Local randomness and spectral independence

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Setting: G: metrizable compact group;

d: bi-invariant metric on G; $\forall g \in G, \rho > 0$, g_ρ: ball of radius ρ centered at g.

$\hat{G} := \{\pi : G \to \mathcal{U}(\mathcal{H}) \mid \pi : \text{irreducible unitary}\}/\sim$

Def. We say G is L-locally random if $\exists C_0 > 0$

$$\forall \pi \in \hat{G}, \; g \in G, \quad \|\pi(g) - I\|_{op} \leq C_0 (\dim \pi)^L d(g, 1).$$

(poly-dimensional Lipschitz)

Basic Properties Suppose G is locally random; then

(1) G has FAb; that means $H \leq_o G \Rightarrow |H/\overline{[H,H]}| < \infty$.

(2) $\forall n \in \mathbb{Z}^+$, $\mathrm{Irr}_n(G) := |\{\pi \in \hat{G} \mid \dim \pi \leq n\}| < \infty$.

(3) $\forall N \trianglelefteq_c G$, G/N is locally random.

(4) If G' is locally random, then $G \times G'$ is locally random.

1-Locally random with respect to some metric.

Important examples.

(1) Compact semisimple real Lie groups. ← highest weight theory + Weyl's formula

(2) Compact p-adic analytic group with perfect Lie $\mathbb{Q}_p$-alg.

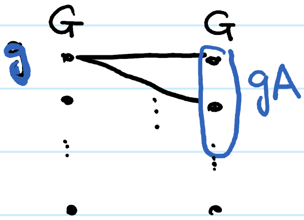
Howe's Kirillov theory

Connection with Quasi-randomness.

<u>Gowers</u>: A finite group G is D-quasi-random if

$$\forall \pi \in \hat{G} \setminus \{1\}, \quad \dim \pi \geq D$$

Using high multiplicity of non-trivial eigenvalues of convolution operators, he proved that, if $|A| \geq |G|/D^{1/3}$, then the bipartite graph statiscally behaves like a quasi-random graph. ⟶ a type of mixing inequality.



(This is similar to a trick used by Sarnak–Xue.)

Based on this, he proved $|A||B||C| \geq |G|^3/D \Rightarrow G = A \cdot B \cdot C$.

. Gowers's quasi-randomness is very effective when G does not many normal subgroups; when it lives in a single scale.

<u>Varjú</u>: A profinite gp G is (c,α)-quasi-random if $\forall \pi \in \hat{G}$,

$$\dim \pi \geq c[G : \ker \pi]^{\alpha}.$$

Connection with quasi-randomness. (Continue.)

As you can see, this definition is indep. of a choice of metric. Having the p-adic analytic groups in mind, borrowing a terminology from number theory, we define the level $\ell(\pi)$ of a rep'n of a profinite gp as follows: $\ell(\pi) := \inf \{ \eta^{-1} \mid 1_\eta \subseteq \ker(\pi) \}$.

$\boxed{\pi \in \hat{G} \Rightarrow \dim \pi \gg \# \pi(G)^{\alpha}}$

If $1_\eta \triangleleft G$ for every $\eta > 0$, then

$$\# \pi(G) \leq |1_{\ell(\pi)+\varepsilon}|^{-1}$$

for every $\varepsilon > 0$.

Dimension Condition $\quad C_0^{-1} \eta^{d_0} \leq |1_\eta| \leq C_0 \eta^{d_0}$

$\boxed{DC(d_0, C_0)}$

Def. G: profinite is called (C,A)-metric quasi-random if (1) $1_\eta \triangleleft G \quad (\forall \eta > 0)$

(2) $\ell(\pi) \leq C (\dim \pi)^A \quad (\forall \pi \in \hat{G})$

Prop. G: profinite, $1_\eta \triangleleft G \ (\forall \eta)$, satisfies DC.

L-locally random $\Leftrightarrow$ metric quasi-random.

Mixing Inequality:

$\eta > 0$, $P_\eta := \dfrac{\mathbb{1}_{B(1,\eta)}}{|B(1,\eta)|}$.

