# Ergodic properties of multiplicative functions and applications

(joint work with Bernard Host)

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# Notation

- $\mathbb{E}_{n\in[N]} a(n) = \frac{1}{N}\sum_{n=1}^{N} a(n), \quad \mathbb{E}_{n\in\mathbb{N}} a(n) = \lim_{N\to\infty} \mathbb{E}_{n\in[N]} a(n).$
- $\mathbb{E}^{\log}_{n\in[N]} a(n) = \frac{1}{\log N}\sum_{n=1}^{N} \frac{a(n)}{n}, \quad \mathbb{E}^{\log}_{n\in\mathbb{N}} a(n) = \lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[N]} a(n).$
- If $n = p_1^{a_1}\cdots p_k^{a_k}$, then $\lambda(n) = (-1)^{a_1+\cdots+a_k}$.
- If $a\colon \mathbb{N}\to\mathbb{U}$ and $N_k\to\infty$ is s.t. all averages below exist, then the corresponding Furstenberg system $(X,\mu,T)$ satisfies

  $$\int T^{n_1}f\cdots T^{n_\ell}f\, d\mu = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N_k]}\, a(n+n_1)\cdots a(n+n_\ell),$$

  for some $f\in L^\infty(\mu)$ for all $\ell\in\mathbb{N}$, $n_1,\ldots,n_\ell\in\mathbb{Z}$.
- If $a=\lambda$, any such system is called a Liouville system.

# Möbius function and primes

Notation: $\mathbb{E}_{n\in[N]} a(n) = \frac{1}{N}\sum_{n=1}^{N} a(n)$.

**Definition (Möbius and von Mangoldt function)**

$\mu(n) = (-1)^k$ if $n$ is a product of $k$ distinct primes, otherwise $\mu(n) = 0$.
$\Lambda(n) = \log p$ if $n = p^k$, and $\Lambda(n) = 0$ elsewhere.

Using the identity

$$\Lambda(n) = -\sum_{d|n} \log(d)\, \mu(d)$$

one can deduce asymptotics for averages of the form

$$\mathbb{E}_{n\in[N]} \Lambda(n)\, a(n) \quad \text{or} \quad \mathbb{E}_{n\in[N]} \Lambda(n+n_1)\cdots\Lambda(n+n_\ell)$$

from estimates of the form

$$\mathbb{E}_{n\in[N]} \mu(n)\, a(n) = O((\log N)^{-A}),$$

where $a \in \ell^\infty(\mathbb{N})$, or the form

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It is a bit more convenient to work with the Liouville function.

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Its signs appear to have "random type behavior". Based on this several well known conjectures have been formulated:

- (Square root cancellation): $\mathbb{E}_{n\in[N]} \lambda(n) = O(N^{-b}), \forall\, b < \frac{1}{2}$ (RH).
- (Chowla conjecture): $(\lambda(n))_{n\in\mathbb{N}}$ forms a normal sequence of $\pm 1$.
- (Sarnak conjecture): $\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \lambda(n) a(n) = 0$ for every $a \in \ell^\infty(\mathbb{N})$ of "low complexity".

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# The Chowla conjecture

## Chowla Conjecture (1965)

If $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$ are distinct, then

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, \lambda(n+n_1)\cdots\lambda(n+n_\ell) = 0.$$

Equivalently: $(\lambda(n))_{n\in\mathbb{N}}$ forms a normal sequence of $\pm 1$, meaning, all length $\ell$ sign patterns appear on the range of $\lambda$ with frequency $1/2^\ell$.

- $\ell = 1$ (PNT): $\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\lambda(n) = 0$.
- $\ell = 2$ (Tao 2015): Proof for logarithmic averages. For all $n_1 \in \mathbb{N}$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n}\lambda(n)\,\lambda(n+n_1) = 0.$$

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# The Sarnak conjecture

The Liouville (and the Möbius) function are expected to not correlate with any bounded sequence of "low complexity".

## Sarnak Conjecture (Dynamical formulation)

Let $Y$ be a compact metric space and $R\colon Y \to Y$ be a continuous 0-entropy transformation. Then for every $g \in C(Y)$ and $y \in Y$

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \lambda(n)\, g(R^n y) = 0.$$

## Sarnak Conjecture (Arithmetic formulation)

If $a\colon \mathbb{N} \to \{-1, 1\}$ satisfies $P_a(\ell) = O(2^{\varepsilon\ell})$ for every $\varepsilon > 0$, then

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($P_a(\ell) = |$ patterns of size $\ell$ of consecutive $\pm 1$ in the range of $a(n)|$.)

