Orbit closures of unipotent flows for hyperbolic manifolds with Fuchsian Ends

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November 2, 2020

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- Special cases were proved earlier by Margulis, Dani-Margulis, Shah.

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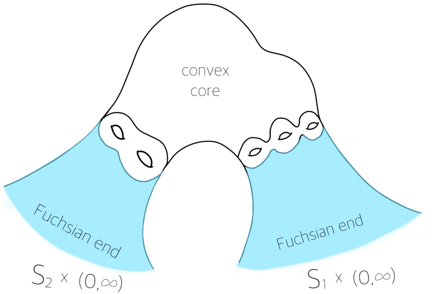

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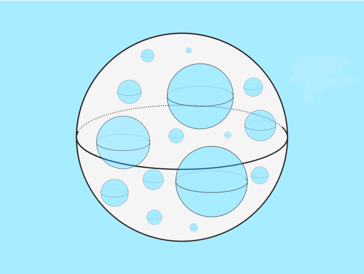

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- $d = 3$ (McMullen-Mohammadi-Oh)

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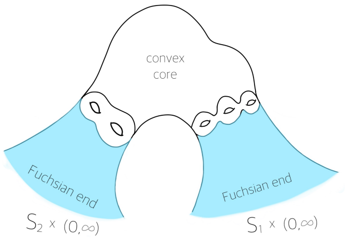

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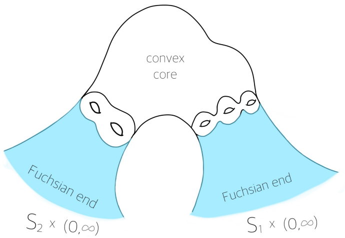

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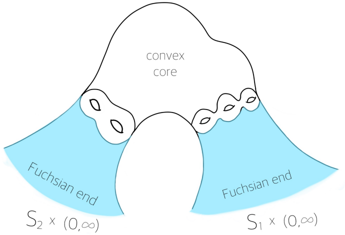

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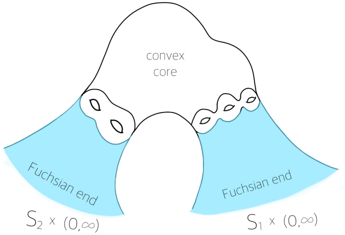

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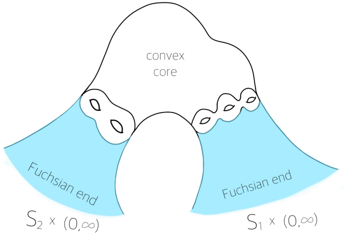

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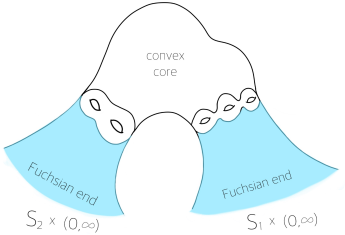

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$H(U)$-orbit closures are relatively homogeneous; for all $x \in RF\, M$, $\overline{xH(U)} = xL \cap RF_+ M \cdot H(U)$. Moreover, L is of the form $H(\hat{U})C$ for $U \subset \hat{U}$ and $C \subset C_G(H(\hat{U}))$.

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Base cases $(1)_0$, $(2)_0$, $(3)_0$ follows from...

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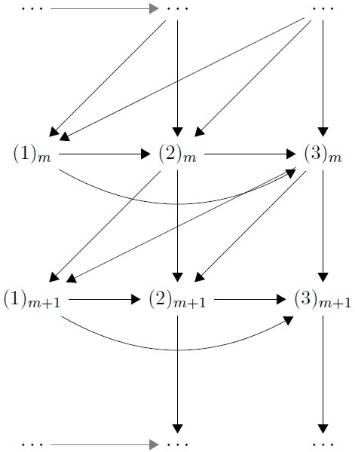

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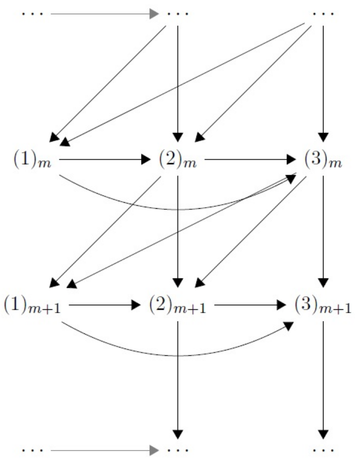

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(For given x_i converging to a generic point x, and $T_i \to \infty$, need $t_i \in [T_i, 2kT_i]$ such that $x_i u_{t_i}$ converges to a generic point.)

The End of part I.