G: L-locally random; $f, F \in L^2(G)$;

$$\|f * F\|_2^2 \le 2\, \|f_\eta * F_\eta\|_2^2 + \eta^{1/(2L)} \|f\|_2^2\, \|F\|_2^2,$$

where $f_\eta := f * P_\eta$ and $C_0 \sqrt{\eta} \le 0.1$.

In the setting of quasi-random finite groups, this type of inequality has been proved by Gowers, and Babai-Nikolov-Pyber. For simple Lie groups (not necessarily compact), a similar inequality was proved by Boutonnet-Ioana-Salehi Golsefidy, Benoist-de Saxcé)

Idea of proof. Parseval's formula (Fourier analysis of compact groups.)

low frequency

$$L(f;D) := \sum_{\substack{\pi \in \hat{G} \\ \dim \pi \le D}} \dim \pi \, \|\hat{f}(\pi)\|_{HS}^2$$

high freq.

$$H(f;D) := \sum_{\substack{\pi \in \hat{G} \\ \dim \pi > D}} \dim \pi \, \|\hat{f}(\pi)\|_{HS}^2$$

$$\boxed{\begin{aligned} \Rightarrow\ & L(f*g;D) \le L(f;D)\, L(g;D) \\ & H(f*g;D) \le \tfrac{1}{D}\, H(f;D)\, H(g;D) \end{aligned}}$$

This is enough for finite quasi-random groups

$$\boxed{L(f;D) \le (1 - C_0 D^{L} \eta)^{-1}\, L(f_\eta;D)}$$

Local randomness is used

A product theorem (for sets with large metric entropy)

For $A \subseteq G$, $\mathcal{N}_\eta(A) :=$ the least number of open balls of radius η that is needed to cover A.

We call $h(A;\eta) := \log \mathcal{N}_\eta(A)$ the metric entropy of A at scale η. It is worth pointing out that under

$\underline{DC(d_0; C_0)}$
$$C_0^{-1}\eta^{d_0} \le |1_\eta| \le C_0\,\eta^{d_0},$$
$$C_0^{-2}\frac{|X_\eta|}{|1_\eta|} \le \mathcal{N}_\eta(X) \le C_0^{2}\frac{|X_\eta|}{|1_\eta|}.$$

Thm. G: L-locally random with coeff. C satisfies $DC(d_0, C_0)$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$\frac{h(A;\eta) + h(B;\eta)}{2} \ge (1-\delta)\, h(G;\eta)$$
$$\Downarrow$$
$$A_\eta B_\eta B_\eta^{-1} A_\eta^{-1} \supseteq 1_{\eta^\varepsilon}$$

$(\eta^\varepsilon \ll_{L, C, C_0, d_0} 1.)$

Random walk. Next we want to study random walks in a locally random group G. Let's recall that for a Borel prob. measure μ, we consider $\{X_i\}_{i=1}^{\infty}$ i.i.d. with prob. law μ. Then $Y_\ell := X_\ell \cdot \cdots \cdot X_2 \cdot X_1$ is distributed with prob. law $\mu^{(\ell)}$, the ℓ-fold convolution of μ. Assuming μ is symmetric, the convolution operator induces

$$T_\mu : L^2(G)^\circ \longrightarrow L^2(G)^\circ, \quad T_\mu(f) = \mu * f.$$

(recall that $(\mu * f)(x) = \int_G f(x'^{-1}x)\, d\mu(x')$.)

Let $\lambda(\mu) := \left\| T_\mu \big|_{L^2(G)^\circ} \right\|_{op}$, and

$$\mathcal{L}(\mu) := -\log \lambda(\mu).$$

Inspired by Lyapunov exponent

We say $\mu \curvearrowright G$ has spectral gap property if

$$\lambda(\mu) < 1 \quad (\text{alternatively } \mathcal{L}(\mu) > 0)$$

Notice that, if μ is symmetric, then

$$\mathcal{L}(\mu^{(\ell)}) = \ell\, \mathcal{L}(\mu)$$

for every positive integer ℓ.

