Exponential mixing of 3D Anosov flows

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Definition

A flow $g : M \to M$ *on a compact Riemannian manifold* M *is a* **Anosov flow** *if there is a continuous splitting*

$$TM = E^s \oplus N \oplus E^u$$

where N *is the flow direction,* E^s *is uniformly contracted by* Dg^1 *and* E^u *is uniformly expanded by* Dg^1*.*

It is known that for any Anosov flow, E^s and E^u integrate to stable foliation W^s and unstable foliaton W^u respectively (but they don't have to be jointly integrable).

The topological and statistical properties of Anosov flow were studied by many authors: Anosov, Bowen, Margulis, Plante, Ratner, Ruelle, Sinai, Smale, etc.
For transitive Anosov flows, the following theorem is well-known.

Theorem (Anosov alternative)

A transitve Anosov flow is either topologically mixing or it is conjugate to a suspension flow over an Anosov diffeomorphism with constant roof function.

We say that g is topologically mixing if for any non-empty open sets $A, B \subset M$, for all sufficiently large $t > 0$, we have $g^t(A) \cap B \neq \emptyset$.

It is also important to study measure-theoretical mixing. For a Anosov flow g, for any Hölder function F, there is a unique measure $\nu_{g,F}$ which maximizes

$$\int F d\mu + h_\mu(g^1).$$

We call $\nu_{g,F}$ an **equilibrium measure** for g with potential F.

1. when $F = 0$, $\nu_{g,F}$ is the entropy maximizing measure (or Bowen-Margulis measure);
2. when $F = -\log|\det(Dg^1|_{E^u})|$, $\nu_{g,F}$ is called the Sinai-Ruelle-Bowen measure

$$\mu = \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} (g^i)_* Leb.$$

In particular, when g is volume preserving, then $\nu_{g,F}$ is the volume.

Theorem (Bowen-Ruelle, 1975)

If a C^2 Anosov flow g is topologically mixing, then g is mixing with respect to equilibrium measures with Hölder potential.

People are interested in the speed of the convergence when A, B are Hölder functions.

Conjecture (Bowen-Ruelle)

If g is topologically mixing, then g is exponentially mixing with respect to Hölder functions and equilibrium measures with Hölder potential.

- The theorem is proved for a more general class of flow, called "Axiom A flow" or "hyperbolic flow". The conjecture was also originally about hyperbolic flow, but counter-examples are found by Ruelle.

Theorem (Tsujii-Z)

A topologically mixing C^∞ 3D Anosov flow is exponentially mixing with respect to any equilibrium measure with Hölder potential.

Progress on Bowen-Ruelle conjecture:

1. (Chernov) Stretched-exponential decay of correlaction;
2. (Dolgopyat) it is true when E^s and E^u are of class C^1 for SRB measure (and all equilibrium measure when $\dim E^u = 1$, or more generally equilibirum measure with doubling property);
3. (Liverani) when g preserves a contact form, exp. mixing w.r.t. volume;
4. (Tsujii) w.r.t. volume in $3D$, when g preserves a volume form, and verifies certain C^3 open and C^∞ dense condition;
5. (Butterley-War) when E^s is $C^{1+\epsilon}$ for SRB measure.

- Other related works: Giulietti-Liverani-Pollicott (zeta function, decay under some pinching condition), Field-Melbourne-Török (super-polynomial rate), Pollicott-Sharp(prime orbit theorem with exponential error terms).
- Usually, E^u and E^s do not need to be smooth. They are always Hölder. But in many cases they are strictly Hölder and exponent can be arbitrarily small (Plante, Hasselblatt, Wilkinson).

Markov partition: $\Pi = \cup_{\alpha \in I} \Pi_\alpha$ where Π_α is a parallelogram defined as follows: there is a local unstable manifold U_α, and a local stable manifold S_α such that $\Pi_\alpha = [U_\alpha, S_\alpha]$. Denote $U = \cup_\alpha U_\alpha$.

Return time: $\tau : \Pi \to \mathbb{R}_+$ (invariance of the stable foliation gives $\tau : U \to \mathbb{R}_+$).

