Minimal Surfaces in Negatively Curved 3-manifolds and Dynamics

- $M = \Gamma \backslash \mathbb{H}^3$ a closed hyperbolic 3-manifold
- $Gr_2(M)$ the Grassmann bundle of tangent 2-planes over M.
- $Gr_2(M)$ has a natural foliation $\mathcal{F}$ by (lifts of) immersed totally geodesic planes.

<u>Thm</u> (Ratner-Shah) Every leaf of $\mathcal{F}$ is either:

i.) A (lift of a) closed properly immersed surface

ii.) Dense in $Gr_2(M)$

- $\pi_1(M)$ has lots of surface subgroups $\pi_1(\Sigma)$, Σ a closed surface (Kahn-Markovic)
- Minimal surfaces locally minimize area, have mean curvature $\equiv 0$.

- For each $\pi_1(\Sigma)$, there is an immersed minimal surface Σ_{min} whose fundamental group injectively includes to (the conjugacy class of) $\pi_1(\Sigma)$.
- Limit set of each $\pi_1(\Sigma)$ is a K-quasiconformal deformation of the equator in $S^2 \cong \partial_\infty(\mathbb{H}^3)$, or K-quasicircle

Def • $S_\epsilon(M) := \{$homotopy classes of π_1-injective immersed surfaces in M with limit sets $(1+\epsilon)$-quasicircles$\}$

• $E(g) :=$

$$\lim_{\epsilon\to 0} \liminf_{L\to\infty} \frac{\log \#\{\mathrm{Area}_g(\Pi) \le 4\pi(L-1) : \Pi \in S_\epsilon(M)\}}{L\log(L)}$$

Thm (Calegari-Marques-Neves) Sectional curvature of g is less than or equal to -1. Then $E(g) \ge E(g_{hyp})$, & equality implies that $g \overset{\text{isom}}{\cong} g_{hyp}$.

<u>Question</u> For negatively curved g, are the immersed minimal surfaces Σ_{min} that include to the $\pi_1(\Sigma) \subset \pi_1(M)$ dense?

<u>Partial answer:</u>

i.) Yes for g suff. close to g_{hyp}

ii.) Possibly not for some neg. curved g.

Main Thm

Let $\{g_t : t \in [0,1]\}$ be a smooth 1-param. family of negatively-curved metrics on M with $g_0 = g_{hyp}$.

Then there exist foliations $\mathcal{F}_t$ of $Gr_2(M)$ for $t \in [0,T) \cap [0,1]$ s.t.:

i.) $\mathcal{F}_0 = \mathcal{F}$

ii.) Leaves of $\mathcal{F}_t$ are (lifts of) immersed minimal planes in (M, g_t)

iii.) $\exists$ homeomorphisms $\Phi_t : Gr_2((M, g_{hyp})) \to Gr_2((M, g_t))$ sending leaves of $\mathcal{F}_0$ to leaves of $\mathcal{F}_t$.

- If $T \leq 1$, then $\exists\, t_n \nearrow T$ and immersed minimal planes S_n in (M, g_{t_n}) that are (projections to M of) leaves of $\mathcal{F}_{t_n}$ s.t. $(*)$ tends to 0 from below as $n \to \infty$:

$$(*) \quad \underbrace{|A_{\Sigma_{t_n}}(x_n)|^2}_{\text{"leaf curvature"}} + \underbrace{Ric_n(v_n, v_n)}_{\text{"ambient curvature"}},$$

for a sequence of points $x_n \in S_n$, where:

a.) v_n is the unit normal vector to S_n at x_n

b.) $A_{\Sigma_{t_n}}$ is the 2^{nd} F.F. of Σ_{t_n}.

c.) Ric_n is the Ricci curvature tensor of g_{t_n}.

Rmks

1. $T \leq 1$ occurs when "leaf curvatures" fail to be bounded by "ambient curvatures".

2. I construct examples of neg.-curved (M, g) which cannot admit foliations as in the Thm.

3. Gromov constructed foliations as above with almost-totally-geodesic leaves (but in all dimensions)

4. Versions of this thm. likely hold in greater generality, e.g. totally geodesic foliations of $Gr_k(M^n)$, M^n a hyp. n-manifold.

