Convex co-compact representations of 3-manifold groups

joint with Mitul Islam (5^{th} year graduate student at Michigan)

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The plan

Outline:

- Background (slide 3)
  - Anosov representations
  - convex co-compact representations
- Results (slide 13)
- Proofs (slide 34)

Part 1: Background

Background: Anosov representations

"Definition:" Suppose:

- G is a semisimple Lie group (e.g. $G = \mathrm{SL}_d(\mathbb{R})$)
- $P \leq G$ is a parabolic subgroup (e.g. the stabilizer of a line)
- Γ is a word hyperbolic group
- $\partial_\infty \Gamma$ is the Gromov boundary of Γ

A representation $\rho : \Gamma \to G$ is P-**Anosov** if there exists an embedding $\xi : \partial_\infty \Gamma \to G/P$ with "good dynamical behavior".

Properties:

1. Discrete image, finite kernel
2. If $X = G/K$ is the symmetric space associated to G and $x_0 \in X$, then the orbit map $\gamma \to \rho(\gamma) \cdot x_0$ is a quasi-isometry
3. Stable under deformations
4. When $G = \mathrm{Isom}(\mathbb{H}^d_{\mathbb{R}})$ and $P \leq G$ is any parabolic, then P-Anosov if and only if convex co-compact
5. Many examples in higher rank

Background: Anosov representations

When $G = \mathrm{PGL}_d(\mathbb{R})$ and $P_1 =$ (stabilizer of a line), then P_1-Anosov representations are often called **projective Anosov representations** since $G/P_1 \cong \mathbb{P}(\mathbb{R}^d)$.

Precise Definition [Tsouvalas 2020?]: Suppose Γ is a word hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}_d(\mathbb{R})$ is called **projective Anosov** if there exists continuous ρ-equivariant embeddings

$$\xi : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^d) \text{ and } \eta : \partial_\infty \Gamma \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$$

such that:

- $\xi(x) + \eta(y) = \mathbb{R}^d$ for all $x, y \in \partial_\infty \Gamma$ distinct,
- if $\gamma_n \to x \in \partial_\infty \Gamma$ and $\gamma_n^{-1} \to y \in \partial_\infty \Gamma$, then

$$\rho(\gamma_n)\ell \to \xi(x)$$

  for all $\ell \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\eta(y))$ (i.e. ℓ is transverse to $\eta(y)$)

Note: The second condition is equivalent to $\rho(\gamma_n) \to T$ in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ where $\mathrm{Im}(T) = \xi(x)$ and $\ker(T) = \eta(y)$.

Theorem [Guichard-Weinhard 2012]: If G is a semisimple Lie group and $P \leq G$ is a parabolic subgroup, then there exists $d > 0$ and an irreducible representation $\phi : G \to \mathrm{PGL}_d(\mathbb{R})$ such that the following are equivalent:

1. $\rho : \Gamma \to G$ is P-Anosov
2. $\phi \circ \rho : \Gamma \to \mathrm{PGL}_d(\mathbb{R})$ is projective Anosov

Background: Anosov representations

Anosov representations seem to be the right class of representations to consider for word hyperbolic groups

1. Flexible - many examples
2. Rigid - can prove theorems about them

Question: How to move beyond the word hyperbolic case?

One proposed solution: Convex co-compact representations in the sense of Danciger-Guéritaud-Kassel

See also: “relative Anosov representations” in the sense of Kapovich-Leeb or Zhu

Background: convex co-compact subgroups

The setup:

- $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a **properly convex domain**, that is a bounded convex open subset of some affine chart
- The **automorphism group** is

$$\mathrm{Aut}(\Omega) = \{g \in \mathrm{PGL}_d(\mathbb{R}) : g\Omega = \Omega\}.$$

- The **Hilbert distance** between $p, q \in \Omega$ is

$$H_\Omega(p,q) = \frac{1}{2} \log \frac{\|p-b\| \, \|q-a\|}{\|p-a\| \, \|q-b\|}$$

Classical Theorem: If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, then:

- (Ω, H_Ω) is a proper geodesic metric space and line segments can be parametrized as geodesics.
- $\mathrm{Aut}(\Omega)$ acts by isometries on (Ω, H_Ω).

