"Global rigidity theorems for actions of higher rank lattices"

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§ Motivation/Settings.

Γ: discrete group., M: closed manifold

(Smooth Riemannian connected cpt without ∂.)

$\Gamma \xrightarrow{\alpha} \mathrm{Diff}(M)$ smooth action

Can we classify?..

Under

- Geometric assumptions:
  Geometry (or topological) properties of M, or invariant measure, ... etc
- Dynamical assumptions:
  (Non-) Uniform hyperbolicity, ...
- Algebraic assumptions:
  property (T), higher rank lattice, $\mathbb{Z}^k$, $k \geq 2$ ...

→ In this talk,

Γ is an irreducible lattice in a higher rank semi-simple real alg. Lie gp without cpt factor G

<u>Note</u> $SL_2\mathbb{Z} < SL_2\mathbb{R}$, $SL_2\mathbb{Z} \approx \mathbb{F}_2$. $\rightsquigarrow$ "flexible".

<u>Example</u>

G	$SL_2\mathbb{R}$	$SL_3\mathbb{R}$	$SL_2\mathbb{R} \times SL_2\mathbb{R}$	
Γ	$SL_2\mathbb{Z}$	$SL_3\mathbb{Z}$	$SL_2\mathbb{Z} \times SL_2\mathbb{Z}$	$SL_2\mathbb{Z}[\sqrt{7}]$.
rank	1	2	2	
irred.	Y	Y	N	Y.

or $\Gamma <_L G_1 \times \cdots \times G_n$, $n \geq 2$. $G_i \cong SL_n\mathbb{R}$, proj. densely to G_i. $\forall i$.

Rmk Heuristic motivation:

"Margulis' superrigidity thm".

$\Gamma < G$ irred. higher rank.

$\Rightarrow$ we can classify all finite dimensional representation.

How about infinite dimensional case, such as $\mathrm{Diff}(M)$ ...

Defn.

1) Uniform hyperbolicity.

· $f \in \mathrm{Diff}^1(M)$: Anosov diffeo. if

$\exists$ Df-inv splitting $TM = E^s \oplus E^u$, $\exists$ $C > 0$, $\lambda \in (0,1)$. s.t.

$\forall n \geq 0$

$$\|Df^n v\| < C\lambda^n \|v\| \quad \forall v \in E^s$$

$$\|Df^{-n} w\| < C\lambda^{-n} \|w\| \quad \forall w \in E^u$$

2) Suspension space : $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ p.m.p. (ergodic)

$\rightsquigarrow$ $G \curvearrowright (G \times X) \curvearrowleft \Gamma$

$$g.(h,x).\gamma = (gh\gamma, \alpha(\gamma^{-1})(x))$$

$\rightsquigarrow$ $G \curvearrowright \left((G \times X)/\Gamma, \ \mu_{Haar} \otimes \mu \right)$ p.m.p. (ergodic).

μ is called induced-irreducible if

$G \curvearrowright \left((G \times X)/\Gamma, \ \mu_{Haar} \otimes \mu \right)$ is irreducible

i.e. $G_{\hat{i}} \curvearrowright \left((G \times X)/\Gamma, \ \mu_{Haar} \otimes \mu \right)$ is ergodic.

- Why do we need?

3) Conjugation / Affine action

§ Main theorems / History

1) $\Gamma \xrightarrow{\alpha} \mathrm{Diff}(M)$ is called **Anosov action** if $\exists \gamma_0 \in \Gamma$ s.t. $\alpha(\gamma_0)$ is an Anosov diffeo.

Katok–Lewis–Zimmer : Anosov vol. pres ergodic C^∞ $SL_n\mathbb{Z} \curvearrowright \mathbb{T}^n$ $n \geq 3$
$\to$ C^∞-conjugate to Affine

Margulis–Qian :
Assume that all simple factor of G has higher rank. $\Gamma <_L G$. $M = N/\Lambda$ Nilmfd.
$\alpha : \Gamma \to \mathrm{Diff}^1(M)$ Anosov action.
& $\exists\, \alpha(\Gamma)$-invariant fully supp. Borel prob msr
$\Rightarrow$ α is C^0-conj. to affine action

Brown–Rodriguez Hertz–Wang :
Assume that all simple factor of G has higher rank.
$\Gamma <_L G$ lattice, $M = N/\Lambda$: nilmanifold.
$\alpha : \Gamma \longrightarrow \mathrm{Diff}^1(N/\Lambda)$ Anosov action
Assume that α lifts to universal cover N.
$\Rightarrow$ α is topologically conjugate to an affine action
If α is C^∞ then conjugacy is C^∞.

$\underline{Q}$. $SL_2\mathbb{Z}[\sqrt{17}]$?

