Distribution of Values of Irrational Forms at Integral Points and Spherical Averages

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Introduction - Values of Integral Points

- Meyer (1884): Any indefinite, integral quadratic form in $n \geq 5$ variables represents 0 over $\mathbb{Z}$.
- Oppenheim Conjecture (Margulis (1986)): Any indefinite, irrational quadratic form in $n \geq 3$ variables takes values at integral points arbitrarily close to 0 or equivalently the set of values is dense in $\mathbb{R}$.
- Birch (1957): For any odd $d \geq 3$ there exists an integer $m_{\mathbb{Z}}(d)$ with the following property. Any integral, homogeneous form of degree d in $m \geq m_{\mathbb{Z}}(d)$ variables represents 0 over $\mathbb{Z}$.
- Schmidt (1980): For any odd $d \geq 3$ there exists an integer $m_0(d)$ with the following property. Any irrational, homogeneous form of degree d in $m \geq m_0(d)$ variables takes values arbitrarily close to 0.
- Note that $m_{\mathbb{Z}}(2) = 5, m_0(2) = 3$. However, for $d \geq 3$ the situation is not well-understood. e.g. $m_0(3) \leq 359\,551\,882$ (Freeman (2000) based on Pitman (1968) and Schmidt) and $m_{\mathbb{Z}}(3) \leq 14$ (Heath-Brown (2007)).
- Mordell (1937): $m_{\mathbb{Z}}(d) \geq d^2 + 1$ (e.g. $m_{\mathbb{Z}}(3) = 10$).

Introduction - Density of Values

- Davenport-Heilbronn (1946): Let F be an indefinite, irrational diagonal form of degree d in $n \geq 2^d + 1$ variables, then $F(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.
- Davenport-Roth (1955): For indefinite, irrational diagonal cubic forms, $n = 8$ is sufficient to obtain density of values at integral points.
- Baker-Brüdern-Wooley (2000's): First 'quantitative' results for indefinite, irrational diagonal cubic forms in $n = 8$ variables and in $n = 7$ with 'heavy Diophantine' restrictions.
- Eskin-Margulis-Mozes (1998, 2005): Quantitative distribution of values of quadratic forms of signature $(p, q) \neq (2, 1)$ or $(2, 2)$. For quadratic forms of signature $(2, 2)$ they obtain quantitative results under 'mild Diophantine' restrictions. Does not rely on the circle method, but instead on 'equidistribution of translates of measures'.

Heuristics

- Let F be a homogeneous form of degree d in m variables.
- Let (a,b) be any interval and set

$$V^F_{(a,b)}(\mathbb{R}) := \left\{ v \in \mathbb{R}^m \,|\, a < F(v) < b \right\},$$
$$V^F_{(a,b)}(\mathbb{Z}) := V^F_{(a,b)}(\mathbb{R}) \cap \mathbb{Z}^m.$$

- Denote by Ω the unit ball in $\mathbb{R}^m$, then
  1. $T\Omega \cap \mathbb{Z}^m$ consists of $\mathcal{O}(T^m)$ points ,
  2. $F(T\Omega \cap \mathbb{Z}^m) \subseteq [-cT^d, cT^d]$ for some $c = c(F,\Omega)$,
  3. $F(V^F_{(a,b)}(\mathbb{Z}) \cap T\Omega) = F(\mathbb{Z}^m \cap T\Omega) \cap [a,b]$.

  One expects

$$\#\big(V_{(a,b)}(\mathbb{Z}) \cap T\Omega\big) \sim c_{F,\Omega}(b-a)T^{m-d}, \text{ as } T \to \infty,$$

  for a constant $c_{F,\Omega}$ depending on F and Ω only,
  but also $\mathrm{vol}\big(V_{(a,b)}(\mathbb{R}) \cap T\Omega\big) \sim c_{F,\Omega}(b-a)T^{m-d}$, as $T \to \infty$.

