A von Neumann algebra valued Multiplicative Ergodic Theorem

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Motivation

Theorem (Oseledets, 1965)

Let

- X *be a closed Riemannian manifold,*
- $\mu =$ *be a Borel probability measure on* X*,*
- $f : X \to X$ *be a diffeomorphism preserving* μ*.*

Then for a.e. $x \in X$*, for every vector* $v \in T_x(X)$*, the* **growth rate**

$$\lim_{n \to +\infty} n^{-1} \log \|D_x(f^n)(v)\|$$

exists.

Theorem (Part of the Classical MET (Oseledets, 1965))

Let

- (X, μ) *be a Borel probability space,*
- $f : X \to X$ *be a measure-preserving transformation,*
- $c : \mathbb{Z} \times X \to \mathrm{GL}(d, \mathbb{R})$ *a measurable cocycle with* $\log \|c(1, x)^{\pm 1}\| \in L^1(X, \mu)$.

Then for a.e. $x \in X$*, for every vector* $v \in \mathbb{R}^d$*, the* **growth rate**

$$\lim_{n \to +\infty} n^{-1} \log \|c(n, x)(v)\|$$

exists.

Convert to a matrix problem

For convenience, choose a measurable field of isomorphisms $\Phi_x : T_x(X) \to \mathbb{R}^d$.

Define the **cocycle** $c : \mathbb{Z} \times X \to \mathrm{GL}(d, \mathbb{R})$

$$c(n, x) = \Phi_{f^n(x)} \circ D_x(f^n) \circ \Phi_x^{-1}.$$

The cocycle is determined by $c(1, \cdot) : X \to \mathrm{GL}(d, \mathbb{R})$ and the cocycle equation

$$c(n + m, x) = c(n, f^m x)c(m, x).$$

Spectral Calculus

Motivation

If a is a matrix, we need to make sense of $|a|, \log|a|, |a|^{1/n}$ etc.

If $d = (d_{ii})$ is a diagonal matrix and $\phi : \mathbb{C} \to \mathbb{C}$ then define

$$\phi(d) = \begin{pmatrix} \phi(d_{11}) & & & \\ & \phi(d_{22}) & & \\ & & \ddots & \\ & & & \phi(d_{dd}) \end{pmatrix}.$$

If $a = a^*$ is self-adjoint then the Spectral Theorem $\Rightarrow$

$$a = udu^{-1} \quad \text{for some unitary } u, \text{ real diagonal matrix } d.$$

If $\phi : \mathbb{R} \to \mathbb{R}$ and a is self-adjoint then define $\phi(a) := u\phi(d)u^{-1}$.

Spectral Calculus

Let $a \in \mathrm{Mat}_{d\times d}(\mathbb{C})$. Then a^*a is self-adjoint (and positive semi-definite). The **matrix absolute value** of a is $|a| = (a^*a)^{1/2} = ud^{1/2}u^{-1}$.

Now we can define

$$\log |a| = u \log(d^{1/2}) u^{-1},$$
$$|a|^{1/n} = ud^{1/2n}u^{-1}.$$

The limit operator

Theorem (The Classical MET)

Let

- (X, μ) *be a Borel probability space,*
- $f : X \to X$ *be a measure-preserving transformation,*
- $c : \mathbb{Z} \times X \to \mathrm{Mat}(d, \mathbb{R})$ *be a measurable cocycle with* $\log \|c(1, x)\| \in L^1(X, \mu)$*.*

Then for a.e. $x \in X$*, there is a* **limit operator** $\Lambda(x)$ *defined by*

$$\lim_{n \to +\infty} n^{-1} \log |c(n, x)| = \log \Lambda(x).$$

Consequences of the Classical MET

Define the **Lyapunov exponents** $\lambda_1(x) \geq \cdots \geq \lambda_d(x)$ to be the eigenvalues of $\log \Lambda(x)$.

Define the **Oseledets subspaces** $E_i(x)$ to be the sum of the eigenspaces of $\lambda_j(x)$ for $j \geq i$.