Littlewood-Paley decomposition for locally random groups.

Classical. Decomposing functions in terms of functions that "live at various scales". This has be done by considering function whose Fourier inverse is supported in only a given frequency interval.

Profinite gps and random walk. Suppose $\{N_i$ is a sequence of open normal subgroups of G such that $\bigcap N_i = 1$. Let

$$\mathcal{H}_0(G) := L^2(G)^{N_1}, \qquad \mathcal{H}_i := L^2(G)^{N_{i+1}} \ominus L^2(G)^{N_i}.$$

$$\Rightarrow \mathcal{E}(G) = \bigoplus_{i=0}^{\infty} \mathcal{H}_i \quad \text{(algebra of matrix coeff.)}$$

is dense in $L^2(G)$.

Having $N_i = 1_{a^i}$ for some $a>1$, we can interpret this decomposition as a Littlewood-Paley decomposition. Notice that for a prob. measure μ on G,

$$\lambda(\mu) = \sup\left\{ \frac{\|\mu * f\|}{\|f\|} \;\middle|\; f \in \mathcal{H}_i \setminus \{0\} \right\}.$$

So we can deal with functions that live at arbitrarily small scales to understand whether or not $\mu \curvearrowright G$ has spectral gap property.

Littlewood-Paley decomposition for locally random groups.

Let $\eta_i := \eta_0^{a^i}$ where η_0 is a fixed small positive number and a is a fixed large number ($\geq 4Ld_0$). Then $\Delta_i(g) := (P_{\eta_{i+1}} - P_{\eta_i}) * g$ provides us with a Littlewood-Paley decomposition.

Def. We say a fuction $f \in L^2(G)$ lives at scale η (with a parameter a) if

(1) $\|f_{\eta^{1/a}}\|_2 \leq \eta^{1/(2a)} \|f\|_2$ (average under larger ball almost zero)

(2) $\|f_{\eta^{a^2}} - f\|_2 \leq \eta^{a/2} \|f\|_2$ (almost invariant under smaller ball)

(SU(d): Bourgain-Gamburd. Simple Lie gps: Boutonnet-Ioana-Salehi Golsefidy.)

Connection with classical point of view:

For $I \subseteq \mathbb{R}^+$, let $\mathcal{H}_I := \bigoplus_{\pi \in \widehat{G}} \mathcal{H}_\pi \subseteq L^2(G)$.

$\mathcal{H}_\pi \to$ matrix coeff. of π

think about as frequencies $\to$ $\dim \pi \in I$

Let $pr_I : L^2(G) \longrightarrow \mathcal{H}_I$ be the orthogonal projection.

Functions that live at small scales and random walk.

- f lives at scale $\eta \Longrightarrow \|\mathrm{pr}\ (f)\|_2^2 \geq (1-8\eta^{1/(2a)})\|f$

where $I_\eta = [O(\eta^{O(1)}), O(\eta^{O(1)})]$. $\left(= \left[\frac{1}{2C_0}\eta^{-1/La}, 2C_0\eta^{-d_0 a^2}\right]\right)$

this is the concept

this one is precise.

- $\forall f \in \mathcal{H}_{I'_\eta}$, lives at scale η where

$I'_\eta = [O(\eta^{O(1)}), O(\eta^{O(1)})]$. $\left(I'_\eta = \left[C_1\eta^{-\frac{d+1}{a}}\ \ C_0^{-1/L}\ \eta^{\frac{-2a^2+a}{2L}}\right]\right)$

Rényi Entropy at scale η. Suppose X is a random variable with prob. law μ. We define its Rényi entropy at scale η as follows $H_2(X;\eta) := \log(1/|I_\eta|) - \log\|\mu_\eta\|_2^2$.

(we also write $H_2(\mu;\eta)$.)

(Recall that Rényi entropy for a random variable with finite support is $-\log\left(\sum_x \mathbb{P}(X=x)^2\right)$.)