Background: convex co-compact subgroups

Example: If

$$\mathbb{B} = \left\{ [x_1 : \cdots : x_{d+1}] \in \mathbb{P}(\mathbb{R}^{d+1}) : x_2^2 + \cdots + x_{d+1}^2 < x_1^2 \right\},$$

then $(\mathbb{B}, H_{\mathbb{B}})$ is the Klein-Beltrami model of real hyperbolic d-space and $\mathrm{Aut}(\mathbb{B}) = \mathrm{PO}(1, d)$.

Background: convex co-compact subgroups

Definition [Danciger-Guéritaud-Kassel]: Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is a discrete group.

- The **limit set** $\mathcal{L}_\Omega(\Lambda) \subset \partial\Omega$ is the set of $x \in \partial\Omega$ where there exists $p \in \Omega$ and $\gamma_n \in \Lambda$ such that $\gamma_n p \to x$.
- The **convex hull** $\mathcal{C}_\Omega(\Lambda) \subset \Omega$ is the convex hull of $\mathcal{L}_\Omega(\Lambda)$ in Ω.
- Λ is **convex co-compact** if $\mathcal{C}_\Omega(\Lambda) \neq \emptyset$ and $\Lambda \backslash \mathcal{C}_\Omega(\Lambda)$ is compact.

Definition: A representation $\rho : \Gamma \to \mathrm{PGL}_d(\mathbb{R})$ is **convex co-compact** if $\ker \rho$ is finite, $\rho(\Gamma)$ is discrete, and there exists a properly convex domain Ω such that $\rho(\Gamma) \leq \mathrm{Aut}(\Omega)$ is convex co-compact.

Background: convex co-compact subgroups

Theorem [D.-G.-K. 2017]: If $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ is convex co-compact, then:

1. any sufficiently small deformation of ρ is convex co-compact
2. if $x \in X = \mathsf{PGL}_d(\mathbb{R})/\mathrm{PO}(d)$, then the orbit map $\gamma \in \Gamma \to \rho(\gamma)x \in X$ is a quasi-isometry

Theorem [D.-G.-K. 2017, Z. 2017 (in irreducible case)]: If Γ is word hyperbolic and $\rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ is convex co-compact, then ρ is projective Anosov.

Theorem [Z. 2017 (in Zariski dense case), D.-G.-K. 2017 (implicit)]: If G is a semisimple Lie group and $P \leq G$ is a parabolic subgroup, then there exists $d > 0$ and an irreducible representation $\phi : G \to \mathsf{PGL}_d(\mathbb{R})$ such that the following are equivalent:

1. $\rho : \Gamma \to G$ is P-Anosov
2. $\phi \circ \rho : \Gamma \to \mathsf{PGL}_d(\mathbb{R})$ is convex co-compact

Theorem [Z. 2017 (in irreducible case)]: If Γ is a one-ended word hyperbolic group which is not commensurable to a surface group, then any projective Anosov representation of Γ is convex co-compact.

Background: convex co-compact subgroups

Questions: What non-word hyperbolic groups can admit convex co-compact representations?

Examples:

- if Γ_1 and Γ_2 admit a convex co-compact representation, then so does $\Gamma_1 * \Gamma_2$ (claimed by D.-G.-K. 2017)
- fundamental groups of certain non-geometric 3-manifolds where every component in the geometric decomposition is hyperbolic (Benoist 2006, Danciger-Ballas-Lee 2018)
- certain Coexter groups (Choi-Lee-Marquis 2016)
- uniform lattices in $\mathsf{SL}_d(\mathbb{R})$, $\mathsf{SL}_d(\mathbb{C})$, $\mathsf{SL}_d(\mathbb{H})$, $\mathsf{SL}_3(\mathbb{O})$

Part 2: Results

Results

Based on: joint work with Mitul Islam (5^{th} year graduate student at Michigan)

- A flat torus theorem for convex co-compact actions of projective linear groups (ArXiv 2019)
- Convex co-compact actions of relatively hyperbolic groups (ArXiv 2019)
- Convex co-compact representations of 3-manifold groups (ArXiv 2020)

General approach: Convex co-compact groups should behave a lot like CAT(0)-groups

- metric balls in the Hilbert distance are convex

But...

