Sub-Riemannian dynamics and local rigidity of higher hyperbolic rank

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Motivation

Question

Which closed Riemannian manifolds have a property that **every geodesic is contained in a (immersed/infinitesimally) geodesic hyperbolic plane of constant curvature -1***? (higher hyperbolic rank)*

(Constructions and examples)

1. *Such a manifold has the universal cover with property: "every geodesic is contained in a totally geodesic hyperbolic plane of constant curvature -1".*
2. *Examples are compact quotients of negatively curved rank one symmetric spaces.*

(Other examples of infinite volume ones)

1. *A construction by C. Connell '03*
2. *A construction by S. Lin and B. Schmidt '17.*

Motivation

It is reasonable to conjecture that all closed Riemannian manifolds of higher hyperbolic rank are locally symmetric.

Evidences:

1. *Hamenstädt '91: true if $\kappa \leq -1$. There is a dynamical analog for hyperbolic rank condition, and for this condition, C. Connell '03 verified the conjecture is true if $\kappa \leq -1$.*
2. *Constantine '08: true if manifold is negatively curved and is of* 0.93^2 *pinched or odd dimension.*
3. *S. Lin '18: true for 3 dimensional manifolds (no condition on curvature).*
4. *Connell-N.-Spatzier '18: true if $-1 \leq \kappa \leq -\frac{1}{4}$.*
5. *Main result of the talk: Connell-N.-Spatzier '21: true for local perturbations of locally symmetric spaces with appropriate conditions.*

Results

Theorem (Connell-N.-Spatzier '21)

Let (M, g_0) be a closed quaternion or Cayley hyperbolic locally symmetric manifold. Then there is an open C^3 neighborhood U of g_0 such that for any $g \in U$, if (M, g) has higher hyperbolic rank and $\kappa_g \geq -1$ then (M, g) is locally symmetric and isometric to (M, g_0).

Theorem (Connell-N.-Spatzier '21)

Let (M, g_0) be a closed complex hyperbolic manifold. There is an open neighborhood U of g_0 in the C^3-topology among C^∞ metrics such that if $g \in U$ and (M, g) has higher hyperbolic rank and sectional curvature $\kappa \geq -1$ then the Liouville measure on SM coincides with the (unique) measure of maximal entropy for the geodesic flow of g on SM.

Local rigidity of hyperbolic rank for constant curvature -1 manifolds follows from our earlier work.

More about history of rank rigidity

Question

Which closed Riemannian manifolds have a property that every geodesic is contained in a (immersed/infinitesimally) geodesic hyperbolic plane of constant curvature -1?

Ask the same question but replace “hyperbolic plane of constant curvature -1” by “Euclidean plane” or “sphere of constant curvature $+1$”.

1. **Euclidean rank:** Ballmann ‘85, Burns-Spatzier ‘87, Eberlein-Heber ‘90, Watkin ‘13.
2. **Spherical rank:** Shankar-Spatzier-Wilking ‘05, Schmidt-Shankar-Spatzier ‘16.

Another motivation

Let M be a closed Riemannian manifold of negative curvature and $\kappa \geq -1$. Here are some dynamical facts:

1. the geodesic flow $\varphi_t : SM \to SM$ is Anosov, that means there is a splitting $TSM = E^u \oplus E^0 \oplus E^s$ with expanding, neutral, and contracting properties.
2. If M has higher hyperbolic rank, analysis about Jacobi fields gives us a splitting $E^u = E^u_{fast} \oplus E^u_{slow}$, corresponding with hyperbolic direction and the perpendicular one. Moreover, we also obtain some extra smoothness of E^u_{slow} along unstable leaves.

Question

If geodesic flow on a manifold admits a splitting of unstable distribution $E^u = E^u_{fast} \oplus E^u_{slow}$ with sufficiently smoothness of the distributions, then is the manifold locally symmetric?

Another motivation

The question can be modified to ask for Anosov diffeomorphisms on nilmanifolds.

Benoist-Foulon-Labourie '92 + Besson-Courtois-Gallot '95 answered a similar question where the distribution E^{cs} for the geodesic flow on a negative closed Riemannian manifold is smooth.

We give partially affirmative answers for perturbations of geodesic flows of locally symmetric spaces or of Anosov automorphisms.