Quadratic Forms

Theorem (Buterus, Götze, H., Margulis). Let Q be a non-degenerate indefinite quadratic form in $n \geq 5$ variables and $\Omega \subset \mathbb{R}^n$ an 'admissible domain'. Then, for any $a < b$ there exist functions $\rho_{Q,b-a}$ and $R_{Q,\Omega,b-a}$ such that for any $T > 0$

$$\#\big(V^{Q}_{(a,b)}(\mathbb{Z}) \cap T\Omega\big) - \mathrm{vol}\big(V^{Q}_{(a,b)}(\mathbb{R}) \cap T\Omega\big) = \frac{T^{n-2}}{|\det Q|^{\frac{1}{2}}} \mathcal{O}_n\Big(\rho_{Q,b-a}(T) + R_{Q,\Omega,b-a}(T)\Big),$$

where $R_{Q,\Omega,b-a}(T) = \mathcal{O}_{Q,\Omega,b-a}(T^{-k})$ as $T \to \infty$ for some $k = k(n) > 0$ and

1. If Q is rational, then $\rho_{Q,b-a}(T) = \mathcal{O}_{Q,b-a}(1)$ as $T \to \infty$,
2. If Q is irrational, then $\rho_{Q,b-a}(T) = o_{Q,b-a}(1)$ as $T \to \infty$,
3. If Q is Diophantine of type (κ, A), then $\rho_{Q,b-a}(T) = \mathcal{O}_{Q,b-a}(T^{-\kappa^*})$ as $T \to \infty$ for some $\kappa^* > 0$ explicitly depending on n and κ only.

Quadratic Forms

Definition. A quadratic form Q is said to be Diophantine of type (κ, A) if for any integer $m \in \mathbb{Z} \setminus \{0\}$ and any integral symmetric matrix $M \in \mathrm{Sym}_n(\mathbb{Z})$ we have

$$\inf_{t \in [1,2]} \|M - tmQ\| \geq \frac{A}{|m|^{\kappa}}.$$

- Almost every quadratic form is Diophantine of some type, e.g. if one ratio consisting of two coefficients of Q is Diophantine, then Q is a Diophantine form.

Determinant Forms

- Let $V = \mathrm{Mat}_n(\mathbb{R})$, $V_{\mathbb{Z}} = \mathrm{Mat}_n(\mathbb{Z})$. We say that a homogeneous form

$$F(v) = \sum_{(i_1,j_1)\leq\cdots\leq(i_n,j_n)} q_{i_1 j_1 \ldots i_n j_n} v_{i_1 j_1} \ldots v_{i_n j_n}$$

  of degree n in n^2 variables is a *determinant form of degree* n if it is of the form $F = \det \circ x$ for some $x \in \mathrm{SL}(V)$.
- The case $n = 2$ corresponds to the case of quadratic forms of signature $(2, 2)$.

Determinant Forms

- Let $\mathrm{G} = \mathrm{SL}(V), \Gamma = \mathrm{SL}(V_{\mathbb{Z}})$ and $\mathrm{X} = \mathrm{G}/\Gamma$.
- $\mathrm{H} = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) \subset \mathrm{G}$ (via the representation $(g, h)v = gvh^t$ for $(g, h) \in \mathrm{H}$ and $v \in V$) is a maximal connected subgroup.
- H is realized as $\mathrm{SL}_n(\mathbb{R}) \otimes \mathrm{SL}_n(\mathbb{R})$, where $\otimes$ denotes the standard Kronecker product.
- The group preserving the form F is $\mathrm{H}_F = x^{-1}\mathrm{H}x$.
- Observation: $\mathrm{H}_F \cdot \Gamma$ is closed in X if and only if F is rational.
- Ratner's Theorem: Since H is maximal, $\mathrm{H}_F \cdot \Gamma$ is dense or closed in X.
- Hence, $F(V_{\mathbb{Z}})$ is dense in $\mathbb{R}$ if F is irrational.

Determinant Forms

- Let $\Omega = \{v \in V_0 \,|\, \|v\| < \rho(\widehat{v}/\|\widehat{v}\|)\}$ where $\widehat{v}$ denotes the adjugate of v, $\rho : \mathrm{S}_V \to \mathbb{R}_{>0}$ is a positive continuous function on the unit sphere $\mathrm{S}_V = \{v \in V \,|\, \|v\| = 1\}$ and $V_0 = \{v \in V \,|\, \mathrm{rk}(v) \geq n-1\}$.