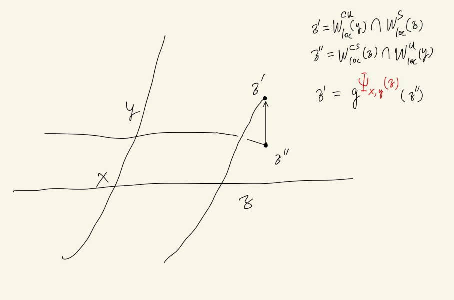
Return map: $\hat{\sigma} : \Pi \to \Pi$. Map on the first coordinate is $\sigma : U \to U$.

Markov property: for each $\alpha \in I$, $\hat{\sigma}(S_\alpha)$ is contained in Π_{β_1} for some $\beta_1 \in I$; and $\hat{\sigma}^{-1}(U_\alpha)$ is contained in Π_{β_2} for some $\beta_2 \in I$.

Measure: let ν_Π be the measure induced by $\nu_{g,F}$, and let ν_U be the projection of ν_Π to U.

Foliation: since $\dim M = 3$, both W^{cs} and W^{cu} are C^{1+}-foliations (Hasselblatt).

Temporal function: for $x \in M$, $z \in W^u_{loc}(x)$ and $y \in W^s_{loc}(x)$, we will study $\Psi_{x,y}(z)$ defined as follows:



The return time function is related to temporal function. For any $k > 0$, any $w \in \sigma^{-k}$ and any $x, z \in Dom(w)$, there is $x^w \in W^s_{loc}(x)$ such that

$$\tau_k \circ w(x) - \tau_k \circ w(z) = \Psi_{x,x^w}(z)$$

Complex RPF operator

Given a Hölder function $f : U \to \mathbb{R}$, we denote a family of bounded linear operators on $C^\theta(U)$ as follows.
For any $a, b \in \mathbb{R}$, for any $\alpha \in I$, for any $x \in U_\alpha$, we set

$$\mathcal{L}_{a,b}u(x) = \sum_{y \in \sigma^{-1}(x)} e^{f(y)+(a+ib)\tau(y)}u(y).$$

There is an argument to transfer the study of exponential mixing of the flow with respect to a Hölder potential function $F \in C^\theta(M)$, to the study of these operators for certain Hölder potential function $f \in C^\theta(U)$.

The main step

Let g be a topologically mixing C^∞ 3D Anosov flow on M.

Proposition (Dolgopyat's estimate)

There exist $C, \kappa, a_0, b_0 > 0$ such that for any a, b with $|a| < a_0$ and $b > |b_0|$, for any $u \in C^\theta(U)$, for any $n > C \ln |b|$ we have

$$\|\mathcal{L}^n_{a,b} u\|_{L^2(\nu_U)} < |b|^{-\kappa} \max(\|u\|_{C^0}, |b|^{-1}\|u\|_{C^\theta}).$$

Consequences:

- exponential mixing.
- Ruell (dynamical) zeta function $\zeta(z) = \prod_\gamma (1 - e^{-z|\gamma|})^{-1}$ only has a single pole on h_{top} on the half-plane $\{Re(z) > h_{top} - \epsilon\}$.
- $N(T) = \#\{\gamma \mid |\gamma| < T\}$ satisfies $N(T) = li(e^{h_{top}} T) + O(e^{(h_{top}-\epsilon)T})$ where $li(x) = \int_0^x \frac{dt}{\log t}$.

We say that a sequence of functions $\{\Lambda^\epsilon : U \to \mathbb{R}_+\}_{\epsilon>0}$ in $L^\infty(U)$ is

• **stable** if there exist $n, \kappa > 0$ such that for all sufficiently small ϵ, we have

$$\Lambda^\epsilon(x) > \epsilon^{-\kappa}, \quad \forall x \in U,$$

and for any integer $m \geq n$,

$$\|Dg^{\tau_m(x)}|_{E^u(x)}\|^{-1}\Lambda^\epsilon(\sigma^m(x))^{-1} \quad < \quad e^{-m\kappa}\Lambda^\epsilon(x)^{-1}, \quad \forall x \in U.$$