Thm. • $\exists\ \mathcal{H}_0 \subseteq L^2(G)$ s.t. $\dim \mathcal{H}_0 \ll 1$

• If $\forall i$, $\exists\ l_i \leq C_2\, h(G;\eta_i)$ s.t.

$$H_2(\mu^{(l_i)};\eta_i) \geq \left(1-\frac{1}{20Ld_0a^3}\right) h(G;\eta_i),$$

then

$$\mathcal{L}(\mu; L^2(G)\ominus\mathcal{H}_0) \geq \frac{1}{40C_2Ld_0a^3}.$$

$\eta_i := \eta_0^{a^i}$

$a \geq 4Ld_0$

$\exists\ 0<\eta_0 \ll 1$

Functions living at small scales and random walk.

(Contraction of functions that live at scale η)

$\forall \eta \le \eta_0$, $\forall g$: lives at scale η, $\exists\, \ell \le C \log(1/\eta)$ s.t.

$$\|\mu^{(\ell)} * g\|_2 \le \eta^c \|g\|_2$$

$$\Downarrow$$

$$\mathcal{L}(\mu) > 0$$

$\left(\exists\, \mathcal{H}_0 ,\ \dim \mathcal{H}_0 \ll 1, \quad \mathcal{L}(\mu; L^2(G) \ominus \mathcal{H}_0) \ge \frac{c}{C} . \right)$

(Not having spectral gap can be detected at arbitrarily small scales.)

(We might come back to this result after we talk about spectrally independent groups.)

Spectrally Independent Groups.

We say two compact groups G_1 and G_2 are spectrally independent if the following holds:

$$\left.\begin{array}{l}\mu:\text{ Borel prob. measure on } G_1\times G_2\\ pr_i(\mu)\curvearrowright G_i \text{ has spectral gap}\end{array}\right\}\Rightarrow \begin{array}{l}\mu\curvearrowright G_1\times G_2 \text{ has}\\ \text{spectral gap.}\end{array}$$

Basic properties.

(1) $\left.\begin{array}{l}G_1 \text{ and } G_2 \text{ are S.I.}\\ N_i \trianglelefteq_c G_i\end{array}\right\}\Rightarrow G_1/N_1$ and G_2/N_2 are S.I.

(2) We say two compact groups are algebraically independent if they do not have non-trivial isomorphic quotients; i.e.

$$G_1/N_1 \simeq G_2/N_2 \Rightarrow G_i = N_i .$$

- G_1 and G_2 are S.I. $\Rightarrow$ G_1 and G_2 are A.I.
- For finite groups, A.I. $\Rightarrow$ S.I.

(3) Two infinite topologically f.g. compact abelian groups are never S.I.

Theorem. $F_i := \mathbb{Q}_{v_i}$ for $v_i \in V(\mathbb{Q})$
$\mathbb{G}_i / F_i$: abs. almost simple
$G_i \subseteq \mathbb{G}_i(F_i)$ compact open
$\mathfrak{g}_1(F_1) \not\simeq \mathfrak{g}_2(F_2)$

$\left.\right\} \overset{?}{\Longrightarrow}$ G_1 and G_2 are S.I.

($\mathbb{Q}_v$ is either $\mathbb{R}$ or $\mathbb{Q}_p$ for some prime p.)

Here is an immediate corollary:

Cor. G_1, G_2: as above;
$\Omega \subseteq G_1 \times G_2$: symmetric; top. gen. set
$\forall g \in \Omega$, $\mathrm{Ad}(g)$: alg. entries

$\left.\right\} \overset{?}{\Longrightarrow}$ $P_\Omega \curvearrowright G_1 \times G_2$ has spectral gap.

(Benoist-de Saxcé, SG.)

As we have pointed out earlier, above mentioned G_i's are locally random; and so $G_1 \times G_2$ is locally random. Hence we can use the mentioned criterion and work with functions that live at a small scale η.

Coupling and Spectral Independence.

m_{G_i}'s : best possible spectral gap.

G_1, G_2 : S.I. $\Leftrightarrow$ any coupling of m_{G_1} and m_{G_2} has spectral gap.

A Borel prob. measure μ on $G_1 \times G_2$ is called a coupling of m_{G_1} and m_{G_2} if $pr_i[\mu] = m_{G_i}$.

