Additive and geometric transversality of fractal sets in the reals and integers

Midwest Dynamics Seminar

Florian K. Richter

Northwestern University

19 October, 2020

Motivation

Goal of today's talk

- Investigate the relative independence between fractal sets that are structured with respect to multiplicatively independent bases.
- We explore this topic in two different regimes: the unit interval $[0,1]$ and the non-negative integers $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$.

Motivating Special Case:

Definition

A base-p restricted digit Cantor set in the non-negative integers $\mathbb{N}_0$ is a set of the form $C_{p,\mathcal{D}} = \left\{\sum_{k=0}^{n} w_k p^k : k \in \mathbb{N}_0,\ w_0, w_1, \ldots, w_k \in \mathcal{D}\right\}$, where $\mathcal{D}$ is a non-empty subset of $\{0,1,\ldots,p-1\}$.

Eg.: The set $C_{3,\{0,2\}}$ is the set of all integers whose ternary expansion only uses the digits 0 and 2.

Motivating question

Suppose $A, B \subset \mathbb{N}_0$ are a base-p and base-q missing digit Cantor sets respectively, where $p, q \in \mathbb{N}$ are multiplicatively independent (i.e. $\log(p)/\log(q) \notin \mathbb{Q}$). Can we show that the sets A and B lie in general position ...

- ... additive combinatorially?
- ... geometrically?

Discrete dimension

What is the right tool to measure the size of missing digit Cantor sets?
Note that missing digit Cantor sets are sparse sets, i.e., they have zero density.

Definition

The upper and lower dimension of a subset of the non-negative integers $A \subset \mathbb{N}_0$ are defined as

$$\overline{\dim}\, A = \limsup_{N\to\infty} \frac{\log\left|A \cap [0, N)\right|}{\log N} \qquad \text{and} \qquad \underline{\dim}\, A = \liminf_{N\to\infty} \frac{\log\left|A \cap [0, N)\right|}{\log N}.$$

If $\overline{\dim}\, A = \underline{\dim}\, A$ then we say that the dimension of A exists and denote this quantity by $\dim A$.

Examples:

- $\dim \mathbb{N}_0 = 1$;
- $\dim(\{p : p \text{ prime}\}) = 1$;
- $\dim(\{n^2 : n \in \mathbb{N}\}) = 1/2$;
- $\dim C_{3,\{0,2\}} = \log(2)/\log(3)$;
- $\dim C_{p,\mathcal{D}} = \log(|\mathcal{D}|)/\log(p)$;

In other words, missing digit Cantor sets have a well-defined fractal dimension. Perhaps we can also use the notion of dimension to measure how $C_{p,\mathcal{D}}$ and $C_{q,\mathcal{E}}$ are positioned with respect to one another.

Additive independence

Let $A, B \subset \mathbb{N}_0$. Borrowing ideas from additive combinatorics, we can use the size of the sumset $A + B$ as a rudimentary measure of the additive structure shared between A and B.

- If A, B are finite then

$$|A| + |B| - 1 \leqslant |A + B| \leqslant |A|\,|B|. \tag{1}$$

  - Equality holds on the left if and only if A and B are arithmetic progressions of the same step size. When $|A + B|$ is near this lower bound, inverse theorems in combinatorial number theory provide additive structural information on the sets A and B.
  - Equality holds on the right if there are no coincidences among sums $a + b$ with $a \in A$ and $b \in B$, a measurement of additive independence.

  Note that $|A + B| \leqslant |A|\,|B|$ implies $\log(|A + B|) \leqslant \log(|A|) + \log(|B|)$.

- If A, B are infinite then we have

$$\dim(A + B) \leqslant \min\big(1, \dim A + \dim B\big). \tag{2}$$

  - If equality holds here then that means there are almost no coincidences among sums $a + b$ with $a \in A$ and $b \in B$.

Heuristic from additive combinatorics

If two fractal sets $A, B \subset \mathbb{N}$ satisfy $\dim(A + B) = \min\big(1, \dim A + \dim B\big)$ then we can think of them as being additively independent.

Geometric transversality

In euclidean geometry, two subspaces V and W of $\mathbb{R}^n$ are said to be transverse if

$$\dim(V+W) = \min\big(n, \dim V + \dim W\big).$$

If we think of fractal subsets of $\mathbb{N}_0$ as "subspaces" then:

Heuristic from geometry

If two fractal sets $A, B \subset \mathbb{N}$ satisfy $\dim(A+B) = \min\big(1, \dim A + \dim B\big)$ then we can think of them as being transverse.