• **tame** if there exist $C, \kappa > 0$ such that for all sufficiently $\epsilon > 0$, for every $x \in U$, for every $y \in (-1,1)$, there exists $R \in C^\theta(-1,1)$ such that

$$\begin{aligned} \|R\|_\theta \quad &\leq \quad C|y|^\kappa, \\ |\epsilon^{-1}\Psi_{x,\Phi^s_x(y)}(\Phi^u_x(\Lambda^\epsilon(x)^{-1}s)) - R(s)| \quad &< \quad \epsilon^\kappa, \quad \forall s \in (-1,1). \end{aligned}$$

• *n*-**adapted to** some subset $\Omega \subset U$ for some integer $n \geq 1$ if there is a constant $C > 0$ such that for all sufficiently small ϵ, for any $x \in \Omega$, for any $v \in \sigma_x^{-n}$, for any $y \in U$ such that $y \in W_g^u(v(x), 4\Lambda(y)^{-1})$, we have

$$\Lambda(x) < C\Lambda(y).$$

• Given a σ-invariant measure ν on U and an integer $n \geq 1$, we say that a subset $\Omega \subset U$ is *n*-**recurrent** with respect to ν if there exist $C, \kappa > 0$ such that for any integer $m > C$ we have

$$\nu(\{x \in U \mid |\{1 \leq j \leq m \mid \sigma^{jn}(x) \in \Omega\}| < \kappa m\}) < e^{-m\kappa}.$$

Given $C > 0$, a sequence of functions $\{\Lambda^\epsilon : U \to \mathbb{R}_+\}_{\epsilon>0}$ and a subset $\Omega \subset U$. We say that C-UNI (short for **uniform non-integrability**) holds on Ω at scales $\{\Lambda^\epsilon : U \to \mathbb{R}_+\}_{\epsilon>0}$ if there exists $\kappa > 0$ such that for every sufficiently small $\epsilon > 0$, for every $x \in U$ with $W^u_g(x, C\Lambda^\epsilon(x)^{-1}) \cap \Omega \neq \emptyset$, there exists $\bar{y} \in (-\varrho_2, \varrho_2)$ such that for any $y \in (\bar{y} - \kappa, \bar{y} + \kappa)$, for any $\omega \in \mathbb{R}/2\pi\mathbb{Z}$, for $J_0 = [0, 1)$ or $(-1, 0]$, there is a sub-interval $J_1 \subset J_0$ with $|J_1| > \kappa$ such that

$$\inf_{s \in J_1} \|\epsilon^{-1}\Psi_{x,y}(\Phi^u_x(\Lambda^\epsilon(x)^{-1}s)) - \omega\|_{\mathbb{R}/2\pi\mathbb{Z}} > \kappa.$$

I. Neither E^u nor E^s for g is $C^{1+\delta}$ for any $\delta > 0$;

I_F. $\int divV_g d\nu_{g,F} \leq 0$.

II. E^u for g is $C^{1+\delta}$ for some $\delta > 0$;

III. E^s for g is $C^{1+\delta}$ for some $\delta > 0$.

We only need consider Case I_F and II.

Proposition

Given a potential function $F \in C^{\theta}(M)$ *for some* $\theta > 0$, *a* C^{∞} *3D Anosov flow* g *in Class* I_F *or II such that* E^s *and* E^u *are not jointly integrable. Then for any* $C_1 > 1$, *for any sufficiently large integer* $n_1 > 0$, *there exist*

- *a subset* $\Omega \subset U$ *which is* n_1*-recurrent with respect to* ν_U;
- *a stable, tame sequence of functions* $\{\Lambda^{\epsilon} : U \to \mathbb{R}_+\}_{\epsilon>0}$ *that is* n_1*-adapted to* Ω

such that C_1*-UNI holds on* Ω *at scales* $\{\Lambda^{\epsilon}\}_{\epsilon>0}$.