Theorem. G_1, G_2 : as above; μ : a coupling of m_{G_1} and m_{G_2}; Then $\exists\, m_0 := m_0(G_1, G_2) \in \mathbb{Z}^+$, $\forall \eta_0$, $\exists\, \eta < \eta_0$ s.t. $\forall f \in L^2(G_1 \times G_2)$ which lives at scale η we have

$$\| \mu_\eta^{(2^{m_0})} * f \|_2 \leq \eta^{O_{G_1,G_2}(1)} \|f\|_2 .$$

$\left(\| f * P_{\eta C_1^2} - f \|_2 \leq \eta^{C_1/2},\ \| f * P_{\eta^{1/C_1}} \|_2 \leq \eta^{1/2 C_1},\ C_1 = O_{G_1,G_2}(1). \right)$

Pf of this theorem startes with a (multi-scale version of a) result of Bourgain and Gamburd which roughly says, if the L^2-norm of convolution $\mu * \nu$ does not drop significantly, then there is an algebraic reason.

Multi-scale Bourgain-Gamburd's L^2-flattening obstruction.

(Weighted version of Tao's non-commutative prod. theorem.)

(Bourgain-Gamburd, Benoist-de Saxcé, de Saxcé-Lindenstrauss, Boutonnet-Ioana-SG)

G: compact group; d: bi-invariant metric; dim. condition

$$H_2(XY;\eta) \le \frac{H_2(X;\eta)+H_2(Y;\eta)}{2} + \log K$$

for some $K \gg 1$, then $\exists\, H \subseteq G$ s.t.

(1) (Approximate structure)

H : $O(K^{O(1)})$-approximate subgroup;

that means $\exists\, X \subseteq G$ symm. s.t. $H\cdot H \subseteq X\cdot H \cap H\cdot X$ and $\#X \ll K^{O(1)}$.

(2) (Metric entropy)

$$|h(H;\eta) - H_2(X;\eta)| \ll \log K$$

(3) (Almost equidistribution)

$$\mathbb{P}(X_\eta \in (xH)_\eta) \ge K^{-O(1)}$$

$$\text{and} \quad \mathbb{P}(Y_\eta \in (Hy)_\eta) \ge K^{-O(1)}$$

for some x, y. Moreover

$$|\{h \in H_\eta \mid \mathbb{P}(X \in (xh)_{3\eta}) \ge \hat{C} K^{-10} 2^{-H_2(X;\eta)}\}| \ge K^{-O(1)} |H_\eta|.$$

Outline of pf of spec. gap for couplings.

Step 1. f: lives at scale η $\Rightarrow$ $\|\mu_\eta^{(2^m)} * f\|_2 \le \eta^{\frac{1}{4L}} \|\mu_\eta^{(2^m)}\|_2$

Mixing inequality

So it is enough to show $\|\mu_\eta^{(2^{m_0})}\|_2 \le \eta^{-\frac{1}{8L}}$ (*)

Step 2. After $O(1)$-times convolution either we get (*)

or an approxi. subgp. (By the mentioned multi-scale BG)

Step 3. Suppose $H_\eta \subseteq G_1 \times G_2$ is the mentioned

approximate subgp, for $K = \eta^{-\delta}$ for some small $\delta(\varepsilon)$.

Since μ is a coupling, $pr_i(H_\eta)$ have large metric

entropy. Hence by the product theorem we get

$$pr_i(\Pi_8 H_\eta) \supseteq 1^{(i)}_{\eta^\varepsilon}.$$

(we will assume $pr_i(H_\eta) \supseteq 1^{(i)}_{\eta^\varepsilon}$.)

Step 4. (Getting local almost homomorphism)

$\alpha_1 := \inf \{ r \in [0,1] \mid (\Pi_8 H_\eta) \cap (G_1 \times \{1\}) \subseteq 1^{(1)}_r \times \{1\} \}$.