- two points can sometimes be joined by infinitely many geodesics in (Ω, H_Ω)
- **Kelly-Straus 1958:** (Ω, H_Ω) is CAT(0) if and only if (Ω, H_Ω) is the Klein-Beltrami model of real hyperbolic space (up to a change of coordinates)

Part 2 (a): A flat torus theorem

- A flat torus theorem for convex co-compact actions of projective linear groups (ArXiv 2019)

Properly embedded simplices

The analog of isometrically embedded flats in CAT(0) spaces seem to be properly embedded simplices

Definition:

- A subset $S \subset \mathbb{P}(\mathbb{R}^d)$ is a **k-dimensional simplex** if there exists $g \in \mathrm{PGL}_d(\mathbb{R})$ such that
$$gS = \left\{[1 : x_1 : \cdots : x_k : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1, \ldots, x_k > 0 \text{ and } \sum x_j < 1\right\}.$$
- S is **properly embedded** in Ω if $S \subset \Omega$ and $\partial S \subset \partial\Omega$.

Proposition: If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded k-simplex, then $(S, H_\Omega) = (S, H_S)$ is isometric to $\mathbb{R}^k$ with the norm
$$\|v\| = \frac{1}{2} \max\left\{\max_{1 \le i \le k} |v_i|, \max_{1 \le i,j \le k} |v_i - v_j|\right\}.$$

Properly embedded simplices

Fact: If $S \subset \mathbb{P}(\mathbb{R}^d)$ is a simplex, then $\mathrm{Aut}(S)$ acts transitively on S.

Proof: Up to a change of coordinates

$$S = \left\{[x_1 : \cdots : x_{k+1} : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1, \ldots, x_{k+1} > 0\right\}.$$

Then the group of diagonal matrices with positive entries acts transitively on S. □

Theorem [Foertsch-Karlsson 2005]: If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, then (Ω, H_Ω) is isometric to a normed vector space if and only if Ω is a simplex.

Note: By Colbois-Verovic 2009: (Ω, H_Ω) is quasi-isometric to a normed vector space if and only if Ω is a convex polygon.

Flat torus theorem

We proved the following analogue of the CAT(0) flat torus theorem of Gromoll-Wolf and Lawson-Yau.

Theorem [Islam-Z. 2019]: Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is convex co-compact. If $A \leq \Lambda$ is a maximal Abelian subgroup of Λ, then there exists a properly embedded simplex $S \subset \mathcal{C}_\Omega(\Lambda)$ such that:

1. S is A-invariant,
2. A acts co-compactly on S, and
3. A fixes each vertex of S.

Moreover, A has a finite index subgroup isomorphic to $\mathbb{Z}^{\dim(S)}$.

Note: When $d = 4$ and Λ acts co-compactly on Ω, the above theorem was established by Benoist (2006) by computing all possible Zariski closures of Abelian subgroups in $\mathrm{PGL}_4(\mathbb{R})$.

Part 2 (b): Relatively hyperbolic groups

- Convex co-compact actions of relatively hyperbolic groups (ArXiv 2019)

Relatively hyperbolic groups

Theorem [Danciger-Guéritaud-Kassel 2017]: Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Lambda \leq \mathrm{Aut}(\Omega)$ is convex co-compact. Then the following are equivalent:

1. $\mathcal{C}_\Omega(\Lambda)$ contains no properly embedded simplices with dimension at least two,
2. $(\mathcal{C}_\Omega(\Lambda), H_\Omega)$ is Gromov hyperbolic,
3. Λ is word hyperbolic.