Results

Theorem (Connell-N.-Spatzier '21)

Let g_0 be a locally quaternionic hyperbolic or Cayley hyperbolic metric on a smooth closed manifold M. Then g_0 is locally rigid within the family of C^2 close metrics whose splittings $E^u_{slow} \oplus E^u_{fast}$ are C^∞ along unstable leaves and are sufficiently uniformly C^1 close to that of g_0.

Theorem (Connell-N.-Spatzier '21)

Let M be a nilmanifold that admits an Anosov automorphism ϕ_0 with unstable leaves are isomorphic to quaternionic Heisenberg group. There is a C^1 open neighborhood U of ϕ_0 in $Diff^\infty(M)$ such that if $\phi \in U$ admits a smooth splitting $E^u_\phi = E^u_{\phi,fast} \oplus E^u_{\phi,slow}$ along unstable leaves with $\dim(E^u_{\phi,slow}) = \dim(E^u_{\phi_0,slow})$, and $E^u_{\phi,slow}$ is sufficiently uniformly C^1 close along unstable leaves to $E^u_{\phi_0,slow}$, then for any invariant ergodic measure μ there is $\lambda_\mu > 0$ such that the unstable Lyapunov exponents of ϕ with respect to μ, are λ_μ and $2\lambda_\mu$ with the same multiplicity as for ϕ_0.

Results

We also get the following result as an application of a tool we introduce to prove the theorems above.

Theorem (Connell-N.-Spatzier '21)

Let $\rho_0 : \Gamma \to \mathrm{Diff}^\infty(S^k)$ for $k = 4n-1$ (resp. $k = 15$) be the projective representation of a cocompact lattice $\Gamma < Sp(n,1)$ (resp. $\Gamma < F_4^{-20}$). Let $\rho : \Gamma \to \mathrm{Diff}^\infty(S^k)$ be a C^1 close perturbation of ρ_0. If ρ preserves a C^∞ distribution E, C^1 close to E_0, then ρ is C^∞ conjugate to ρ_0.

A tool: Sub-Riemannian dynamics

A smooth manifold N is called a *sub-Riemannian manifold* if

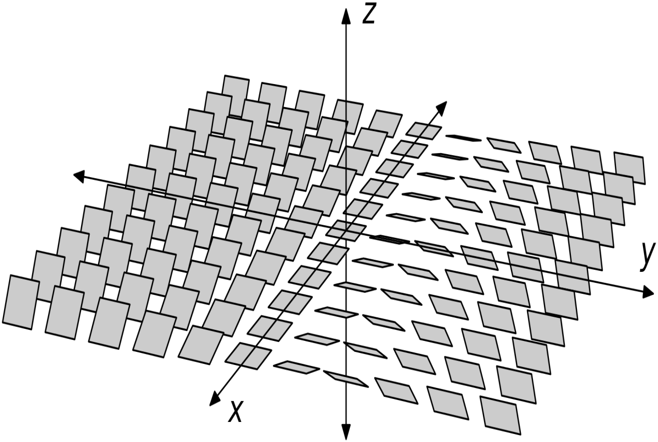
1. N is equipped with a smooth distribution E, called a *horizontal distribution*, satisfying Hörmander's condition; that is, vector fields tangent to E and their brackets generate TN, and
2. E is endowed with a smooth Riemannian metric $\langle \cdot, \cdot \rangle_x$.

The Carnot-Carathéodory metric d_C on N between a pair of points p and q is defined as the infimum of length of curves tangent to E from p to q. By a theorem of Chow '39, the Carnot-Carathéodory metric d_C is finite on connected components of N.

Example: 3-dim Heisenberg group

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

The (right invariant) horizontal plane/distribution at (x, y, z) is spanned by $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$.



A tool: Sub-Riemannian dynamics

Tangent spaces, defined appropriately, of a sub-Riemannian manifold have nice structures. We use *tangent cone* instead of tangent space.

Mitchell '85: tangent cone of a sub-Riemannian manifold at a *generic* point is a graded nilpotent Lie group.

Lemma (Built on Margulis-Mostow '95)

If $f : N \to N$ is a C^1 diffeomorphism and let p is a generic point of a sub-Riemannian manifold N. Then f induces a graded nilpotent Lie group automorphism $f_* : TC_pN \to TC_{f(p)}N$, where TC_pN denotes the tangent cone of N at p. We call f_* is the Carnot derivative of f at p.