Similarly define α_2. Based on the upper bound on $|H_\eta|$

and a suitable open function theorem, $\min \alpha_i \ll \eta^{O_{G_i}(1)}$

$\forall g \in pr_1(H_\eta)$, let $f(g) \in G_2$ be s.t. $(g, f(g)) \in H_\eta$.

Then $f: 1^{(1)}_{\eta^\varepsilon} \longrightarrow G_2$ is a local $\eta^{O_{G_1}(1)}$-almost homomorphism

(1) $d(f(g^{-1}), f(g)) \leq \eta^{O(1)}$

(2) $d(f(g_1 g_2), f(g_1) f(g_2)) \leq \eta^{O(1)}$.

<u>Step 5</u>. Getting a Lie alg. homomorphism.

(<u>Kazhdan</u>: almost hom close to hom; but here f is local.)

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To prove the mentioned spectral independence, we need to find couplings that are "close" to $\mu^{(\ell)}$.

Notice that since $pr_i[\mu] \curvearrowright G$ have spec. gap property, for large enough ℓ, $pr_i[\mu^{(\ell)}]$ can be viewed as perturbations of the Haar measures m_{G_i} (at least after looking at its η-smoothing.):

A Coupling of m_{G_i}'s close to perturbations of m_{G_i}'s.

G_1, G_2 : L - locally random ;

Dimension cond. : $C_0^{-1} \rho^{d_0^{(i)}} \leq |1_\rho| \leq C_0 \rho^{d_0^{(i)}}$

Discretizable : $\forall N \gg 1$, $\exists \{X_i\}_{i=1}^{N}$ s.t.

(1) $|X_i| = 1/N$, (2) $\operatorname{diam}(X_i) \ll N^{-1/d_0^{(i)}}$, $X_i \supseteq$ a ball of radius $O(N^{-1/d_0^{(i)}})$

(Compact Lie groups are discretizable
Gigante – Leopardi.)

μ : symmetric Borel prob. measure on $G_1 \times G_2$

$\max\{\lambda(\mathrm{pr}_1[\mu]; G_1), \lambda(\mathrm{pr}_2[\mu]; G_2)\} = \lambda < 1$.

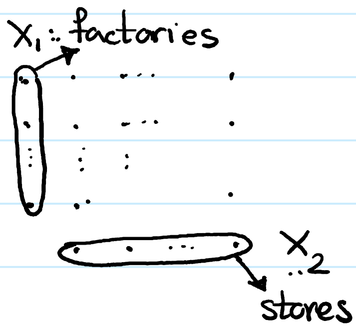
Then, $\forall \eta \in (0,1)$, $\exists$ a symm. coupling υ^{η} of m_{G_i}'s:

$\forall f \in L^2(G_1 \times G_2)$, $\|f * P_\eta - f\|_2 \leq \eta^{C_1} \|f\|_2$ implies

$$\|\upsilon^{\eta} * f - \mu^{(\ell)} * f\|_2 \leq \eta^{C_2} \|f\|_2$$

for $\ell = C_3 \cdot \log_\lambda \eta$, $C_2^{-1}, C_3 \ll_{d_0^{(i)}, C_1, L} 1$, and $\eta \ll \rho$.

Step 1 (Transportation problem)



ν: prob. measure on $X_1 \times X_2$

$$\left| pr_i[\nu](x) - \frac{1}{|X_i|} \right| \le \frac{1}{|X_1 \times X_2|^c} \Rightarrow$$

$\exists$ a coupling μ of $\mathcal{P}_{X_1}$ and $\mathcal{P}_{X_2}$ s.t.

$$|\mu(x_1, x_2) - \nu(x_1, x_2)| \le \frac{1}{|X_1 \times X_2|^{c-1}}.$$

Step 2. Discretize; use step 1 and almost invariance of f at scale η to get the desired result.

Reduction of S.I. to a statement about couplings.

G_1, G_2: L-locally random; Dim. Condition; discretizable;
$\forall$ symm. coupling ν of m_{G_i}'s,
f lives at scale $\eta \Rightarrow \|\nu * f\|_2 \le \eta^{O(1)} \|f\|_2$

$\Rightarrow$ G_1 and G_2 are S.I.

Thank you!