Theorem (Fromm, H., Oh). Let F be a an irrational determinant form in $n \geq 3$ variables. Then, for any interval (a, b),

$$\#\left(V^F_{(a,b)}(\mathbb{Z}) \cap T\Omega\right) \sim \lambda_{F,\Omega}(b-a)T^{n(n-1)}, \text{ as } T \to \infty,$$

where $\lambda_{F,\Omega} = \lim_{T\to\infty} \frac{\mathrm{vol}(V^F_{(a,b)}(\mathbb{R}) \cap T\Omega)}{(b-a)T^{n(n-1)}}$.

Spherical Averages

- Suppose X is a space with a base point point o and $(\mathcal{A}_r)_{r>0}$ is a family of 'mean value' operators.
- For a function f, $(\mathcal{A}_r f)(x)$ is the 'mean value' of f along the 'sphere of radius r' centered at x.
- Under certain conditions, if f is a positive function satisfying an inequality of the form

  $$(A_{r_0} f)(x) \leq cf(x) + b, \text{ for all } x \in \mathrm{X},$$

  for some $r_0 > 0$, then the 'mean values' of f based at o (i.e. $(\mathcal{A}_r f)(o)$ for all $r > 0$) can be controlled.
- Usually $c < 1$ is necessary and yields a bound for $\sup_{r>0}(A_r f)(o)$.
- If $c \geq 1$ one cannot expect $\sup_{r>0}(\mathcal{A}_r f)(o)$ to be bounded. However, if the underlying 'mean value' operators are 'better understood', e.g. if the spherical functions can be related to the above inequality, then one can obtain growth estimates for $(\mathcal{A}_r f)(o)$.

Geometry of Numbers

- Here $\mathrm{X} = \mathrm{G}/\Gamma$ space of (G-)lattices in $\mathbb{R}^m$
- The function f will be related to the height function α. If $\Delta \in \mathrm{X}$ is a lattice, then

$$\alpha_i(\Delta)^{-1} = \text{ covolume of smallest } i \text{ dimensional sublattice of} \Delta,$$
$$\alpha(\Delta) = \sup_{0 \le i \le m} \alpha_i(\Delta)$$

- Lipschitz principle: let $f : \mathbb{R}^m \to \mathbb{R}_{>0}$ be a sufficiently fast decreasing function, then $\widetilde{f} \le c(f)\alpha$, where

$$\widetilde{f}(\Delta) = \sum_{v \in \Delta \setminus \{0\}} f(v), \ \Delta \in \mathrm{X}.$$

- Siegel's mean value theorem: let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous and compactly supported, then

$$\int_{\mathrm{X}} \widetilde{f} \, \mathrm{d}\mu_{\mathrm{X}} = \int_{\mathbb{R}^m} f \, \mathrm{d}v.$$

Analytic Counting Method

- $\mathrm{H} = \mathrm{SL}_2(\mathbb{R}), \mathrm{K} = \mathrm{SO}(2)$ embedded in $\mathrm{G} = \mathrm{Sp}_{2n}(\mathbb{R}), \Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$ $(n \geq 1)$, $a_t = \mathrm{diag}(\mathrm{e}^{t/2}, \mathrm{e}^{-t/2})$,

Theorem (BGHM). For any $\beta > \frac{2}{n}$ and any symplectic lattice $\Delta \in \mathrm{G}/\Gamma$

$$\int_{\mathrm{K}} \alpha^{\beta}(a_t k \Delta)\,\mathrm{d}k \ll_{\beta} (\mathrm{e}^{t/2})^{\beta n - 2} \alpha(\Delta)$$

- Assume that the eigenvalues of Q are bounded below by 1 in absolute value. Let $h(v, \zeta) = f(v) g(\zeta)$ be an appropriately smooth approximation of $\chi_{\Omega \times [a,b]}$ and set

$$L_T(h) := \sum_{n \in \mathbb{Z}^d} h\left(\frac{n}{T}, Q(n)\right) - \int_{\mathbb{R}^n} h\left(\frac{x}{T}, Q(x)\right) \mathrm{d}x$$