$\underline{\text{Example}}$

G	$SL_2\mathbb{R}$	$SL_3\mathbb{R}$	$SL_2\mathbb{R}\times SL_2\mathbb{R}$	
Γ	$SL_2\mathbb{Z}$	$SL_3\mathbb{Z}$	$SL_2\mathbb{Z}\times SL_2\mathbb{Z}$	$SL_2\mathbb{Z}[\sqrt{17}]$.
rank	1	2	2	
irred.	Y	Y	N	Y.
(T)	N	Y	N	N.

Nevertheless !?

$\underline{\text{Thm A.}}$ (L.) $\Gamma <_L G$ irred. higher rank lattice.
$\alpha : \Gamma \longrightarrow \mathrm{Diff}^1(N/\Lambda)$ Anosov action.
Assume that $\exists$ fully support $\alpha(\Gamma)$-induced irred. Borel prob msr μ.
$\Rightarrow$ α is top'l conj. to an affine action.
If α is C^∞ then conjugacy is C^∞.

2) On the other hand, it needs Anosov action.
How about "weaker" unif. hyp. ?

- $f \in \mathrm{Diff}^1(M)$ admits a dominated splitting

if $\exists$ Df-inv. splitting $TM = E \oplus F$

$\exists\, C > 0,\ \lambda \in (0,1)$ s.t. $\forall n \geq 0$

$$\frac{\| Df^n(v) \|}{\| Df^n(w) \|} < C\lambda^n \qquad \forall v \in E,\ w \in F \quad \|v\| = \|w\| = 1.$$

* Anosov diffeo, partial hyp. diffeo. admits a dominated splitting.

Thm B. (L.) $\Gamma < SL_n\mathbb{R}$ lattice $n \geq 3$.

M^n: n-dim mfd.

$\alpha : \Gamma \longrightarrow \mathrm{Diff}^1_{vol}(M^n)$ vol-preserving C^1-action

Assume $\exists\, \gamma_0 \in \Gamma$ s.t. $\alpha(\gamma_0)$ admits a dominated splitting.

$\Rightarrow$ M^n is homeomorphic to torus and

α is top'l conj. to an affine action on torus.

If α is C^∞, vol-pres. then conjugacy is C^∞.

Rmk Similar thms hold for actions on M^{2n} by $\Gamma < Sp(2n, \mathbb{R})$ and $\Gamma < SO(n,n)$
$n \geq 2$ $\qquad$ $n \geq 5$.

* History / Motivation for Thm B

① <u>Conjecture</u> Γ : irred. higher rank lattice

$\alpha : \Gamma \curvearrowright M$ C^∞ action. Assume that

$\exists\, \gamma_0 \in \Gamma$ s.t. $\alpha(\gamma_0)$ is a partial hyp. diffeo.

$\Rightarrow$ α is C^∞ conj. to an "alg." action.

② <u>Thm</u> (Brown, Fisher, Hurtado)

$\Gamma \curvearrowright M^{n-1}$ vol preserving C^∞ action $\Rightarrow$ finite.

previously,

- Goetze - Spatzier proved C^∞-Cartan Γ-action. is C^∞-conj. to an algebraic action.

- Feres proved similar theorem under connection preserving.

⋮

§ Main ingredients.

I Cocycle :

$\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ Leb-space , H: gp.

$\beta : \Gamma \times X \longrightarrow H$ is called <u>cocycle</u>

if $\forall \gamma_1, \gamma_2 \in \Gamma$, a.e. $x \in X$,

$$\beta(\gamma_1\gamma_2, x) = \beta(\gamma_1, \alpha(\gamma_2)(x))\, \beta(\gamma_2, x)$$

- $\beta_1, \beta_2 : \Gamma \times X \to H$, $\beta_1 \sim \beta_2 \iff$
  $\exists\, \phi : X \to H$ msble $\beta_1(\gamma, x) = \phi(\gamma . x)^{-1} \beta_2(\gamma, x)\, \phi(x)$.

Cohomologous

<u>example</u>

1) Derivative Cocycle

$\Gamma \xrightarrow{\alpha} \mathrm{Diff}^1(M)$, $\dim M = d$ $TM \cong M \times \mathbb{R}^d$

msble trivialization

$\Rightarrow \mathcal{D} : \Gamma \times M \to GL_d\mathbb{R}$,

$\mathcal{D}(\gamma, x) := D_x\, \alpha(\gamma)$. is a cocycle.
(Chain rule!!)

2) Actions on torus : $\Gamma \overset{\alpha}{\curvearrowright} \mathbb{T}^n$.