Note: When Λ acts co-compactly on Ω, the above theorem was established by Benoist (2004).

Question [D.-G.-K. 2017]: Under what conditions is Λ relatively hyperbolic with respect to a collection of virtually Abelian subgroups?

Relatively hyperbolic groups

Theorem [Islam-Z. 2019]: Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \mathrm{Aut}(\Omega)$ is convex co-compact, and $\mathcal{S}_{max}$ is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two. Then the following are equivalent:

1. $\mathcal{S}_{max}$ is closed and discrete in the local Hausdorff topology induced by H_Ω,
2. $(\mathcal{C}_\Omega(\Lambda), H_\Omega)$ is a relatively hyperbolic space with respect to $\mathcal{S}_{max}$,
3. $(\mathcal{C}_\Omega(\Lambda), H_\Omega)$ is a relatively hyperbolic space with respect to a family of properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two,
4. Λ is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank at least two.

Note: Similar to results of Hruska-Kleiner (2005) for CAT(0)-groups, but the proofs are different.

Relatively hyperbolic groups

Theorem [Islam-Z. 2019]: Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Lambda \leq \mathrm{Aut}(\Omega)$ is convex co-compact, and $\mathcal{S}_{max}$ is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Lambda)$ of dimension at least two.

If $\mathcal{S}_{max}$ is closed and discrete in the local Hausdorff topology induced by H_Ω, then:

1. If $S \in \mathcal{S}_{max}$, then $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S and contains a finite index subgroup isomorphic to $\mathbb{Z}^{\dim S}$.
2. Λ has finitely many orbits in $\mathcal{S}_{max}$ and if $\{S_1, \dots, S_m\}$ is a set of orbit representatives, then Λ is a relatively hyperbolic group with respect to
$$\{\mathrm{Stab}_\Lambda(S_1), \dots, \mathrm{Stab}_\Lambda(S_m)\}.$$
3. If $A \leq \Lambda$ is an infinite Abelian subgroup of rank at least two, then there exists a unique $S \in \mathcal{S}_{max}$ with $A \leq \mathrm{Stab}_\Lambda(S)$.
4. If $S_1, S_2 \in \mathcal{S}_{max}$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and $\partial S_1 \cap \partial S_2 = \emptyset$.
5. If $\ell \subset \overline{\mathcal{C}_\Omega(\Lambda)} \cap \partial\Omega$ is a non-trivial line segment, then there exists $S \in \mathcal{S}_{max}$ with $\ell \subset \partial S$.
6. If $x \in \overline{\mathcal{C}_\Omega(\Lambda)} \cap \partial\Omega$ is not a C^1-smooth point of $\partial\Omega$, then there exists $S \in \mathcal{S}_{max}$ with $x \in \partial S$.

Part 2 (c): 3-manifold groups

- Convex co-compact representations of 3-manifold groups (ArXiv 2020)

Theorem [Benoist 2006]: If M is a closed irreducible orientable 3-manifold and M admits a convex real projective structure, then either

1. M is geometric with geometry $\mathbb{R}^3$, $\mathbb{R} \times \mathbb{H}^2$, or $\mathbb{H}^3$,
2. M is non-geometric and every component in the geometric decomposition is hyperbolic.

Recall, a convex real projective structure on a manifold M is a homeomorphism $M \cong \Lambda \backslash \Omega$ where

- $\widetilde{M} \cong \Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain (note: $d = \dim M + 1$)
- $\pi_1(M) \cong \Lambda \leq \mathrm{Aut}(\Omega)$ acts freely and properly discontinuously on Ω

If M is closed, then $\Lambda \curvearrowright \Omega$ acts co-compactly and so $\pi_1(M) \xrightarrow{\sim} \Lambda \leq \mathrm{PGL}_d(\mathbb{R})$ is a convex co-compact representation.

Question: Which 3-manifold groups admit convex co-compact representations?