Analytic Counting Method

- Via Fourier analysis, the main term to be estimated is

$$\int_{\mathscr{I}} \left|\vartheta(\mathfrak{Z}_{T,\tau}, v/T)\right| \mathrm{d}\tau,$$

where $\mathscr{I} \asymp (T^{-1}, T^{\delta})$ $(\delta > 0)$

$$\vartheta\big(\mathfrak{Z}, \xi\big) = \sum_{x \in \mathbb{Z}^n} \exp\left\{\pi \mathrm{i} \mathfrak{Z}(x) + 2\pi \mathrm{i} \langle x, \xi \rangle\right\}, \ \mathfrak{Z} \in \mathcal{H}_d, \xi \in \mathbb{R}^n$$

$$\mathfrak{Z}_{T,\tau} = \tau Q + \frac{\mathrm{i}}{T^2} Q_+ \in \mathcal{H}_d, (Q_+ = (Q^2)^{\frac{1}{2}})$$

- One can show (via Poisson summation and the Lipschitz principle) that

$$\int_{\mathscr{I}} \left|\vartheta(\mathfrak{Z}_{T,\tau}, v/T)\right| \mathrm{d}\tau \ll_Q \frac{T^{n/2}}{|\det Q|^{\frac{1}{2}}} \int_{\mathrm{K}} \alpha^{\frac{1}{2}}(a_t k \Delta_Q)\, \mathrm{d}k,$$

where $t = 2\log(T)$ and Δ_Q is a symplectic lattice for which $\alpha(\Delta_Q) \ll |\det Q|^{\frac{1}{2}}$

Dynamical Counting Method

- $V = \mathrm{Mat}_n(\mathbb{R})$, $V_{\mathbb{Z}} = \mathrm{Mat}_n(\mathbb{Z})$, V_0 as before, S_V the unit sphere
- $\mathrm{G} = \mathrm{SL}(V)$, $\Gamma = \mathrm{SL}(V_{\mathbb{Z}})$, $\mathrm{X} = \mathrm{G}/\Gamma$,
- $\mathrm{H} = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) \subset \mathrm{G}$,
- $b_t = \mathrm{diag}(\mathrm{e}^{-t/2}, \ldots, \mathrm{e}^{-t/2}, \mathrm{e}^{t(n-1)/2})$, $a_t = (b_t, b_t) \in \mathrm{H}$.
- $\mathrm{K} = \mathrm{SO}(n) \times \mathrm{SO}(n)$, $\mathrm{M} = \mathrm{SO}(n-1) \times \mathrm{SO}(n-1)$

Theorem. Suppose $n \geq 3$ and $0 < s < 2$, then for any lattice $\Delta \in V$

$$\sup_{t>0} \int_{\mathrm{K}} \alpha^s(a_t k \Delta) \mathrm{d}k < \infty.$$

The upper bound is uniform as Δ varies over compact sets in the space of lattices.

- This is not true for $n = 2$!

Dynamical Counting Method

- Let F be a determinant form of degree n defined by $x \in \mathrm{G}$,
- We can approximate $\chi_{[1/2,1]\partial\Omega\times[a,b]}$ by a continuous function of compact support $h(xv,\zeta)$ be contained in $V_0 \times \mathbb{R}$

$$\sum_{v\in V_{\mathbb{Z}}} \chi_{[1/2,1]\partial\Omega\times[a,b]}(\mathrm{e}^{-t}v, F(v)) \approx \sum_{v\in V_{\mathbb{Z}}} h(\mathrm{e}^{-t}x\cdot v, F(v))$$

Functions on $V_0 \times \mathbb{R}$ can be approximated by finite linear combinations of functions of the form $J_f(\mathrm{M}\cdot\kappa^0(v),\zeta)\nu(\widehat{v}/\|\widehat{v}\|)$, where

