_\Gamma$, build a measure on $X \cup \partial X$

$$\nu^s = \frac{1}{P(s)} \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} \mathcal{D}_{\gamma o} \, .$$

When $s \to \delta_\Gamma$, get $\nu$ on $\partial X$ as any weak limit of $\nu^s$. The measure $\nu$ on $\partial X$ is quasi-invariant under $\Gamma$.

The unit tangent bundle satisfies $T^1 X \simeq \partial X \times \partial X \setminus Diag \times \mathbb{R}$
Build a $\Gamma$-invariant product equivalent to $\nu \times \nu \times dt$
Get *Bowen-Margulis measure* $m_{BM}$ on $T^1 X / \Gamma$ (ergodic, mixing...)

# Strategy of the proof

**Step 1 :** Twisted Poincaré series $A(s) = \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} \rho(\gamma)$. It has a critical exponent $\delta_\rho \in [\delta_{\Gamma'}, \delta_\Gamma]$ such that for $s > \delta_\rho$, $A(s) \in \mathcal{B}(\ell^2(\Gamma/\Gamma'))$.

**Step 2 :** Build a twisted Patterson-Sullivan measure $a^\rho$ on $\partial X$ with (nonzero) values in $\mathcal{B}(\ell^2(\Gamma/\Gamma'))$, by taking limits of

$$\frac{1}{\|A(s)\|} \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} \rho(\gamma) \mathcal{D}_{\gamma o} \,.$$

**Step 3 :** When $\delta_\rho = \delta_\Gamma$ and $\delta_\Gamma^\infty < \delta_\Gamma$, get absolute continuity of $a^\rho$ w.r.t. the classical Patterson-Sullivan measure $\nu$.

**Step 4 :** By an ergodicity argument, deduce that $a^\rho = \Psi \,.\nu$ where $\Psi \in \mathcal{B}(\ell^2(\Gamma/\Gamma'))$ is a "multiplicative constant".

**Step 5 :** By construction of $a^\rho$ and $\nu$, $\Psi$ "takes values" in the set of almost invariant vectors.

## Step 1 : The twisted Poincaré series

Study $A(s) = \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} \rho(\gamma)$.

The Hilbert space $\mathcal{H} = \ell^2(\Gamma/\Gamma', \mathbb{R})$ has a partial order compatible with the norm : $\phi \geq 0$ if for all $y \in \Gamma/\Gamma'$, $\phi(y) \geq 0$.

Define the associated positive cone $\mathcal{H}_+$. A bounded operator on $\mathcal{H}$ is positive if it preserves $\mathcal{H}_+$. All $\rho(\gamma)$ are positive.

The Poincaré series $A(s)$ is bounded if $\exists M > 0$, s.t. for all finite $S \subset \Gamma$, $\| \sum_{\gamma \in S} e^{-sd(o,\gamma o)} \rho(\gamma) \| \leq M$. The critical exponent

$$\delta_\rho = \inf\{s \in \mathbb{R}, A(s) \text{ is bounded}\}$$

is well defined. Easy to check that

$$\delta_{\Gamma'} \leq \delta_\rho \leq \delta_\Gamma \, .$$

The assumption $\delta_{\Gamma'} = \delta_\Gamma$ is only used to guarantee that $\delta_\rho = \delta_\Gamma$.

## Step 2 : The twisted Patterson-Sullivan measure

Let $\mathcal{H} = \ell^2(\Gamma/\Gamma')$. Use a nonprincipal ultrafilter $\omega : \mathcal{P}(\mathbb{N}) \to \{0,1\}$. Build a larger Hilbert space $\mathcal{H}_\omega = \lim_\omega \mathcal{H}$, and extend $\rho$ to $\rho_\omega : \Gamma \to \mathcal{U}(\mathcal{H}_\omega)$. We still have a partial order on $\mathcal{H}_\omega$.
A sequence $\Phi = (\phi_n)$ of almost invariant vectors in $\mathcal{H}^{\mathbb{N}}$ becomes an invariant vector $\Phi$ under $\rho_\omega$ on $\mathcal{H}_\omega$.
Choose $s_n \to \delta_\rho$. Define

$$a_n^\rho = \frac{1}{\|A(s_n)\|} \sum_{\gamma \in \Gamma} e^{-s_n d(o,\gamma o)} \rho(\gamma) \ \mathcal{D}_{\gamma o} \,.$$

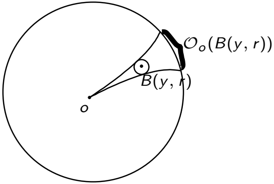
For $f \in C(X \cup \partial X)$, $\int f \, da_n^\rho$ belongs to $\mathcal{B}(\mathcal{H})$, with norm uniformy bounded in $n$.
Define $a^\rho : C(X \cup \partial X) \to \mathcal{B}(\mathcal{H}_\omega)$ positive, linear, continuous by

$$a^\rho(f) := \lim_\omega \int f \, da_n^\rho \in \mathcal{B}(\mathcal{H}_\omega) \,.$$

Nonzero measure on $\partial X$ with values in $\mathcal{B}(\mathcal{H}_\omega)$

# Step 3 : Absolute continuity (I)

A shadow is $\mathcal{O}_o(B(y,r)) = \{\xi \in \partial X, (o\xi) \cap B(y,r) \neq \emptyset\}$.