3-manifold groups

Theorem [Islam-Z. 2020]: Suppose M is a closed irreducible orientable 3-manifold. If $\rho : \pi_1(M) \to \mathsf{PGL}_d(\mathbb{R})$ is a convex co-compact representation, then either

1. M is geometric with geometry $\mathbb{R}^3$, $\mathbb{R} \times \mathbb{H}^2$, or $\mathbb{H}^3$,
2. M is non-geometric and every component in the geometric decomposition is hyperbolic.

In each case we can describe the structure of examples.

The structure of $\mathbb{R}^3$ and $\mathbb{R} \times \mathbb{H}^2$ examples

In this case, convex co-compact representations come from convex real projective structures.

Proposition: Suppose M is a closed 3-manifold with $\mathbb{R}^3$ or $\mathbb{R} \times \mathbb{H}^2$ geometry. If

- $\rho : \pi_1(M) \to \mathsf{PGL}_d(\mathbb{R})$ is a convex co-compact representation and
- $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain where $\Lambda := \rho(\pi_1(M)) \leq \mathsf{Aut}(\Omega)$ is convex co-compact,

then there exists a four dimensional linear subspace $V \subset \mathbb{R}^d$ such that

$$\mathcal{C}_\Omega(\Lambda) = \Omega \cap \mathbb{P}(V).$$

Moreover,

1. If M has $\mathbb{R}^3$ geometry, then $\mathcal{C}_\Omega(\Lambda)$ is a properly embedded simplex in Ω,
2. If M has $\mathbb{R} \times \mathbb{H}^2$ geometry, then $\mathcal{C}_\Omega(\Lambda)$ is a properly embedded cone in Ω with strictly convex base.

In both cases, $M \cong \Lambda \backslash \mathcal{C}_\Omega(\Lambda)$.

The structure of $\mathbb{H}^3$ examples

Using work in D.-G.-K. 2017 and Z. 2017:

Proposition: Suppose M is a closed 3-manifold with $\mathbb{H}^3$ geometry. If

- $\rho : \pi_1(M) \to \mathsf{PGL}_d(\mathbb{R})$ is a convex co-compact representation and
- $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain where $\Lambda := \rho(\pi_1(M)) \leq \mathsf{Aut}(\Omega)$ is convex co-compact,

then ρ is projective Anosov. Moreover, if $\xi : \partial_\infty \pi_1(M) \to \mathbb{P}(\mathbb{R}^d)$ is the Anosov boundary map, then

- $\mathrm{Image}(\xi) = \partial_{\mathsf{i}} \, \mathcal{C}_\Omega(\Lambda)$,
- $\partial_{\mathsf{i}} \, \mathcal{C}_\Omega(\Lambda)$ contains no non-trivial line segments
- every point in $\partial_{\mathsf{i}} \, \mathcal{C}_\Omega(\Lambda)$ is a C^1 point of $\partial\Omega$.

Notation: $\partial_{\mathsf{i}} \, \mathcal{C}_\Omega(\Lambda) := \overline{\mathcal{C}_\Omega(\Lambda)} \cap \partial\Omega$ is the ideal boundary

Non-geometric examples

Suppose

- M is non-geometric and every component in the geometric decomposition is hyperbolic,
- $\rho : \pi_1(M) \to \mathrm{PGL}_d(\mathbb{R})$ is convex co-compact,
- $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain where $\Lambda := \rho(\pi_1(M)) \leq \mathrm{Aut}(\Omega)$ is convex co-compact, and
- $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$.

Dahmani's combination theorem (2003): $\pi_1(M)$ is relatively hyperbolic with respect to a collection of subgroups virtually isomorphic to $\mathbb{Z}^2$ (namely the fundamental groups of the Klein bottles and tori in the geometric decomposition).

Non-geometric examples - structure of the domain

Let $\mathcal{S}_{max}$ denote the collection of **all** properly embedded simplices in $\mathcal{C}$ of dimension at least two.