Main theme

Suppose $A, B \subset \mathbb{N}_0$ are a base-p and base-q missing digit Cantor sets respectively, where $p, q \in \mathbb{N}$ are multiplicatively independent (i.e. $\log(p)/\log(q) \notin \mathbb{Q}$). Can we show that the sets A and B lie in general position ...

$$\left.\begin{array}{l} \bullet\ \text{... additive combinatorially?} \\ \bullet\ \text{... geometrically?} \end{array}\right\} \implies \dim(A+B) = \min(1, \dim A + \dim B)$$

Special Case of our main result (Glasscock-Moreira-R., 2020)

Let p and q be multiplicatively independent. If $A, B \subset \mathbb{N}_0$ are a base-p and base-q missing digit Cantor sets respectively, then $\dim(A+B) = \min\big(1, \dim A + \dim B\big)$.

History and background

Leitmotiv in Number Theory

If p and q are multiplicatively independent then "base-p structure and base-q-structure are orthogonal".

"base-p structure" could mean

- base digit restrictions (missing digit Cantor sets, SFTs)
- invariance under certain transformations ($T_p : x \mapsto px \bmod 1$)
- limits/attractors of "p-systems" (iterated function systems)

Conjecture (Alaoglu-Erdős 1944, dates back to Ramanujan)

There is no non-integer x such that 2^x and 3^x are both rational.

Conjecture (Mendés-France 1980, attributed to Mahler)

If $(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ is not eventually periodic then at least one of $\sum_{n=1}^{\infty} a_n 2^{-n}$ or $\sum_{n=1}^{\infty} a_n 3^{-n}$ is transcendental.

Theorem (Cassels 1959, answering a question by Steinhaus)

Suppose p is multiplicatively independent from 3. Then almost every point in Cantor's middle thirds set is normal in base-p.

Conjecture (Erdős 1979)

For all but finitely many $n \in \mathbb{N}$ the ternary expansion of 2^n contains at least one 2.

Furstenberg's results and conjectures I

In the language of fractal geometry and dynamical systems, Furstenberg established a number of conjectures and results that explore the relationship between multiplicative structures with respect to different bases.

Definition

A set $X \subseteq [0, 1]$ is called $\times p$-invariant if it is closed and invariant under the map $T_p \colon x \mapsto px \bmod 1$.

Examples:

- Cantor's middle thirds set is $\times 3$-invariant.
- The prime gap set (in the $[0, 1]$-interval) consists of all numbers $x \in [0, 1]$ whose binary expansion has a prime-number of 0s between any two 1s; It is $\times 2$-invariant.

Theorem (Furstenberg, 1967)

If $X \subseteq [0, 1]$ is simultaneously $\times 2$- and $\times 3$-invariant then either $|X| < \infty$ or $X = [0, 1]$.

The measure-theoretic analogue of the above theorem is a famous open conjecture in ergodic theory.

Furstenberg's $\times 2$ $\times 3$ Conjecture

If μ is an ergodic Borel probability measure on $[0, 1]$ that is simultaneously $\times 2$- and $\times 3$-invariant then μ is either atomic or the Lebesgue measure.

Furstenberg's results and conjectures II

Theorem (Hochman-Shmerkin 2012, originally conjectured by Furstenberg)

Let $X \subseteq [0,1]$ be a closed and $\times 2$-invariant set, and let $Y \subseteq [0,1]$ be a closed and $\times 3$-invariant set. Then

$$\dim_{\mathrm{H}}(X+Y) = \min\big(1, \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y\big).$$

Theorem (Shmerkin 2019, Wu 2019, originally conjectured by Furstenberg)

Let $X \subseteq [0,1]$ be a closed and $\times 2$-invariant set, and let $Y \subseteq [0,1]$ be a closed and $\times 3$-invariant set. Then

$$\dim_{M}(X \cap Y) \leqslant \max\big(0, \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y - 1\big).$$

Conjecture (Furstenberg)

Let $x \in \mathbb{R}\backslash\mathbb{Q}$ and define $X = \overline{\{T_2^n x : n \in \mathbb{N}\}}$ and $Y = \overline{\{T_3^n x : n \in \mathbb{N}\}}$. Then $\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y \geqslant 1$.

Think of it this way: If the binary expansion of an irrational number is "simple" then its ternary expansion must be "complex".