Proposition

Given a potential function $F \in C^\theta(M)$ *for some* $\theta > 0$, *a* C^∞ *3D Anosov flow* g *in Class* I_F *or II such that* E^s *and* E^u *are not jointly integrable. There exists* $C_1 > 1$ *such that if the conclusion of the previous proposition is satisfied for* C_1 *and all sufficiently large* n_1, *then Dolgopyat's estimate holds.*

Template approximation

We introduce a family of coordinate charts on M: for each $x \in M$, there is $\iota_x : (-10, 10)^3 \to M$ so that the following holds:

- $\iota_x(z, 0, 0)$ is unstable normal coordinate chart, $\iota_x(0, y, 0)$ is stable normal coordinate chart and $t \mapsto \iota_x(z, y, t)$ parametrizes the flow.
- From chart ι_x to chart $\iota_{g^1(x)}$, the map g^1 writes

  $$g_x(z, y, t) = (g_{x,1}(z, y), g_{x,2}(z, y), t + \psi_x(z, y)).$$

  Then $\partial_y \psi_x(\cdot, 0)$ and $\partial_z \psi_x(0, \cdot)$ are polynomials of degree K; and $\partial_z g_{x,1}(0, \cdot)$ and $\partial_y g_{x,2}(\cdot, 0)$ are both constant functions.

Under this coordinate system, in each chart ι_x, W^{cu} is almost parallel to the plane $y = 0$(near $y = 0$).

Under chart ι_x, $E^s(z,0,0)$ writes

$$\mathbb{R}(*, 1, \varphi_x^{u,s}(z));$$

and $E^u(0,y,0)$ writes

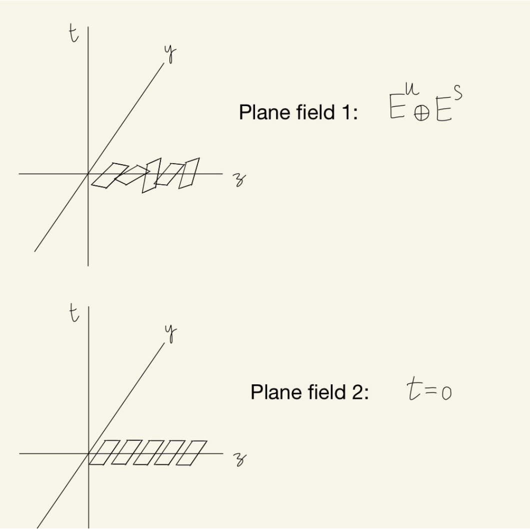
$$\mathbb{R}(1, *, \varphi_x^{s,u}(y)).$$

We define

$$\mathcal{T}_x^s = \{c\varphi_x^{u,s} + P \mid c \in \mathbb{R}, P \in Poly^K, P(0) = 0\}.$$

Define $\mathcal{T}_x^u$ is a similar way. We define

$$\begin{aligned} \mathcal{T}_{x,n} \quad = \quad & \{h_1\varphi_{g^n(x)}^{u,s} + h_2\varphi_x^{s,u} + Q \mid h_1, h_2 \in \mathbb{R}, \\ & Q \in Poly^{K,K}, Q(\cdot, 0) = Q(0, \cdot) = 0\}. \end{aligned}$$



- $\varphi_x^{u,s} = \tan \angle(PF1, PF2)$.

Proposition

If there is $x \in M$ such that $\varphi_x^{u,s} \in Poly^K$, then $\varphi_y^{u,s} \in Poly^K$ for all $y \in M$. In this case, g is in Class II.

Proof.

$\varphi_x^{u,s} \in Poly^K \implies \varphi_y^{u,s} \in Poly^K$ for y in an open set of $W_g^u(g^1(x)) \implies \varphi_z^{u,s} \in Poly^K$ for z in a dense subset of M. Prove by continuity of $x \mapsto \mathcal{T}_x^s$.

In this case, $E^u \oplus E^s$ is C^∞ on each W_g^u. Since $E^s \oplus N$ is C^{1+} everywhere, E^s is C^{1+} on each W_g^u. But E^s is C^∞ on each W_g^{cs}. We conclude by Journé's lemma. □

Proposition (Template approximation I)