1. ν is a positive and continuous function on the sphere S_V and $\widehat{v} = \mathrm{adj}(v)$ denotes the adjugate of v.
2. $\kappa^0(v) = \mathrm{diag}(\kappa_1(v),\ldots,\kappa_{n-1}(v))$ and $\kappa_1(v) \geq \cdots \geq \kappa_{n-1}(v)$ denotes the first $n-1$ singular values of v.
3. f is a continuous function of compact support on $V_+ = \{v \in V \mid \langle \widehat{v}e_n, e_n\rangle > 0\}$

Dynamical Counting Method

1. $J_f(\mathrm{M}\cdot r,\zeta) := \frac{1}{\det^2(r)} \int_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} \int_{\mathrm{M}} f\begin{pmatrix} m\cdot r & x_2 \\ x_1^t & x_{nn}\end{pmatrix} \mathrm{d}m\,\mathrm{d}x_1\mathrm{d}x_2$, where in the integral $x_{nn} = x_{nn}(r,\zeta,m,x_1,x_2)$ is determined by requiring

$$\det\begin{pmatrix} m\cdot r & x_2 \\ x_1^t & x_{nn}\end{pmatrix} = \zeta.$$

- Let $h(v,\zeta) = J_f(\mathrm{M}\cdot\kappa^0(v),\zeta)\nu(\widehat{v}/\|\widehat{v}\|)$ with f and ν as before,
- A direct calculation shows there is $T_0 > 0$ such that for any t with $\mathrm{e}^t > T_0$, and any $v \in V$ with $\|v\| > T_0$,

$$h(\mathrm{e}^{-t}\cdot v, \det(v)) \approx c_{\det}\mathrm{e}^{n(n-1)t}\int_{\mathrm{K}} f(a_t k\cdot v)\nu(\widehat{k}^t\cdot e_{nn})\,\mathrm{d}k$$

  where $c_{\det} = \omega_{n-1}^2$ and ω_{n-1} denotes the volume of the unit sphere in $\mathbb{R}^n$.
- Summing up (*) over $v \in V_{\mathbb{Z}}$, exchanging summation and integral yields

$$\mathrm{e}^{-n(n-1)t}\sum_{v\in V_{\mathbb{Z}}} h(\mathrm{e}^{-t}x\cdot v, F(v)) \approx c_{\det}\int_{\mathrm{K}} \widetilde{f}(a_t kx)\nu(\widehat{k}^t\cdot e_{nn})\,\mathrm{d}k,$$

Dynamical Counting Method

Theorem (FHO). $\mathrm{G}, \Gamma, \mathrm{X}, \mathrm{H}, \mathrm{K}, \{a_t\}$ as above. Let ϕ be a continuous function on X. Assume that for some s, $0 < s < 2$ and some $C > 0$,

$$|\phi(\Delta)| < C\alpha(\Delta)^s, \text{ for all } \Delta \in \mathrm{X}.$$

Let $x_0 \in \mathrm{X}$ be a unimodular lattice such that $\mathrm{H} \cdot x_0$ is not closed. Let ν be any continuous function on K, then

$$\lim_{t\to\infty} \int_{\mathrm{K}} \phi(a_t k \cdot x_0)\nu(k)\,\mathrm{d}k = \int_{\mathrm{K}} \nu\,\mathrm{d}k \int_{\mathrm{X}} \phi\,\mathrm{d}\mu_{\mathrm{X}}.$$

- This is due to Shah for ϕ compactly supported and continuous.
- For ϕ as above, we 'truncate' ϕ via appropriate bump functions g_r to sets of the form $A(r) = \{x \in \mathrm{X} \mid \alpha(x) > r\}$ so that $\phi - \phi g_r$ is continuous and compactly supported.