Can we make sense of Furstenberg's results and conjectures in the integers

For $p \in \mathbb{N}$ define $\Phi_p \colon \mathbb{N}_0 \to \mathbb{N}_0$ and $\Psi_p \colon \mathbb{N}_0 \to \mathbb{N}_0$ as

$$\Phi_p \colon n \mapsto \lfloor n/p \rfloor \qquad \text{and} \qquad \Psi_p \colon n \mapsto n - p^{k_n} \lfloor n/p^{k_n} \rfloor,$$

where $k_n = \lfloor \log n / \log p \rfloor$.

Remark

- Φ_p is the map that "forgets" the last digit in p-adic representation:

$$\Phi_p(a_0 + a_1 p + \ldots + a_k p^k) = a_1 + a_2 p + \ldots + a_k p^{k-1}.$$

- Ψ_p is the map that "forgets" the leading digit in p-adic representation:

$$\Psi_p(a_0 + a_1 p + \ldots + a_k p^k) = a_0 + a_1 p + \ldots + a_{k-1} p^{k-1}.$$

Definition

A set $A \subseteq \mathbb{N}_0$ is called $\times p$-invariant if $\Phi_p A \subset A$ and $\Psi_p A \subset A$.

Eg.: Any base-p missing digit Cantor set in $\mathbb{N}_0$ is $\times p$-invariant.

For any closed and shift-invariant set $X \subseteq \{0, 1, \ldots, p-1\}^{\mathbb{N}_0}$ the corresponding language set is defined as

$$\mathcal{L}(X) = \{(w_0, w_1, \ldots, w_{k-1}) : \ (w_0, w_1, \ldots) \in X, \ k \in \mathbb{N}_0\}.$$

Any such language set naturally embeds into the integers via

$$A_X = \{w_0 + w_1 p + \ldots + w_{k-1} p^{k-1} : (w_0, w_1, \ldots, w_{k-1}) \in \mathcal{L}(X)\}.$$

Eg.: The prime gap set (in the integers) consists of all numbers $n \in \mathbb{N}_0$ whose binary expansion has a prime-number of 0s between any two 1s. For instance, this set contains $18 = (10010)_2$ but not $5 = (101)_2$.

Observation

Any set constructed this way is $\times p$-invariant.

Proposition

A set $A \subseteq \mathbb{N}_0$ is $\times p$-invariant if and only if $A = A_X$ for some subshift X of $\{0, 1, \ldots, p-1\}^{\mathbb{N}_0}$. Moreover, the discrete dimension $\dim A$ exists and equals $h_{top}(X)/\log(p)$.

Main results

Let p and q be multiplicatively independent.

Theorem (Furstenberg, 1967)

If $X \subseteq [0,1]$ is simultaneously $\times p$- and $\times q$-invariant then either $|X| < \infty$ or $X = [0,1]$.

We have the following discrete analogue of the above theorem.

Theorem (Glasscock-Moreira-R., 2020)

If $A \subseteq \mathbb{N}_0$ is simultaneously $\times p$- and $\times q$-invariant then either $|A| < \infty$ or $A = \mathbb{N}_0$.

Theorem (Hochman-Shmerkin 2012)

Let $X \subseteq [0,1]$ be a $\times p$-invariant set, and let $Y \subseteq [0,1]$ be a $\times q$-invariant set. Then

$$\dim_{\mathrm{H}}(X+Y) = \min\big(1, \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y\big).$$

Theorem (Glasscock-Moreira-R., 2020)

Let $A \subseteq \mathbb{N}_0$ be $\times p$-invariant, and let $Y \subseteq \mathbb{N}_0$ be a $\times q$-invariant. Then

$$\dim(A+B) = \min\big(1, \dim A + \dim B\big).$$

Thank you

Ideas behind the proof

Main idea:

- To move from the discrete to the continuous regime, consider the sets

$$A_N = \frac{A \cap \{0, 1, \dots, N-1\}}{N} \qquad \text{and} \qquad B_N = \frac{B \cap \{0, 1, \dots, N-1\}}{N}.$$

  Let $X = \lim_{k\to\infty} A_{p^k}$ and $Y = \lim_{k\to\infty} A_{q^k}$, where the limits are taken with respect to the Hausdorff topology on $[0, 1]$. For this to work, we need a result which is proved using topological dynamics (see Theorem 1 below).