By results in Islam-Z. 2019:

- $(\mathcal{C}, H_\Omega)$ is relatively hyperbolic with respect to $\mathcal{S}_{max}$.
- $\mathcal{S}_{max}$ is closed and discrete in the local Hausdorff topology.
- Every line segment in $\partial_{\mathrm{i}} \mathcal{C}$ is contained in the boundary of a simplex in $\mathcal{S}_{max}$.
- If $x \in \partial_{\mathrm{i}} \mathcal{C}$ is not a C^1-smooth point of $\partial\Omega$, then there exists $S \in \mathcal{S}_{max}$ with $x \in \partial S$.

And

- If $S \in \mathcal{S}_{max}$, then S is two dimensional, $\mathrm{Stab}_\Lambda(S)$ acts co-compactly on S, and $\mathrm{Stab}_\Lambda(S)$ is virtually isomorphic to $\mathbb{Z}^2$.
- If $A \leq \Lambda$ is an Abelian subgroup with rank at least two, then A is virtually isomorphic to $\mathbb{Z}^2$ and there exists a unique $S \in \mathcal{S}_{max}$ such that $A \leq \mathrm{Stab}_\Lambda(S)$.

Recall: $\Lambda = \rho(\pi_1(M))$, $\mathcal{C} = \mathcal{C}_\Omega(\Lambda)$, and $\partial_{\mathrm{i}} \mathcal{C} = \overline{\mathcal{C}} \cap \partial\Omega$.

Non-geometric examples - equivariant boundary maps

Leeb 1995: We can assume that M is a non-positively curved Riemannian manifold

Hruska-Kleiner 2005: If $\pi_1(M)$ acts geometrically on a CAT(0) space X, then there exists an equivariant homeomorphism $\widetilde{M}(\infty) \to X(\infty)$.

Question: Does there exists a ρ-equivariant homeomorphism $\widetilde{M}(\infty) \to \partial_{\mathsf{i}}\,\mathcal{C}$?

Recall: In the hyperbolic/Anosov case there exists a ρ-equivariant homeomorphism $\partial_\infty \pi_1(M) \to \partial_{\mathsf{i}}\,\mathcal{C}$.

Non-geometric examples - equivariant boundary maps

Question: Does there exists a ρ-equivariant homeomorphism $\widetilde{M}(\infty) \to \partial_{\mathrm{i}} \mathcal{C}$?

Answer: No.

Non-geometric examples - equivariant boundary maps

Let:

- $\widetilde{M}(\infty)/\sim$ denote the quotient of $\widetilde{M}(\infty)$ obtained by identifying points which are in the geodesic boundary of the same flat
- $\partial_{\mathrm{i}}\,\mathcal{C}\,/\sim$ denote the quotient of $\partial_{\mathrm{i}}\,\mathcal{C}$ obtained by identifying points which are in the boundary of the same simplex in $\mathcal{S}_{max}$.

Theorem [Tran 2013]: $\widetilde{M}(\infty)/\sim$ is the Bowditch boundary of $\pi_1(M)$.

Theorem [Islam-Z. 2020]: Any ρ-equivariant quasi-isometry $\widetilde{M} \to \mathcal{C}$ extends to a ρ-equivariant homeomorphism

$$\widetilde{M}(\infty)/\sim \longrightarrow \partial_{\mathrm{i}}\,\mathcal{C}\,/\sim.$$

Note: Can also be derived from a recent general result of Weisman (2020) about convex co-compact representations of relatively hyperbolic groups.

Non-geometric examples - dynamics

Using the identification $\widetilde{M}(\infty)/\sim \longrightarrow \partial_{\mathrm{i}}\,\mathcal{C}\,/\sim$ we can prove:

Theorem [Islam-Z. 2020]: $\Lambda = \rho(\pi_1(M))$ acts minimally on $\partial_{\mathrm{i}}\,\mathcal{C}$.

Corollary: The geodesic flow associated to $\Lambda\backslash\,\mathcal{C}$ is topologically transitive.