Idea of proof

1. $|A|^2 + Ric(\nu,\nu) < 0$ at a point on a minimal $\Sigma \implies$ small tubular nbd. of Σ has a mean-convex foliation

$\implies$ there can't be another min. surface at small Hausdorff distance from Σ.

2. Construct leaves by solving asymptotic plateau problems in universal cover
   - Use 1. to show solutions to asy. Plateau are unique $\to$ rules out gaps in the foliation.

- Inspired by work of Uhlenbeck on minimal surfaces in quasi-Fuchsian manifolds (hyp. structures on $\Sigma \times \mathbb{R}$ with quasicircle limit sets.)

(Will describe more in extra time.)

Applications

- Let $\{\pi_1(\Sigma_n)\}$ a sequence of surface subgroups of $\pi_1(M)$ whose limit sets tend to a circle $C \subset \partial_\infty \mathbb{H}^3$. Assume $C = \partial_\infty P$ and the projection of P to M is dense.

- Let g be a metric on M to which Main Thm. applies to construct a foliation. $\{\Sigma_n\}$ the corresponding g-minimal immersed surfaces.

<u>Thm</u> (density) If $U \subset Gr_2(M)$ open, then $\exists N$ s.t. $n \geq N \Longrightarrow \overline{\Sigma}_n \cap U \neq \phi$

<u>Thm</u> (Quantitative version)

Assume (M, g_{hyp}) has no properly immersed totally geodesic surfaces.

Then weak-$*$ limits of the Σ_n assign positive measure to every open set in $Gr_2(M)$

<u>Thm</u> (totally geodesic) Assume (M, g_{hyp}) has infinitely many (closed) properly immerse totally geodesic surfaces Σ_n.

Let Σ_n' be the corresponding minimal surfaces in (M, g). Then weak-$*$ limits of Σ_n' have full support in $Gr_2(M)$.

Proof (totally geodesic)

- The Σ_n equidistribute in $Gr_2(M, g_{hyp})$ (Mozes-Shah)

- Use conjugating map Φ from main thm transfer this to (M, g) (following Calegari-Marques-Neves)

Questions about foliations $\mathcal{F}_g$ from Main Thm

i.) Possible weak-$*$ limits of the ξ_n'?

ii.) Regularity of conjugating maps in directions transverse to leaves?

iii.) • Foliated geodesic flow on unit tangent bundle of $\mathcal{F}_g$, Horocycle foliation.

Model: $\Gamma \backslash PSL_2(\mathbb{C})$

- Lots of measures natural to foliated geodesic flow. Harmonic measures. Measure classification in this setting? Statements about the measures that distinguish the constant curvature case?

Extra Time

Uhlenbeck Theory of Almost-Fuchsian Manifolds (also Huang, Wang, ...)

- Hyperbolic metric g_{hyp} on $\Sigma_g \times \mathbb{R}$.
- Assume $\exists$ minimal embedded $\Sigma \subset (\Sigma_g \times \mathbb{R}, g_{hyp})$ with principal curvatures less than 1 in abs. value.
- Then $(\Sigma_g \times \mathbb{R}, g_{hyp})$ is quasi-Fuchsian, and Σ is the unique minimal surface.

- $(\Sigma_g \times \mathbb{R}, g_{hyp})$ is determined by the conformal class of Σ and a quadratic differential that encodes the 2^{nd} FF of Σ.

	Almost Fuchsian	Variable Negative Curvature
Model	$\cosh^2 t\, g_{hyp} + dt^2$	Totally geodesic foliation $\mathcal{F}$ of $Gr_2((M, g_{hyp}))$
"convex" minimal surfaces	Unique minimal surface Σ, in the middle	Unique minimal plane in universal cover at finite distance from initial totally geodesic plane
Mechanism for uniqueness	Explicitly write down the metric, global mean-convex foliation	Foliation structure, local mean-convex foliations.

Examples of metrics for which $Gr_2(M)$ cannot admit foliations as in Main Thm.