Dynamical Counting Method

- Hence, for $h \approx \chi_{[1,2]\partial\Omega\times[a,b]}$

$$\begin{aligned} e^{-n(n-1)t} \sum_{v \in V_{\mathbb{Z}}} h(e^{-t} x \cdot v, F(v)) &\approx c_{\det} \int_{\mathrm{K}} \widetilde{f}(a_t k x) \nu(\widehat{k}^t \cdot e_{nn}) \, \mathrm{d}k \\ &\approx c_{\det} \int_{\mathrm{K}} \nu \, \mathrm{d}k \int_{\mathrm{X}} \widetilde{f} \, \mathrm{d}\mu_{\mathrm{X}} \\ &= c_{\det} \int_{\mathrm{K}} \nu \, \mathrm{d}k \int_{V} f \, \mathrm{d}v \end{aligned}$$

- On the other hand, a direct computation shows

$$e^{-n(n-1)t} \int_V h(e^{-t} x \cdot v, F(v)) \, \mathrm{d}v \approx c_{\det} \int_{\mathrm{K}} \nu \, \mathrm{d}k \int_V f \, \mathrm{d}v$$

Spherical Averages *I*

- Suppose $\mathrm{H} = \mathrm{SL}_2(\mathbb{R}), \mathrm{K} = \mathrm{SO}(2)$ embedded in $\mathrm{G} = \mathrm{Sp}_{2n}(\mathbb{R})$, $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$ $(n \geq 1)$, $\mathrm{X} = \mathrm{G}/\Gamma$, $a_t = \mathrm{diag}(\mathrm{e}^{t/2}, \mathrm{e}^{-t/2})$ and $\beta > \frac{n}{2}$.
- For a lattice $\Delta \in \mathrm{G}/\Gamma$ let $f_0(h) := \alpha_n(h\Delta)^\beta, h \in \mathrm{H}$. Fact $\alpha \asymp_n \alpha_n$ on the space of symplectic lattices X.
- Define the mean value operator on H (K\H)

$$(\widetilde{\mathcal{A}}_r f)(h) = \int_{\mathrm{K}} f(a_r kh)\, \mathrm{d}k,$$

- ... reinterpreted in the language of the upper half-space $\mathbb{H}$

$$(\mathcal{A}_r f)(z) = \int_{\partial B_r(z)} f\, ds,$$

  where $\partial B_r(z)$ is the sphere of radius r centered at z, $(\mathrm{d}s)^2 = \frac{(dx)^2+(dy)^2}{y^2}$.
- Eigenfunctions of $\mathcal{A}_r$ are the spherical functions $\{\tau_\lambda\}_{\lambda\in\mathbb{C}}$. $\tau_\lambda(z) = \tau_\lambda(r)$ whenever $r = d(z,i)$ and here indexed such that $\tau_\lambda = \tau_{2-\lambda}$ and $\tau_\lambda(r) \asymp (e^{r/2})^{\lambda-2}$ as $r \to \infty$ if $\lambda > 2$.

Spherical Averages *I*

Lemma. Let $f \in C(\mathbb{H})$ be a positive function. Suppose there is $\lambda > 2, C, c > 1$ and $r_0 > 0$ such that

$$\text{For all } z \in \mathbb{H} : \ (\mathcal{A}_{r_0} f)(z) \leq C\tau_\lambda(r_0) f(z),$$
$$\text{For all } z \in \mathbb{H}, \ \text{ for all } w \in B_{r_0}(z) : f(w) \leq cf(z),$$

then $\mathcal{A}_r f(i) \ll_{C,c,\lambda} \tau_\lambda(r) f(i)$.

Spherical Averages *II*

- Suppose $V = \mathrm{Mat}_n(\mathbb{R}),\ V_{\mathbb{Z}} = \mathrm{Mat}_n(\mathbb{Z})$,
- $\mathrm{G} = \mathrm{SL}(V), \Gamma = \mathrm{SL}(V_{\mathbb{Z}}), \mathrm{X} = \mathrm{G}/\Gamma$,
- $\mathrm{H} = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) \subset \mathrm{G}$,
- $b_t = \mathrm{diag}(\mathrm{e}^{-t/2}, \dots, \mathrm{e}^{-t/2}, \mathrm{e}^{t(n-1)/2}), a_t = (b_t, b_t) \in \mathrm{H}$.
- $\mathrm{K} = \mathrm{SO}(n) \times \mathrm{SO}(n)$.
- For a lattice $\Delta \in \mathrm{G}/\Gamma$ let $f_0(h) := \alpha^s(h\Delta), h \in \mathrm{H}$.
- Define the averaging operator on H (K\H)