Assume α lifts to $\Gamma \overset{\tilde{\alpha}}{\curvearrowright} \mathbb{R}^n$. Let

$\tilde{\alpha}(\gamma)(x+\lambda) = \tilde{\alpha}(\gamma)(x) + \rho(\gamma)(\lambda)$ $\quad \rho : \Gamma \to Aut(\mathbb{Z}^n)$

$\Rightarrow \quad \beta : \Gamma \times \mathbb{T}^n \longrightarrow SL_n\mathbb{R} \ltimes \mathbb{R}^n$

$\beta(\gamma, [x]) := \left(\rho(\gamma), \rho(\gamma)^{-1}(\tilde{\alpha}(\gamma)(x)) - x \right)$.

is a conti. cocycle and

" $\alpha(\gamma)[x] = \beta(\gamma, [x])[1, x]$ "

$$\begin{array}{ccc} \mathbb{T}^n & & SL_n\mathbb{Z} \ltimes \mathbb{R}^n / SL_n\mathbb{Z} \ltimes \mathbb{Z}^n \\ \Psi & & \Psi \\ [x] & \longleftrightarrow & [1, x] \end{array}$$

Ⅱ cocycle superrigidity

→ (Real) Motivation; Zimmer's cocycle superrigidity thm

$\boxed{\begin{array}{l}\text{Thm (ZCSR; Fisher–Margulis)} \\ \Gamma < G \text{ irred. lattice}, \quad \Gamma \overset{\alpha}{\curvearrowright} (X,\mu) \text{ ergodic pmp} \\ \text{all simple factors have higher rank}, \quad \left(\begin{array}{l} G = \mathbb{G}(\mathbb{R}), \\ \mathbb{G}\text{: alg. simply conn} \end{array} \right) \\ \beta : \Gamma \times X \to GL_d\mathbb{R} \text{ cocycle, } L^1\text{-integrable} \\ \quad \text{i.e. } \forall \gamma, \quad \ln\|\beta(\gamma,-)\| \in L^1(X,\mu). \\ \Rightarrow \exists \text{ msble } \phi : X \to GL_d\mathbb{R}, \ \exists \pi : G \to GL_d\mathbb{R} \text{ hom.} \\ \quad \exists K : \Gamma \times X \to O(d) \text{ cocycle s.t.} \\ \qquad 1) \ \beta(\gamma,x) = \phi(\alpha(\gamma)(x))^{-1}\pi(\gamma)K(\gamma,x)\phi(x) \quad \text{a.e.} \\ \qquad 2) \ K(\Gamma \times X) \text{ commutes w/ } \pi(G). \end{array}}$

⇝ Lyapunov exp. comes from π.

Rmk How can we have "cpt error" K here?

Unlike Margulis' superrigidity, we need ...

Prop T : (T) gp. A: amenable

$T \curvearrowright (X,\mu)$ ergodic p.m.p

$$\underbrace{\beta : T \times X \to A \text{ cocycle} \Rightarrow \beta \sim K : T \times X \to \overset{\text{cpt}}{\overset{\downarrow}{K}} < A}_{\text{"cocycle version" of } \rho: T \to A \rightsquigarrow \rho(T) \text{ cpt.}}$$

For $SL_2\mathbb{Z}[1/7]$: A priori, no reason to have "bdd error"

even we may not have error as a cocycle *.

G: alg. simply conn. $G = \mathbb{G}(\mathbb{R})$

Thm C. (L.) Dynamical Superrigidity.

Γ: irred. higher rank lattice in G, $\Gamma \curvearrowright (X, \mu)$ induced-irred.

$\beta : \Gamma \times X \to GL_d\mathbb{R}$ cocycle, L^2-integrable

i.e. $\forall \gamma \in \Gamma$, $\ln \|\beta(\gamma, -)\| \in L^2(X, \mu)$.

$\Rightarrow$ $\exists$ msble $\phi : X \to GL_d\mathbb{R}$, $\exists$ $\mathcal{E} : \Gamma \times X \to A$: cocycle

$\exists$ $\pi : G \to GL_d\mathbb{R}$ hom. (A: Amenable)

s.t.

- $\beta(\gamma, x) = \phi(\gamma.x)^{-1}\pi(\gamma)\mathcal{E}(\gamma, x)\phi(x)$
- $\pi(G)$ commutes A.
- Lyapunov spectrum of β for $\forall \gamma \in \Gamma$ = Lyapunov spectrum of π

up to finite extension

§ Ideas of Proofs.

can read exp. growth from repn

Superrigidity $\longleftrightarrow$ Measurable dyn. data

hyperbolicity $\longleftrightarrow$ conti. dyn data $\longrightarrow$ Thm A & B

different exp growth makes conti. splitting.

Thm C, ZCSR.