Then

- (invariance) $\lambda_i(f(x)) = \lambda_i(x)$, $E_i(f(x)) = c(1,x)E_i(x)$,
- (growth rates) For all $v \in E_i(x) \setminus E_{i+1}(x)$,

$$\begin{aligned} \lambda_i(x) &= \lim_{n\to+\infty} n^{-1} \log \|c(n,x)v\| \\ &= \lim_{n\to+\infty} n^{-1} \log \|\Lambda^n(x)v\|. \\ \det(\Lambda(x)) &= \lim_{n\to\infty} \det(|c(n,x)|)^{1/n}. \end{aligned}$$

Special case: the Pointwise Ergodic Theorem

Theorem (Birkhoff, 1932)

Let

- (X, μ) *be a probability space,*
- $f : X \to X$ *a measure-preserving transformation,*
- $\phi \in L^1(X, \mu)$ *an integrable observable.*

Then the ergodic averages

$$\frac{1}{n}\sum_{i=0}^{n-1} \phi \circ f^i$$

converge pointwise a.e. to an f-invariant function. If f is ergodic, the limit is the constant $\int \phi \, d\mu$.

Proof.

Wlog ϕ is real-valued. Set $d = 1$ and apply the MET to $c(1, x) = \exp(\phi)$. □

Special case: powers of a single matrix

Theorem

Let a be a $d \times d$ matrix. Then

$$\lim_{n\to\infty} |a^n|^{1/n}$$

exists.

Proof.

Apply the MET with $c(n, x) = a^n$, or apply the Jordan Decomposition Theorem. □

A potpourri of proofs

1. (Oseledets, 1965) reduces to the lower triangular case via conjugation.
2. (Raghunathan, 1979) uses multi-linear algebra to reduce to Furstenberg-Kesten's 1960 Theorem that $n^{-1} \log \|c(n, x)\|$ converges.
3. (Kaimanovich, 1989) uses the non-positively curved geometry of the space of positive definite matrices.
4. (Walters, 1993) uses compactness of the projective space $\mathbb{RP}^{d-1}$.

Infinite dimensions?

Can the MET be extended to operators on an infinite-dimensional Hilbert space?

Let a be a bounded operator on Hilbert space. Does $\lim_{n\to\infty} |a^n|^{1/n}$ exist?

Voiculescu's counterexample

Define $a : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by
$a(e_n) = k_n e_{n+1}$

$$k_n = \begin{cases} 1 & \text{if } 2^i \leq n < 2^{i+1}, i \text{ even}, \\ 2 & \text{if } 2^i \leq n < 2^{i+1}, i \text{ odd}. \end{cases}$$

Then

$$|a^n|^{1/n} e_1 = \left(\prod_{i=1}^{n} k_i \right)^{1/n} e_1$$

oscillates between $2^{1/3} e_1$ and $2^{2/3} e_1$.

So $|a^n|^{1/n}$ does not converge in the weak, strong or operator norm topologies.

Some infinite-dimensional METs

Infinite-dimensional METs were established by Ruelle (1982), Mañé (1983), Thiellen (1987), Schaumlöffel (1991), Karlsson-Margulis (1999), Lian and Lu (2010), González-Tokman and Quas (2015) and Blumenthal (2016).

All of these require some **quasi-compactness assumptions** on the cocycle $c(n, x)$. So the Oseledets subspaces are finite-dimensional and the Lyapunov spectrum is discrete.

Goal: Prove statements of the following form:

Let

- (X, μ) be a standard Borel probability space,
- $f : X \to X$ be a measure-preserving transformation,
- $\mathcal{H}$ be a separable Hilbert space,
- G be a **group** of linear operators acting on $\mathcal{H}$,
- $c : \mathbb{Z} \times X \to G$ a measurable cocycle satisfying a log first moment condition.

Then for a.e. $x \in X$, there is a limit operator $\Lambda(x)$ defined by

$$\lim_{n \to +\infty} n^{-1} \log |c(n, x)| = \log \Lambda(x)$$

where the limit is in **?? topology**. Moreover, we'd like a Lyapunov distribution, Osedelets subspaces and vector growth rates.

Outline

1. Example 1: exponentiated Hilbert Schmidt operators
2. Example 2: the abelian case
3. The general case

Example: exponentiated Hilbert Schmidt operators

- $\mathcal{H}$ = separable Hilbert space.
- $B(\mathcal{H}) = \{a : \mathcal{H} \to \mathcal{H} : \ \|a\| < \infty\}$.
- $\{e_i\}_{i \in \mathbb{N}}$= an ON basis on $\mathcal{H}$.
- $a \in B(\mathcal{H})$ is **Hilbert-Schmidt** if $\|a\|_2^2 := \sum_i \|ae_i\|^2 < \infty$.
- $\mathrm{L}^2(B(\mathcal{H})) = \{a \in B(\mathcal{H}) : \ \|a\|_2 < \infty\}$.
- Let $\mathrm{GL}^2(B(\mathcal{H})) = \{a \in B(\mathcal{H}) : \ \log|a| \in \mathrm{L}^2(B(\mathcal{H}))\}$.