$$(\mathcal{A}_r f)(h) = \int_{\mathrm{K}} f(a_r k h)\,\mathrm{d}k,\ r > 0.$$

Spherical Averages *II*

Lemma (Eskin, Margulis, Mozes). Let H, K, $\{a_t\}_t$ and $\mathcal{A}_r$ as above. Let f be a strictly positive left K-invariant function on H such that $\log(f)$ is uniformly continuous with respect to a left-invariant uniform structure on H. Then, there exists $0 < c < 1$, such that for any $r_0 > 0$ and $b > 0$ there is $B = B(r_0, b) < \infty$ with the following property: If

$$(\mathcal{A}_{r_0} f)(h) < cf(h) + b, \text{ for all } h \in \mathrm{KAK},$$

then

$$(\mathcal{A}_t f)(e) < B \text{ for all } t > 0.$$

Spherical Averages *II*

Lemma. Let $n \geq 3$ and $1 \leq i < n^2$. Then, for any s, $0 < s < 2$ and any $c > 0$ there is $t > 0$ and $\omega > 1$ such that for any lattice Λ in V

$$\int_{\mathrm{K}} \alpha_i^s(a_t k \cdot \Lambda)\,\mathrm{d}k < \frac{c}{2}\alpha_i^s(\Lambda) + \omega^2 \max_{0<j\leq\min\{n^2-i,i\}} \left(\sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)}\right)^s.$$

This inequality together with the previous key lemma implies

$$\sup_{t>0} \int_{\mathrm{K}} \alpha^s(a_t k \Lambda)\,\mathrm{d}k < \infty,$$

for $0 < s < 2$.

Spherical Averages *II*

Lemma (FHO). Suppose $n \geq 3$ and let $1 \leq i < n^2$. Denote by ρ_i the i-th exterior representation of H in $W_i = \wedge^i V \cong \wedge^i \mathbb{R}^{n^2}$ and let $Q(i) := \{v_1 \wedge \cdots \wedge v_i \,|\, v_1, \ldots, v_i \in V\} \subset W_i$. Then, for any s, $0 < s < 2$

$$\lim_{t \to +\infty} \sup_{w \in Q(i), \|v\|=1} \int_{\mathrm{K}} \frac{\mathrm{d}k}{\|\rho_i(a_t k) w\|^s} = 0,$$

Note true for $n = 2$! When $i = 2$, the Lie algebra of K is 'too small' and the result holds only for $0 < s < 1$.

Spherical Averages *II*

Lemma. Let W be a finite dimensional real inner product space, A a self-adjoint linear transformation of W, K a closed connected subgroup of the orthogonal group $\mathrm{O}(W)$, Q a closed subset of the unit sphere. Assume that the eigenvalues of A are $\lambda_1 < \cdots < \lambda_M$ and denote by W^i the eigenspace corresponding to λ_i for each $1 \leq i \leq M$. Assume there is m with $2 \leq m \leq M - 1$ such that $\lambda_m \geq 0$ and such that the following conditions are satisfied

1. $\mathrm{K}w \not\subseteq \bigoplus_{j=1}^{m} W^j$ for any $w \in Q$;
2. For every $1 \leq i \leq m-1$ and any non-zero $w \in \bigoplus_{j=1}^{i} W^j \setminus \bigoplus_{j=1}^{i-1} W^j$ there exists an l-dimensional subspace L_w of $\mathrm{Lie}(\mathrm{K})$ such that $Xw \notin \bigoplus_{j=1}^{i} W^j$ for every non-zero $X \in \mathrm{Lie}(\mathrm{K})$;

Then, for any $0 < s < l$

$$\lim_{t\to\infty} \sup_{w \in Q} \int_{\mathrm{K}} \frac{\mathrm{d}k}{\|\mathrm{e}^{tA} k w\|^s} = 0.$$

Thank you!