The classical Patterson-Sullivan measure $\nu$ is a weak limit of $\frac{1}{P(s)} \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} \mathcal{D}_{\gamma o}$.

Sullivan's Shadow lemma says that $\nu(\mathcal{O}_o(B(\gamma o, r)) \asymp e^{-\delta_\Gamma d(o,\gamma o)}$.

A half-Shadow lemma for $a^\rho$ : $\|a^\rho(\mathcal{O}_o(B(\gamma o, r))\| \leq e^{-\delta_\Gamma d(o,\gamma o)}$.

# Step 3 : Absolute continuity (II)

Absolute continuity on shadows
$\|a^\rho(\mathcal{O}_o(B(\gamma o, r))\| \leq \nu(\mathcal{O}_o(B(\gamma o, r))$.

The entropy gap allows to show that points lying in infinitely many shadows have full $\nu$ and $a^\rho$-measure.

Only but crucial place where we need the entropy gap.

By a Vitali type argument, we deduce that $0 \neq a^\rho << \nu$.

For all $\phi \in \mathcal{H}$, $a^\rho.\phi << \nu$.
There exists a Radon-Nikodym derivative $D(\phi) \in L^\infty(X \cup \partial X, \mathcal{H})$, such that

$$\int f \; d(a^\rho.\phi) = \int f \; D(\phi)\, d\nu \,.$$

# Step 4 : Ergodicity

$*$ The map $\phi \in \mathcal{H} \to D(\phi) \in L^\infty((\partial X, \nu), \mathcal{H}_\omega)$ is linear and satisfies

$$\rho(\gamma) \circ D(\phi) \circ \gamma^{-1} = D(\phi)\,.$$

$*$ The map $(\xi, \eta) \in (\partial X)^2 \to < D(\phi)(\xi), D(\phi)(\eta) >_{\mathcal{H}_\omega} \in \mathbb{R}$ is a $\Gamma$-invariant real-valued map.

$\to$ The measure $\nu \times \nu$ on $\partial X \times \partial X$ is ergodic w.r.t. the $\Gamma$-action

*Hint*: $T^1X \simeq \partial X \times \partial X \times \mathbb{R}$. The PS measure $\nu$ on $\partial X$ allows to build the Bowen-Margulis measure $m_{BM} \sim \nu \times \nu \times dt$ on $T^1X/\Gamma$. By Hopf argument, when $X$ is a $CAT(-1)$-space, it is an ergodic invariant measure for the geodesic flow. Also true for $X$ Gromov-hyperbolic (Bader-Furman strategy).

$\to$ We deduce $D(\phi)$ is $\nu \times \nu$-a.s. constant.

# Step 5 : Conclusion

We already know that (for any $\phi \in \mathcal{H}_\omega$, say with $\|\phi\| = 1$)

$$\rho(\gamma) \circ D(\phi) \circ \gamma^{-1} = D(\phi)\,.$$

Moreover, as a map defined on $X \cup \partial X$, it is a.s. constant.

Therefore, for all $\gamma \in \Gamma$, we get the equality in $\mathcal{H}_\omega$

$$\rho(\gamma).D(\phi) = D(\phi)$$

We got it !! $D(\phi)$ is our $\rho$-invariant vector in $\mathcal{H}_\omega$.

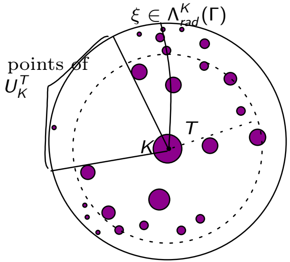
# More on the entropy gap

Show that $\nu$ and $a^\rho$ are supported on $\Lambda_{rad}(\Gamma)$ (same proof).

$$\Lambda_{rad}(\Gamma) \supset \Lambda_{rad}^K(\Gamma) = \{\xi \in \Lambda(\Gamma), [o\xi) \text{ returns i.o. in } \Gamma.K\}$$

Define

$$U_K(T) = \{y \in X \cup \partial X, [oy) \text{ does not return in } K \text{ until time } T\}.$$




Entropy gap $\delta_\Gamma^\infty < \delta_\Gamma$ allows to show $\nu(U_K^T) \leq e^{(\delta_{\Gamma_K} - \delta_\Gamma)T}$

Deduce

$$\nu(\cap_{T>0} U_K^T) = 0 \quad \text{and} \quad \nu(\Gamma.(\cap_{T>0} U_K^T)) = 0$$

# The "easy" direction

Kesten Criterion : any random walk on $\Gamma/\Gamma'$ has spectral radius $= 1$.

Build a sequence of random walks w.r.t. uniform spherical measures on the spheres $S(e, n)$.

Barta's trick : estimate spectral radius on positive functions.

Estimate uniformly from above their spectral radius by $\exp(n(\delta_{\Gamma'} - \delta_\Gamma))$.

Roblin needed $\Gamma' \triangleleft \Gamma$. We remove this assumption, but use $\delta_\Gamma^\infty < \delta_\Gamma$

Thank you!