

Commensurators & Arithmeticity of Thin Groups

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Setting • G semisimple Lie gp [finite center,
no capt factors]

- $\Gamma < G$ discrete subgp

Ex

- $G = SL(n, \mathbb{R})$
- $\Gamma = SL(n, \mathbb{Z})$

Def Commensurator of Γ in G :

$$\text{Comm}_G(\Gamma) := \left\{ g \in G \mid g\Gamma g^{-1} \cap \Gamma \begin{matrix} \subseteq \Gamma \\ \subseteq g\Gamma g^{-1} \end{matrix} \text{ finite index} \right\}$$

Commensurators of Lattices

Thm (Borel, '61).

$$\Gamma = G(\mathcal{O}_k) \implies G(k) \subset \Delta$$

arithmetic lattice rational pts.

Pf for $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$

$SL(n, \mathbb{Q})$ commensurates $\Gamma = SL(n, \mathbb{Z})$

Con: Γ arithmetic \Rightarrow dense comm.

Thm (Margulis '74).

$\Gamma \subseteq G$ irred. lattice } $\Rightarrow \Gamma$ is arithmetic
 $\Lambda \subseteq G$ dense

Con: Γ nonarithmetic $\Rightarrow \Delta = \Gamma^2$
(up to
finite index)

Q (Shalom, '11) Margulis' Thm for Zariski-dense gps?
 $\left. \begin{array}{l} \Gamma \subset G \text{ inned., discrete Zariski-dense} \\ \text{dense comm. } \Delta \subset G \end{array} \right\} \Rightarrow \Gamma \text{ is arithmetic lattice}$

History

- [Greenberg, '74] Yes for $G = \mathrm{SL}(2, \mathbb{R})$, Γ fin. gen
 - [Mj '11, uses ^{Leininger-}Long-Rid '11] Yes for $G = \mathrm{SL}(2, \mathbb{C})$, Γ fin. gen
 - [Koburda-Mj '19] Yes if G real, rk 1
 - Open for all other G
 - Open for Γ inf. gen.
- $\Gamma \trianglelefteq \Delta$ normal in arithmetic lattice
 + assumptions on Δ/Γ
 e.g. i abelian.

Thm (Fisher-Mj-rL, '20): $\Gamma \subset G$ discrete, \mathbb{Z} -dense dense comm

(1) G has (T) $\Rightarrow \Gamma$ is arithmetic

(2) G has rk 1:

(i) $\Gamma \subset \Delta$ for some arithmetic Δ

(ii) Δ commensurates $\Gamma \Rightarrow \Gamma$ is arithmetic

Applications

1) Commensurated subgps of lattices

Q: $\Lambda \subset G$ irred. lattice.

Which $\Gamma \subset \Lambda$ are commensurated by Λ ?

Rmks

1) Classifies $\Lambda \rightarrow M$ where M loc. comp, tot disc

\Rightarrow Classifies $\Lambda \curvearrowright X$

2) $\text{rk } G \geq 2 \xRightarrow{\text{Margulis}} \Lambda$ is S -arithmetic

Conj (Margulis - Zimmer, ~ '78).

$\left. \begin{array}{l} \text{rk } G \geq 2 \\ \Gamma \subset \Lambda \text{ commensurated} \end{array} \right\} \Rightarrow \Gamma \text{ is } S' \text{-arithmetic}$
for some $S' \subset S$

Ex Commensurated subgps of $SL(n, \mathbb{Z}[1/p])$:

M-Z conj \Rightarrow finite or finite index or $SL(n, \mathbb{Z})$.

History: Proven for

1) [Venkatararamana '87] G simple / \mathbb{Q}

2) [Shalika - Willis '13] G split / \mathbb{Q}

Thm (Fischer-Mj-vL, '20) Write $G = G_{\infty} \times G_{\text{fin}}$

$M-Z$ conj is true if

• $S \neq \emptyset$ (i.e. $G_{\text{fin}} \neq 1$).

and • G_{∞} has Property (T) or $\text{rk } 1$

Thm (Fischer '20)

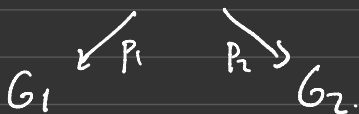
$M-Z$ conj is true if • $S = \emptyset$

• \exists factor w/ (T).

Combine: $M-Z$ conj is true if G_{∞} is simple

2) Irreducible discrete subgps:

Def $\Theta \subset G_1 \times G_2$ irreducible if $\pi_i(\Theta) \subset G_i$
dense



Q: \exists discrete irred. free gps? e.g. in $SL(n, \mathbb{R}) \times SL(n, \mathbb{Q}_p)$?
(Benoist) surface gps?

Note: Suppose G_2 p -adic
 \cup
 K_2 max'lly cmt

Conclusion:

$\Theta \subset G_1 \times G_2$ discrete irred. \rightsquigarrow $\Gamma \subset G_1$ discrete w/ dense comm.

Cor (Fisher-Mj-vL '20)

Let: $\Theta \subset G_1 \times G_2$ discrete irred.

Suppose: G_2 p -adic and G_1 has (T)

\Rightarrow Θ is an S -arithmetic lattice

Cor No irred. free/surface gps

Proofs: $\Gamma \subset G$ discrete, Zariski-dense
 $\Delta \subset G$ dense commensurator of Γ

(1) $B \subset G$ Borel

Then $\Gamma \curvearrowright G/B \times G/B$ is ergodic

Idea: Consider $X // \Gamma$ (space of ergodic comp's)

\leadsto factor $X \longrightarrow \hat{X}$
 $\sigma \qquad \qquad \sigma$
 $\wedge \qquad \qquad \wedge$

Two miraculous properties of \hat{X} : 1) G -factor
2) Γ -invt prob. measure.

② Pf if G has (T) :

Note: $\Gamma \curvearrowright G/MA$ ergodic $\implies MA \curvearrowright G/\Gamma$ ergodic

Thm (Mautner '47) Either:

(1) $A \curvearrowright G/\Gamma$ is tot. dissipative

OR (2) $\Gamma < G$ is weakly co-amenable