For all sufficiently large $K > 1$, there exist $\delta_0, \eta_0 \in (0, 1/2)$, $C_2 > 0$, and a sequence $\{D_n > 0\}_{n \geq 1}$ with $\lim_{n \to \infty} D_n = 0$ such that for all sufficiently small $\epsilon > 0$, for any $x \in M$, for any integer $n \geq 1$ satisfying $\|Dg^n|_{E^s(x)}\|, \|Dg^n|_{E^u(x)}\|^{-1} < \epsilon$, there exist $R \in \mathcal{T}_{x,n}$, $\varkappa \in \{\pm 1\}$, and functions $a_2, \cdots, a_K : (-10, 10) \to \mathbb{R}$ satisfying

$$|a_i(y)| \leq C'|y| \sum_{m=0}^{(1-\eta_0)n} \|Dg^m|_{E^s(x)}\| \|Dg^{n-m}|_{E^u(g^m(x))}\|^{-i}$$

such that for any $y \in (-\varrho_1, \varrho_1)$

$$\begin{aligned} |\Psi_x(\Lambda_n(x)^{-1}\varkappa z, y) - R(z, y) - \sum_{i=2}^{K} a_i(y) z^i| &< C_2((\epsilon|y|)^{1+\delta_0} + \epsilon^2), \\ \|R(\cdot, y)\|_{(-10,10)} &< D_n |y|^{\delta_0}. \end{aligned}$$

Illustration of the idea

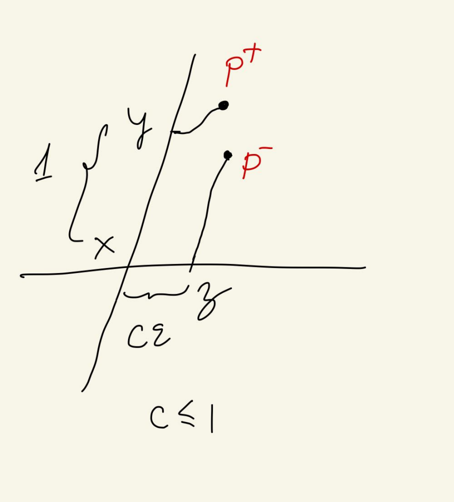
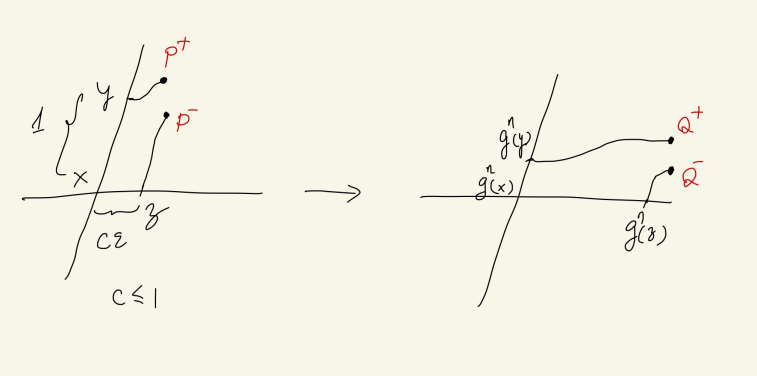


Illustration of the idea



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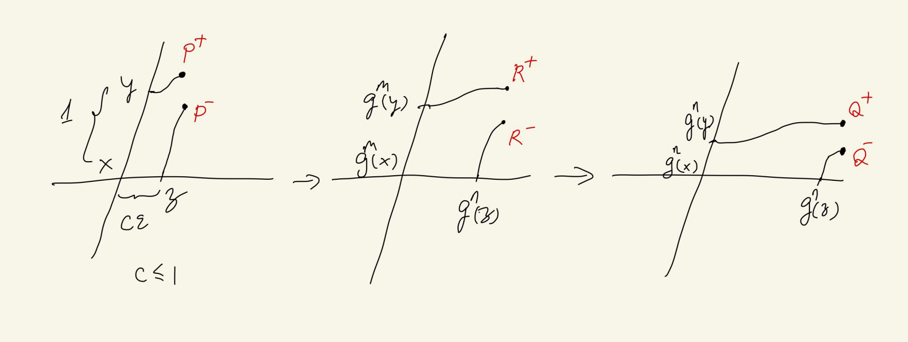