Theorem (Karlsson-Margulis, 1999)

The model statement holds with $G = \mathrm{GL}^2(B(\mathcal{H}))$ *and convergence in the* $L^2(B(\mathcal{H}))$ *topology.*

Hilbert-Schmidt operators are compact. So G consists of operators of the form **compact + unitary**. This statement was covered by Ruelle (1982).

Example: exponentiated Hilbert Schmidt operators

The **canonical trace** on $B(\mathcal{H})$ is defined by

$$\tau_{B(\mathcal{H})}(a) = \sum_i \langle ae_i, e_i \rangle$$

for any $a \in B(\mathcal{H})$ for which this is absolutely summable.

From the trace, one can derive notions of: **inner product** between two operators, **dimension** of subspaces of $\mathcal{H}$, **determinant** and **spectral measure**.

Example: uses of the canonical trace

Inner products between operators

$\langle a, b \rangle = \tau_{B(\mathcal{H})}(b^* a)$.

Dimension

For $S \subset \mathcal{H}$, $\dim(S) = \tau_{B(\mathcal{H})}(\mathrm{proj}_S)$.

Determinants

$\det(a) = \exp(\tau_{B(\mathcal{H})}(\log |a|))$ whenever this is well-defined.

Spectral measures

The spectral measure of a, when well-defined, is the unique measure μ_a on $\mathbb{C}$ satisfying

$$\tau_{B(\mathcal{H})}(a^n) = \int z^n \, d\mu_a(z).$$

Continuous spectrum?

Is there an MET that allows for continuous Lyapunov spectrum?

Example: the abelian case

Let $M = \mathrm{L}^\infty([0,1])$. $M \curvearrowright \mathrm{L}^2([0,1])$ by multiplication.

Let $G = \mathrm{GL}^2(M) = \{\exp(a) : \ a \in \mathrm{L}^2([0,1])\}$.

Pointwise Ergodic Theorem $\Rightarrow$ the model statement holds with $G = \mathrm{GL}^2(M)$ and convergence in the $\mathrm{L}^2(B(\mathcal{H}))$ topology.

This allows for continuous Lyapunov spectrum.

Convergence does not occur in the operator-norm topology in general.

The trace in the abelian case

Trace

For $a \in M = \mathrm{L}^\infty([0,1])$, $\tau(a) := \int_0^1 a(x)\,dx$.

Inner products between operators

$\langle a, b\rangle = \tau(b^* a) = \int_0^1 a(x)\overline{b(x)}\,dx$.

Dimension

If $Y \subset [0,1]$, then $\mathrm{L}^2(Y) \subset \mathrm{L}^2([0,1])$ and

$$\dim(\mathrm{L}^2(Y)) = \tau(\mathrm{proj}_{\mathrm{L}^2(Y)}) = \tau(1_Y) = \mathrm{Leb}(Y).$$

The trace in the abelian case

Determinants

$\det(a) = \exp(\tau(\log|a|)) = \exp\left(\int_0^1 \log|a(x)|\, d\,\mathrm{Leb}(x)\right).$

Spectral measures

The spectral measure of *a* is its distribution. It's also the unique measure on $\mathbb{C}$ with

$$\tau(a^n) = \int z^n \, d\mu_a(z).$$

Theorem (B.-Hayes-Lin, the vN-algebra MET)

Let

- (X, μ) *be a Borel probability space,*
- $f : X \to X$ *be a measure-preserving transformation,*
- (M, τ) *a* **semi-finite tracial von Neumann algebra***,*
- $c : \mathbb{Z} \times X \to \mathrm{GL}^2(M, \tau)$ *a measurable cocycle with* $\log \|c(1, x)\|_2 \in L^1(X, \mu)$.

Then for a.e. x the **drift***, defined by*

$$D(x) = \lim_{n \to \infty} \frac{\| \log |c(n, x)| \|_2}{n}$$

exists. For a.e. x with $D(x) > 0$ there is a **limit operator** $\Lambda(x)$ *defined by*

$$\lim_{n \to +\infty} n^{-1} \log |c(n, x)| = \log \Lambda(x).$$

A semi-finite tracial von Neumann algebra is a sub-algebra of $B(\mathcal{H})$ equipped with a **trace**.