A tool: Sub-Riemannian dynamics

Comparison between Carnot derivative and ordinary derivative. One connection is the following result.

Lemma

If $f : N \to N$ is a C^∞ diffeomorphism and $f(p) = p$ where p is a generic point. Suppose that $f_* : TC_pN \to TC_pN$ is a homothety. Then there is $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$ such that Lyapunov exponents (without multiplicity) of D_pf are $\lambda, 2\lambda, \ldots, k\lambda$.

Here Lyapunov exponents are just log of modulus of eigenvalues of the matrix D_pf.
Application: get versions of the two lemma above for foliations and apply them to unstable foliations. The outcome of this application is a control on Lyapunov exponents of geodesic flows or Anosov diffeomorphisms.

Outline main idea of proofs

For the theorem about local rigidity of hyperbolic rank:
We consider the case perturbation of quaternion locally symmetric spaces.

Theorem (Connell-N.-Spatzier '21)

Let (M, g_0) *be a closed quaternion or Cayley hyperbolic locally symmetric manifold. Then there is an open* C^3 *neighborhood* U *of* g_0 *such that for any* $g \in U$*, if* (M, g) *has higher hyperbolic rank and* $\kappa_g \geq -1$ *then* (M, g) *is locally symmetric and isometric to* (M, g_0)*.*

Outline main idea of proofs

1. Using the assumption that -1 is an extremal curvature, deduce the fact that the splitting $E^u = E^u_{fast} \oplus E^u_{slow}$ is C^∞ along unstable leaves.
2. Using frame flow and Brin-Pesin group to show the local stability of hyperbolic rank for perturbations.

Lemma

Brin-Pesin groups, as subgroups of orthogonal groups, for frame flow on perturbed manifolds cannot be smaller than the Brin-Pesin group of the unperturbed manifold. As a consequence, hyperbolic rank does not decrease for perturbations.

Outline main idea of proofs

3. Slow distribution E^u_{slow} is horizontal and generic in unstable leaves, and thus each unstable leave is a sub-Riemannian manifold.
4. Apply sub-Riemannian dynamics to conclude geodesic flow on the perturbed manifold have the same Lyapunov spectra as of the locally symmetric space.
5. Apply a spectra rigidity by Butler '19 to conclude the perturbed manifold is locally symmetric.

Outline main idea of proofs

For the theorem about local rigidity of projective action:

Theorem (Connell-N.-Spatzier '21)

Let $\rho_0 : \Gamma \to \mathrm{Diff}^\infty(S^k)$ for $k = 4n-1$ (resp. $k = 15$) be the projective representation of a cocompact lattice $\Gamma < Sp(n,1)$ (resp. $\Gamma < F_4^{-20}$). Let $\rho : \Gamma \to \mathrm{Diff}^\infty(S^k)$ be a C^1 close perturbation of ρ_0. If ρ preserves a C^∞ distribution E, C^1 close to E_0, then ρ is C^∞ conjugate to ρ_0.

Outline main idea of proofs

We let X be the quaternionic symmetric space or Cayley plane.

1. Consider the suspension $(X \times \partial X)/\Gamma$, which is diffeomorphic to the unit tangent bundle over $X/\rho_0(\Gamma)$. There is a new flow that is C^1-close to the geodesic flow on $X/\rho_0(\Gamma)$.
2. The new flow is dominated and admit a splitting of unstable distribution into slow and fast ones. The unstable slow distribution projects to a $\rho(\Gamma)$-invariant distribution on ∂X and thus coincide with the $\rho(\Gamma)$-invariant distribution E.

Outline main idea of proofs

3. E is C^1-closed to E^0 thus tangent cones of unstable leaves exist and are asymmetric.
4. Relate Carnot derivative and ordinary derivative, we obtain the Lyapunov spectra of the new flow is proportional to the one of geodesic flow.
5. A result of Butler '19 show that there is a C^∞ orbit equivalence between two flows.
6. Projecting to ∂X we get a C^∞ conjugation between $\rho_0(\Gamma)$ and $\rho(\Gamma)$ action.