- There exists a finite interval $I \subset [0, \infty)$ such that for all $N \in \mathbb{N}$ there exists $\sigma \in I$ and $k, j \in \mathbb{N}$ such that

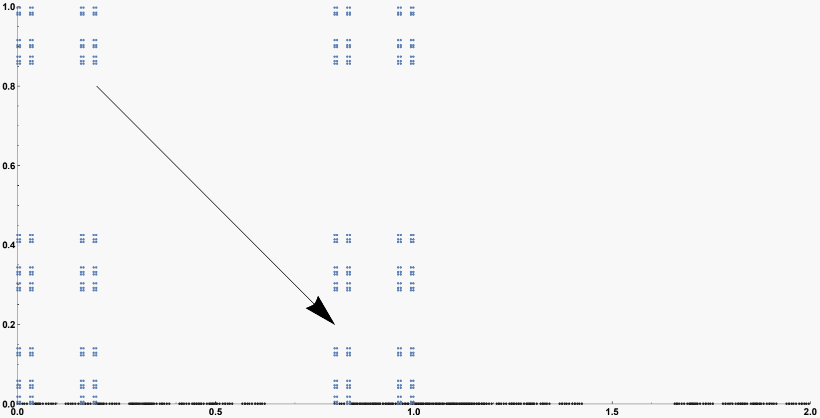
$$A_N + B_N \;\approx\; A_{p^k} + \sigma B_{q^j} \;\approx\; X + \sigma Y.$$

  ($k = \lfloor \log(N)/\log(p) \rfloor$, $j = \lfloor \log(N)/\log(q) \rfloor$, and $\sigma = q^{\{k\theta\}}$, where $\theta = \log(p)/\log(q)$.)

- Prove an extension of the Hochman-Shmerkin theorem that exhibits "uniformity in σ at every finite scale."

Ideas behind the proof

– The sumset $A_{p^k} + \sigma B_{q^j} \approx X + \sigma Y$ is a projection of $A_{p^k} \times B_{q^j} \approx X \times Y$ to $\mathbb{R}$



Ideas behind the proof

Theorem

Let $A \subset \mathbb{N}_0$ be $\times p$-invariant (i.e. $\Phi_p(A) \subset A$ and $\Psi_p(A) \subset A$). Then there exists $A' \subset A$ such that $\dim A' = \dim A$ and $\Phi_p(A') = A'$ and $\Psi_p(A') = A'$.

Follows from the fact that for every subshift $X \subset \{0, 1, \ldots, p-1\}^{\mathbb{N}}$ contains a subshift $X' \subset X$ with the property that $h_{top}(X) = h_{top}(X')$ and every point in X' is recurrent.
The discrete Hausdorff content, defined for $X \subseteq \mathbb{R}^d$ to be

$$\mathcal{H}^{\gamma}_{\geqslant\rho}(X) = \inf\left\{\sum_{i\in I} \delta_i^{\gamma} : X \subseteq \bigcup_{i\in I} B_i,\ B_i \text{ open ball of diameter } \delta_i \geqslant \rho\right\}.$$

For the current discussion, it is helpful to know that for compact sets X,

$$\dim_H X = \sup\left\{\gamma \geqslant 0 : \lim_{\rho\to 0^+} \mathcal{H}^{\gamma}_{\geqslant\rho}(X) > 0\right\}; \tag{3}$$

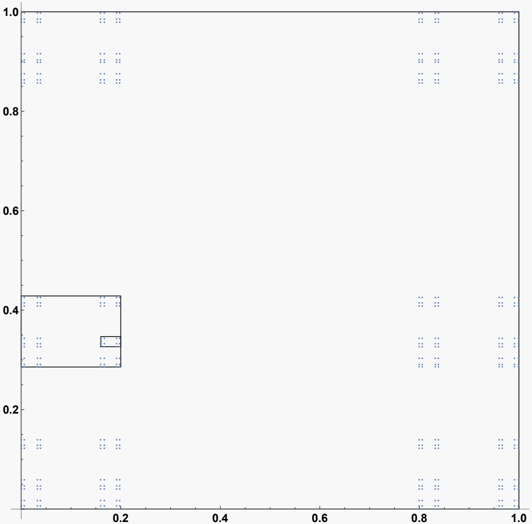
Theorem (G.-Moreira-Richter 2020)

Let p and q be multiplicatively independent and $X, Y \subseteq [0,1]$ be $\times p$- and $\times q$-invariant. Put $\gamma = \min(\dim X + \dim Y, 1)$. For all compact $I \subseteq \mathbb{R}\backslash\{0\}$ and all $\varepsilon > 0$,

$$\lim_{\rho\to 0^+} \inf_{\sigma\in I} \mathcal{H}^{\gamma-\varepsilon}_{\geqslant\rho}(X + \sigma Y) > 0,$$

where $\mathcal{H}^{\gamma-\varepsilon}_{\geqslant\rho}$ is the "discrete Hausdorff content" at scale ρ.

Self-similarity



Rotation of directions