Trace $\rightsquigarrow$ **inner products** between operators, **dimensions**, **determinants** and **spectral measures**.

Example

- Let $\Gamma =$ be a discrete group.
- Let $\lambda : \Gamma \to B(\ell^2(\Gamma))$ be the left-regular representation.
- Let $L\Gamma = \overline{\mathrm{algebra}(\lambda(\Gamma))}^{SOT}$ be the **group von Neumann algebra**.
- For $g \in \Gamma$, $\tau(\lambda_g) = 1$ if $g = 1_\Gamma$ and $\tau(\lambda_g) = 0$ otherwise.
- Extend τ to $L\Gamma$ by linearity and continuity.

von Neumann algebras

Let

- $\mathcal{H}$ be a separable Hilbert space,
- $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$.

A **von Neumann algebra** is a subalgebra $M \subset B(\mathcal{H})$ satisfying:

- (adjoint-closed) $a \in M \Rightarrow a^* \in M$;
- (identity) $I \in M$;
- M is closed in the Strong Operator Topology (SOT).

If $\mathcal{F} \subset B(\mathcal{H})$ is any subset, then there is a unique smallest von Neumann algebra containing $\mathcal{F}$.

Traces

Let $M_+ \subset M$ be the positive operators on M.

A **trace** on M is a map $\tau : M_+ \to [0, \infty]$ with

1. $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in M_+$;
2. $\tau(\lambda x) = \lambda\tau(x)$ for all $\lambda \in [0, \infty)$, $x \in M_+$;
3. $\tau(x^* x) = \tau(xx^*)$ for all $x \in M$.

We will always assume τ is

- **faithful**: $\tau(x^* x) = 0 \Rightarrow x = 0$;
- **normal**: $\tau(\sup_i x_i) = \sup_i \tau(x_i)$ for every increasing net $(x_i)_i$ in M_+;
- **semi-finite**: $\forall x \in M_+$, $\exists y \in M_+$ with $0 < y < x$ and $0 < \tau(y) < \infty$.

Theorem (Karlsson-Margulis (1999))

Let

- $f : (X, \mu) \to (X, \mu)$ *be a prob-meas-preserv.,*
- (Y, d) *a complete CAT(0) space,* $y_0 \in Y$*,*
- $L : X \to \mathrm{Isom}(Y, d)$ *be measurable with finite first moment:* $\int_X d(y_0, L(x)y_0)\, d\mu(x) < \infty$.

Then for a.e. $x \in X$*, the* **drift** *exists:*

$$\lim_{n\to\infty} \frac{d(y_0, L(x)L(fx)\cdots L(f^{n-1}x)y_0)}{n} = D(x).$$

Moreover, for a.e. x *with* $D(x) > 0$*,* $\exists$ *a* **geodesic ray** $\gamma_x(\cdot)$ *starting at* y_0 *that sub-linearly tracks the 'cocycle random walk':*

$$\lim_{n\to\infty} \frac{1}{n} d\Big(\gamma_x(Dn), L(x)L(fx)\cdots L(f^{n-1}x)y_0\Big) = 0.$$

From Karlsson-Margulis to the Classical MET

- $\mathcal{P}(d,\mathbb{R}) = \{a \in \mathrm{GL}(d,\mathbb{R}) : \; a > 0\}$.
- $M_{sa}(d,\mathbb{R}) = \{a \in \mathrm{Mat}(d,\mathbb{R}) : \; a = a^*\}$.
- $\mathcal{P}(d,\mathbb{R}) = \exp(M_{sa}(d,\mathbb{R}))$.

- The tangent space to $p \in \mathcal{P}(d,\mathbb{R})$ is $T_p(\mathcal{P}(d,\mathbb{R})) = M_{sa}(d,\mathbb{R})$.
- $\langle A, B\rangle_p = \mathrm{trace}(p^{-1}Ap^{-1}B)$ is an inner product on $T_p(\mathcal{P}(d,\mathbb{R}))$.
- This gives a Riemannian metric on $\mathcal{P}(d,\mathbb{R})$.

- $\mathrm{GL}(d,\mathbb{R}) \curvearrowright \mathcal{P}(d,\mathbb{R})$ transitively and isometrically by $a \cdot p := apa^*$.
- Every geodesic ray from the identity I has the form $t \mapsto \exp(ta)$ for some self-adjoint $a \in M_{sa}(d,\mathbb{R})$.
- Obtain the classical MET from Karlsson-Margulis by setting $Y = \mathcal{P}(d,\mathbb{R})$, $y_0 = I$, $L(x) = c(1,x)^*$.

