Commensurators & Arithmeticity of Thin Groups

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Setting • G semisimple Lie gp [finite center, no cmpt factors]

• $\Gamma \subset G$ discrete subgp

Ex • $G = SL(n, \mathbb{R})$

• $\Gamma = SL(n, \mathbb{Z})$.

Def Commensurator of Γ in G:

$$\mathrm{Comm}_G(\Gamma) := \left\{ g \in G \;\middle|\; g\Gamma g^{-1} \cap \Gamma \begin{array}{l} \leq \Gamma \\ \leq g\Gamma g^{-1} \end{array} \text{finite index} \right\}$$

Commensurators of Lattices

Thm (Borel, '61).

$$\underset{\text{arithmetic lattice}}{\Gamma = G(\mathcal{O}_k)} \implies \underset{\text{rational pts.}}{G(k) \subset \Delta}$$

Pf for $G = SL(n,\mathbb{R})$ and $\Gamma = SL(n,\mathbb{Z})$

$SL(n,\mathbb{Q})$ commensurates $\Gamma = SL(n,\mathbb{Z})$

Cor: Γ arithmetic $\Rightarrow$ dense comm.

Thm (Margulis '74).

$$\left.\begin{array}{l}\Gamma \subseteq G \text{ irred. lattice} \\ \Delta \subseteq G \text{ dense}\end{array}\right\} \Rightarrow \Gamma \text{ is arithmetic}$$

Cor: Γ nonarithmetic $\Longrightarrow$ $\underline{\Delta} = \underline{\Gamma}$

(up to finite index)

Q (Shalom, '11) Margulis' Thm for Zariski-dense gps?

$\Gamma \subset G$ irred., discrete Zariski-dense, dense comm. $\Delta \subset G$ $\Big\} \Longrightarrow$ Γ is arithmetic lattice

History

- [Greenberg, '74] Yes for $G = SL(2,\mathbb{R})$, Γ fin. gen
- [Mj '11, uses Leininger-Long-Reid '11] Yes for $G = SL(2,\mathbb{C})$, Γ fin. gen
- [Koberda-Mj '19] Yes if: G real, rk 1, $\Gamma \trianglelefteq \Lambda$ normal in arithm lattice + assumptions on Λ/Γ e.g.: abelian.
- Open for all other G
- Open for Γ inf. gen.

Thm (Fisher-Mj-vL, '20): $\Gamma \subset G$ discrete, Z-dense, dense comm

(1) G has (T) $\Longrightarrow$ Γ is arithmetic

(2) G has rk 1:

(i) $\Gamma \subset \Lambda$ for some arithmetic Λ

(ii) Λ commensurates Γ $\Rightarrow$ Γ is arithmetic

Applications

1) Commensurated subgps of lattices

Q: $\Lambda < G$ irred. lattice.
Which $\Gamma < \Lambda$ are commensurated by Λ?

Rmks

1) Classifies $\Lambda \to H$ where H loc cmpt, tot disc

$\Rightarrow$ Classifies $\Lambda \curvearrowright X$

2). $\mathrm{rk}\, G \geq 2 \overset{\boxed{\text{Margulis}}}{\Longrightarrow} \Lambda$ is S-arithmetic

Conj (Margulis – Zimmer, ~ '78).

$$\left.\begin{array}{l} \mathrm{rk}\, G \geq 2 \\ \Gamma < \Lambda \text{ commensurated} \end{array}\right\} \Rightarrow \begin{array}{l} \Gamma \text{ is } S'\text{-arithmetic} \\ \text{for some } S' \subset S \end{array}$$

Ex Commensurated subgps of $SL(n, \mathbb{Z}[1/p])$:

M-Z conj $\Rightarrow$ finite or finite index or $SL(n, \mathbb{Z})$.

History: Proven for

1). [Venkataramana '87] G simple / $\mathbb{Q}$

2) [Shalom–Willis '13] G split / $\mathbb{Q}$

Thm (Fisher-Mj-vL, '20) Write $G = G_\infty \times G_{fin}$

M-Z conj is true if

- $S \neq \emptyset$ (i.e. $G_{fin} \neq 1$).

and • G_∞ has Property (T) or rk 1

Thm (Fisher '20)

M-Z conj is true if
- $S = \emptyset$
- $\exists$ factor w/ (T).

Combine: M-Z conj is true if G_∞ is simple

2) Irreducible discrete subgps:

Def $\Theta \subset G_1 \times G_2$ *irreducible* if $p_i(\Theta) \subset G_i$ dense

$$\begin{tikzcd} & G_1 \times G_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G_1 & & G_2. \end{tikzcd}$$

Q: $\exists$ irred. discrete free gps? e.g. in $SL(n,\mathbb{R}) \times SL(n, \mathbb{Q}_p)$?
(Benoist) surface gps?

Note: Suppose G_2 p-adic
$\cup$
K_2 max'l cmpt

Conclusion:

$\Theta \subset G_1 \times G_2$ discrete irred. $\rightsquigarrow$ $\Gamma \subset G_1$ discrete w/ dense comm.

Cor (Fisher–Mj–vL '20).

Let: $\Theta \subset G_1 \times G_2$ discrete irred.
Suppose: G_2 p-adic and G_1 has (T)
$\Rightarrow$ Θ is an S-arithmetic lattice

Cor No irred. free/surface gps

<u>Proofs</u>: $\Gamma \subset G$ discrete, Zariski-dense
$\Delta \subset G$ dense commensurator of Γ

(1) $B \subset G$ Borel

Then $\Gamma \curvearrowright G/B \times G/B$ is ergodic

<u>Idea</u>: Consider $X /\!\!/ \Gamma$ (space of ergodic comp'ts)

$\rightsquigarrow$ factor
$$\begin{array}{ccc} X & \longrightarrow & \hat{X} \\ \circlearrowleft & & \circlearrowleft \\ \Delta & & \Delta \end{array}$$

<u>Two miraculous properties of</u> $\hat{X}$:
1) G-factor
2) Γ-invt prob. measure.

(2) Pf if G has (T):

Note: $\Gamma \curvearrowright G/MA$ ergodic $\Longrightarrow$ $MA \curvearrowright G/\Gamma$ ergodic

Thm (Margulis '97) Either:

(1) $A \curvearrowright G/\Gamma$ is tot. dissipative

OR (2) $\Gamma < G$ is weakly co-amenable