Note: If $\Lambda \leq \mathrm{PGL}_d(\mathbb{R})$ is strongly irreducible, then using a result of Blayac (2020) the Corollary can be upgraded to topologically mixing.

What is the geodesic flow?

- Let $\mathcal{G}_\Omega$ denote the space of unit speed geodesic lines in Ω which parametrize line segments.
- The geodesic flow $\phi_t : \mathcal{G}_\Omega \to \mathcal{G}_\Omega$ is defined by $\phi_t(\gamma) = \gamma(\cdot + t)$
- Let $\mathcal{G}_\Omega(\Lambda)$ denote the subset of $\mathcal{G}_\Omega$ whose image is contained in $\mathcal{C}_\Omega(\Lambda)$.
- ϕ_t descends to the compact quotient $\Lambda\backslash\,\mathcal{G}_\Omega(\Lambda)$

Part 3: Proofs of the 3-manifold results

Key lemma

Theorem [Islam-Z. 2020]: Suppose

- $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain,
- $\Lambda \leq \mathsf{Aut}(\Omega)$ is convex co-compact,
- $A \leq \Lambda$ is an infinite Abelian subgroup, and
- $C_\Lambda(A)$ is the centralizer of A in Λ.

If

$$V := \mathrm{Span}\left\{ v \in \mathbb{R}^d \setminus \{0\} : [v] \in \overline{\mathcal{C}_\Omega(\Lambda)} \text{ and } a[v] = [v] \text{ for all } a \in A \right\},$$

then $\Omega \cap \mathbb{P}(V)$ is a non-empty $C_\Lambda(A)$-invariant properly convex domain in $\mathbb{P}(V)$ and the quotient $C_\Lambda(A)\backslash\Omega \cap \mathbb{P}(V)$ is compact.

Corollary: $C_\Lambda(A)$ is virtually the fundamental group of a closed aspherical $(\dim V - 1)$-manifold.

Corollary: If N is the normalizer of A in Λ, then $C_\Lambda(A)$ has finite index in N.

Proof - main result

Theorem [Islam-Z. 2020]: Suppose M is a closed irreducible orientable 3-manifold. If $\rho : \pi_1(M) \to \mathrm{PGL}_d(\mathbb{R})$ is a convex co-compact representation, then either

1. M is geometric with geometry $\mathbb{R}^3$, $\mathbb{R} \times \mathbb{H}^2$, or $\mathbb{H}^3$,
2. M is non-geometric and every component in the geometric decomposition is hyperbolic.

Proof sketch: In non-geometric case either

1. every component in the geometric decomposition is hyperbolic or
2. there exists a Seifert fibered component in the geometric decomposition

Suppose for a contradiction that there exists a Seifert fibered component S. Let $\langle h \rangle$ denote the infinite cyclic subgroup in $\pi_1(S)$ generated by a regular fiber. Then

- $C_{\pi_1(S)}(h)$ has finite index in $\pi_1(S)$,
- $C_{\pi_1(S)}(h) = C_{\pi_1(M)}(h)$,
- $\pi_1(S)$ is viturally isomorphic to $\mathbb{Z} \times \mathrm{F}_m$

But by centralizer result $C_{\pi_1(M)}(h)$ is virtually the fundamental group of a closed aspherical manifold.

So $\mathbb{Z} \times \mathrm{F}_m$ is virtually the fundamental group of a closed aspherical manifold. Contradiction.