From Karlsson-Margulis to the vN algebra MET

- Let (M, τ) be a semi-finite tracial von Neumann algebra.
- Let $\mathcal{P}^\infty \subset M^\times$ be the set of positive definite operators with bounded inverses.
- The tangent space to $p \in \mathcal{P}^\infty$ is $T_p(\mathcal{P}^\infty) = M_{sa}$.
- $\langle a, b \rangle_p = \tau(p^{-1} a p^{-1} b)$ is an inner product on $T_p(\mathcal{P}^\infty)$.
- $M^\times \curvearrowright \mathcal{P}^\infty$ transitively and isometrically by $a \cdot p := apa^*$.

Theorem (Andruchow-Larotonda, 2006)

- *$\mathcal{P}^\infty$ is non-positively curved.*
- *Geodesics in $\mathcal{P}^\infty$ are $t \mapsto \exp(ta)$ for $a \in M_{sa}$.*

But $\mathcal{P}^\infty$ is incomplete! Identifying the completion was stated as an open problem in [Conde-Larotonda, 2010].

The completion

- Let $\mathrm{GL}^2(M,\tau) = \{L \in \mathrm{L}^0(M,\tau) : \log(|L|) \in \mathrm{L}^2(M,\tau)\}$.
- Let $\mathcal{P} \subset \mathrm{GL}^2(M,\tau)$ be the positive definite log-square integrable operators.
- Define $d_{\mathcal{P}}(a,b) = \| \log a^{-1/2} b a^{-1/2} \|_2$.

Theorem (B.-Hayes-Lin)

- $\mathrm{GL}^2(M,\tau)$ *is a group.*
- $(\mathcal{P}, d_{\mathcal{P}})$ *is the metric completion of* $\mathcal{P}^\infty$*. It is CAT(0).*
- $\mathrm{GL}^2(M,\tau) \curvearrowright \mathcal{P}$ *transitively and isometrically by* $a \cdot p := apa^*$*.*
- *Geodesics in* $\mathcal{P}$ *are* $t \mapsto \exp(ta)$ *for* $a \in L^2(M,\tau)_{sa}$*.*

(the above) + (Karlsson Margulis) $\Rightarrow$ the vN-algebra MET.

Moreover, . . .

The **Lyapunov distribution** at $x \in X$ is the spectral measure $\mu_{\Lambda(x)}$.

It is invariant: $\mu_{\Lambda(x)} = \mu_{\Lambda(fx)}$.

The **Oseledets subspaces** are $\mathcal{H}_s(x) = 1_{(-\infty,s]}(\log \Lambda(x))\mathrm{L}^2(M,\tau)$.

They are invariant: $\mathcal{H}_s(fx) = L(x)\mathcal{H}_s(x)$.

Theorem (Asymptotic behavior of determinants)

$\lim_{n\to\infty} \det(L(f^{n-1}x)\cdots L(x))^{1/n} = \det \Lambda(x).$

Theorem

For a.e. $x \in X$ and every vector $v \in L^2(M, \tau)$,

$$\begin{aligned} & \inf\left\{\liminf_{n\to\infty} \|c(n,x)v_n\|_2^{1/n} : \lim_n v_n = v\right\} \\ = & \inf\left\{\limsup_{n\to\infty} \|c(n,x)v_n\|_2^{1/n} : \lim_n v_n = v\right\} \\ = & \lim_{n\to\infty} \|\Lambda(x)^n v\|_2^{1/n}. \end{aligned}$$

Even in the abelian case $M = \mathrm{L}^\infty([0,1])$, there are counterexamples to the claim

$$\lim_{n\to\infty} \|c(n,x)v\|_2^{1/n} = \lim_{n\to\infty} \|\Lambda(x)^n v\|_2^{1/n}.$$

Further directions

- Applications to infinite-dimensional Hamiltonian flows? to infinite-dimensional non-uniform hyperbolic dynamics?
- local stable/unstable manifold theory?
- Ruelle's entropy inequality? Pesin's entropy formula?
- Can the main results be extended to non-invertible dynamics, non-invertible operators, log integrable operators, type III vN-algebras, other Banach spaces?

The study of von Neumann algebras was initiated by von Neumann and Murray in a series of 4 papers from (1936-1943) totaling around 300 pages.

- F. J. Murray and J. Von Neumann, On rings of operators, Ann. of Math. (2) 37 (1936), no. 1, 116–229.
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