Construction of Λ^ϵ in Class I

Definition

Given a sufficiently small $\epsilon > 0$, for any $x \in M$, we let

- *$k^\epsilon(x)$ be the smallest integer $n \geq 1$ such that $\|Dg^n|_{E^s(x)}\|, \|Dg^n|_{E^u(x)}\|^{-1} < \epsilon$;*
- *the* matching time of order ϵ *at x, denoted by $\varsigma^\epsilon(x)$, be the smallest integer $n \geq k^\epsilon(x)$ satisfying that there is $\varkappa \in \{\pm 1\}$ such that for every $y \in (-1, 1)$, there exists $\varphi \in \mathcal{T}^s_{g^n(x)}$ such that*

$$\begin{aligned} \|\Psi_x(\Lambda_n(x)^{-1}\varkappa\cdot, y) - \varphi\|_{(-2,2)} &\leq C_3((\epsilon|y|)^{1+\delta_5} + \epsilon^2), \\ \|\varphi\|_{(-2,2)} &\leq \max(\epsilon|y|^{\delta_5/2}, C_3\epsilon|y|). \end{aligned}$$

For every $x \in U$, the matching scale of order ϵ *at x is defined by*

$$\Lambda^\epsilon(x) = \sup_{y \in W^s_g(x,1)} \Lambda_{\varsigma^\epsilon(y)}(x).$$

Proposition

For some sufficiently large $C_1 > 1$, there exist $\kappa_3, \kappa_4 > 0$ and an integer $n_1 > 0$ such that for any a with $|a|$ sufficiently small, for any b with $|b|$ sufficiently large, for any $u \in C^\theta(U)$, there is a sequence of functions $\{H_n\}_{0 \le n \le \lfloor \ln |b| \rfloor}$ in $C^0(U, \mathbb{R}_+)$ such that $H_0 \le \max(\|u\|_{C^0}, |b|^{-1}\|u\|_\theta$, and

- *for any $0 \le n \le \ln |b|$ we have*

$$|\widetilde{\mathcal{L}}^{C \ln |b| + nn_1} u(x)| \le H_n(x), \quad \forall x \in U;$$

- *for any $1 \le n \le \ln |b|$ there is a subset $\Omega_n \subset U$ such that*

$$H_n^2(x) \le \begin{cases} (1-\kappa_4)\mathcal{M}^{n_1} H_{n-1}^2(x), & \text{if } x \in \Omega_n, \\ \mathcal{M}^{n_1} H_{n-1}^2(x), & \text{otherwise}; \end{cases}$$

- *for any $\frac{1}{2} \ln |b| \le n \le \ln |b|$, we have*

$$\nu_U(\{x \in U \mid |\{1 \le j \le n \mid \sigma^{jn_1}(x) \in \Omega_j\}| < \kappa_3 n\}) < e^{-n\kappa_3}.$$

We can deduce Dolgopyat's estimate from the previous proposition. Indeed, we define a U-valued random process X by $\{X_n(x) = \sigma^{nn_1}(x)\}_{n\geq 0}$ where x has distribution ν_U, and consider the $\mathbb{R}$-valued random process G defined by

$$G_0(x) = H_0^2(x), \quad G_{m+1}(x) = \begin{cases} (1-\kappa_4)G_m(x), & \text{if } X_{m+1} \in \Omega_{m+1}, \\ G_m(x), & \text{otherwise.} \end{cases}$$

By (2), we have $\mathbb{E}(G_m \mid X_m) \geq H_m^2(X_m)$. By (3) we only need to consider x such that

$$|\{1 \leq j \leq n \mid \sigma^{jn_1}(x) \in \Omega_j\}| \geq \kappa_3 n.$$

But for such x, we have $G_N(x) \leq (1-\kappa_4/2)^{\kappa_3 L} G_0(x)$. We conclude the proof by (1).

It remains to construct Ω_n, H_n for each u. We construct them inductively using the hypotheses (stable, tame, n_1-adapted, C_1-UNI and recurrence):

- stableness and tameness allow us to control the Hölder regularity of $\mathcal{L}^{nn_1}u$ in terms of the C^0 norm of H_{n-1}.
- adaptedness and UNI property allow us to control pointwise $\mathcal{L}^{nn_1}u$ by H_n of the form $H_n = \mathcal{M}^{n_1}(P_n H_{n-1})$ where P_n has valued in $[0,1]$ and is away from 1 in many places in (or near) Ω (this subset is Ω_n). This cancellation mechanism, in a similar form, is already in Dolgopyat's paper.
- Ω_n is "dense "and "thick "in a subset containing Ω. Recurrence property allow us to verify (3) by comparing the iterations of σ^{n_1} with a coin-flipping process.