Proof - structure of $\mathbb{R}^3$ or $\mathbb{R} \times \mathbb{H}^2$ manifolds

Proposition: Suppose M is a closed 3-manifold with $\mathbb{R}^3$ or $\mathbb{R} \times \mathbb{H}^2$ geometry. If

- $\rho : \pi_1(M) \to \mathrm{PGL}_d(\mathbb{R})$ is a convex co-compact representation and
- $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain where $\Lambda := \rho(\pi_1(M)) \leq \mathrm{Aut}(\Omega)$ is convex co-compact,

then there exists a four dimensional linear subspace $V \subset \mathbb{R}^d$ such that

$$\mathcal{C}_\Omega(\Lambda) = \Omega \cap \mathbb{P}(V).$$

Moreover,

1. If M has $\mathbb{R}^3$ geometry, then $\mathcal{C}_\Omega(\Lambda)$ is a properly embedded simplex in Ω,
2. If M has $\mathbb{R} \times \mathbb{H}^2$ geometry, then $\mathcal{C}_\Omega(\Lambda)$ is a properly embedded cone in Ω with strictly convex base.

In both cases, $M \cong \Lambda \backslash \mathcal{C}_\Omega(\Lambda)$.

Proof sketch: Almost immediate from structure of centralizers

Proof - structure of non-geometric examples

Suppose

- M is non-geometric and every component in the geometric decomposition is hyperbolic,
- $\rho : \pi_1(M) \to \mathrm{PGL}_d(\mathbb{R})$ is convex co-compact,
- $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain where $\Lambda := \rho(\pi_1(M)) \leq \mathrm{Aut}(\Omega)$ is convex co-compact,
- $\mathcal{C} := \mathcal{C}_\Omega(\Lambda)$, and
- $\partial_{\mathrm{i}} \mathcal{C} := \overline{\mathcal{C}} \cap \partial\Omega$.

Theorem: Any ρ-equivariant quasi-isometry $\widetilde{M} \to \mathcal{C}$ extends to a ρ-equivariant homeomorphism

$$\widetilde{M}(\infty)/\sim \longrightarrow \partial_{\mathrm{i}} \mathcal{C} /\sim.$$

Theorem: $\Lambda = \rho(\pi_1(M))$ acts minimally on $\partial_{\mathrm{i}} \mathcal{C}$.

Corollary: The geodesic flow associated to $\Lambda \backslash \mathcal{C}$ is topologically transitive.

Proof - structure of non-geometric examples

Theorem: Any ρ-equivariant quasi-isometry $\widetilde{M} \to \mathcal{C}$ extends to a ρ-equivariant homeomorphism

$$\widetilde{M}(\infty)/\sim \longrightarrow \partial_{\mathrm{i}}\, \mathcal{C}\,/\sim.$$

Key tool:

Relative Fellow Traveller Property [Druțu-Sapir]: Suppose (X, dist) is relatively hyperbolic with respect to $\mathcal{Y}$. For $\alpha \geq 1$, $\beta \geq 0$, then there exists $L = L(\alpha, \beta) > 0$ with the following property: if $\gamma : [a, b] \to X$ and $\sigma : [a', b'] \to X$ are (α, β)-quasi-geodesics with the same endpoints, then there exist partitions

$$a = t_0 < t_1 < \cdots < t_{m+1} = b$$
$$a' = t'_0 < t'_1 < \cdots < t'_{m+1} = b'$$

where for all $0 \leq i \leq m$

$$\mathrm{dist}(\gamma(t_i), \sigma(t'_i)) \leq L$$

and either

1. $\mathrm{dist}^{\mathrm{Haus}}(\gamma|_{[t_i, t_{i+1}]}, \sigma|_{[t'_i, t'_{i+1}]}) \leq L$ or
2. $\gamma|_{[t_i, t_{i+1}]}, \sigma|_{[t'_i, t'_{i+1}]} \subset \mathcal{N}(Y; L)$ for some $Y \in \mathcal{Y}$.

Proof - structure of non-geometric examples

Theorem: $\Lambda = \rho(\pi_1(M))$ acts minimally on $\partial_{\mathrm{i}}\,\mathcal{C}$.

Proof - structure of non-geometric examples

Corollary: The geodesic flow associated to $\Lambda \backslash \mathcal{C}$ is topologically transitive.

Proof sketch: Modify the proof that the geodesic flow on a closed NPC manifold is topologically transitive if and only if the fundamental group acts minimally on the geodesic boundary.

The End