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# Some known cases of the Sarnak conjecture

$\lim_{N\to\infty} \mathbb{E}_{n\in[N]} \lambda(n)\, g(R^n y) = 0$ holds when $(Y, S)$ comes from:

- Rational rotations (PNT in arithmetic progressions), Irrational rotations (Vinogradov-Davenport 1937)
- Nilsystems (Green, Tao 2012)
- Horocycle flows (Bourgain, Sarnak, Ziegler 2013) and more general homogeneous dynamics (Peckner 2015)
- Some rank one transformations (Bourgain 2013, Abdalaoui, Lemańczyk, de la Rue, 2014, Ferenczi, Mauduit 2015)
- Various substitutions (Mauduit, Rivat 2015, Deshouillers, M. Drmota, C. Müllner 2015, Ferenczi, Kułaga-Przymus, Lemańczyk, Mauduit 2016)
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- Some systems of number theoretic origin (Green 2012, Bourgain 2013)
- And there are many other results...

Almost all proofs start by using variants of Vinogradov's bilinear method. One needs to show that a large class of $g \in C(Y)$

$$\lim_{N\to\infty} \mathbb{E}_{n\in[N]}\, g(R^{pn}y)\, \overline{g(R^{qn}y)} = 0$$

for all $y \in Y$ and distinct primes $p, q$. But there are limits to this approach... Many systems cannot be handled this way.

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# The Sarnak conjecture for ergodic weights

Notation: $\mathbb{E}^{\log}_{n\in\mathbb{N}} a(n) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{a(n)}{n}$.

**Theorem (F., Host 2018)**

*Let* $a\colon \mathbb{N} \to \mathbb{U}$ *be a* 0*-entropy sequence that is* **ergodic.** *Then*

$$\mathbb{E}^{\log}_{n\in\mathbb{N}} \lambda(n)\, a(n) = 0.$$

- Assumptions apply when $(Y, R)$ is a 0-entropy uniquely ergodic system and $a(n) = g(R^n y)$ for some $g \in C(Y)$ and $y \in Y$.
- Subsequently we extended this result to a larger class of multiplicative functions $f\colon \mathbb{N} \to \mathbb{U}$ that are called strongly aperiodic.
- These results follow from structural results of certain measure preserving systems associated to multiplicative functions.

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# Chowla averages with totally ergodic weights

**Theorem (F., Host 2019)**

*If $f_1, \ldots, f_\ell \colon \mathbb{N} \to \mathbb{U}$ are arbitrary multiplicative functions and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then*

$$\mathbb{E}^{\log}_{n\in\mathbb{N}}\, e^{2\pi i n\alpha}\, f_1(n+n_1)\cdots f_\ell(n+n_\ell) = 0$$

*for all $n_1, \ldots, n_\ell \in \mathbb{N}$.*

- The result follows by showing that a certain measure preserving system does not have irrational spectrum.
- The weight $(e^{2\pi i n\alpha})$ can be replaced with any 0-entropy, totally ergodic, zero-mean sequence.

# Chowla averages along deterministic sequences

**Theorem (Tao 2015 and Tao-Teräväinen 2018)**

*For $\ell = 2$ ($n_1 \neq n_2$) and all odd $\ell$ we have*

$$\mathbb{E}^{\log}_{n \in \mathbb{N}} \lambda(n + n_1) \cdots \lambda(n + n_\ell) = 0.$$

For odd $\ell$ proof uses an ergodic decomposition result of A. Le for sequences of the form $A(p) = \int T^{n_1 p} f \cdots T^{n_\ell p} f \, d\mu$, $p \in \mathbb{P}$.

**Theorem (F. 2019)**

*Let $a \colon \mathbb{N} \to \mathbb{N}$ be a 0-entropy and totally ergodic sequence, for example $a(n) = [n\sqrt{2}]$. For $\ell = 2$ ($n_1 \neq n_2$) and all odd $\ell$ we have*

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For odd $\ell$ proof uses an ergodic decomposition result of A. Le for sequences of the form $A(p) = \int T^{n_1 p} f \cdots T^{n_\ell p} f \, d\mu$, $p \in \mathbb{P}$.

**Theorem (F. 2019)**

*Let $a \colon \mathbb{N} \to \mathbb{N}$ be a 0-entropy and totally ergodic sequence, for example $a(n) = [n\sqrt{2}]$. For $\ell = 2$ ($n_1 \neq n_2$) and all odd $\ell$ we have*

$$\mathbb{E}^{\log}_{n \in \mathbb{N}} \lambda(a(n + n_1)) \cdots \lambda(a(n + n_\ell)) = 0.$$

- If the Chowla conjecture holds, then $\lambda$ is a normal sequence and hence (by Kamae, Weiss 70's) so is $\lambda \circ a$.

# Furstenberg systems of sequences

## Furstenberg Correspondence Principle

Let $a\colon \mathbb{N} \to \mathbb{U}$ and $N_k \to \infty$ integers. Then there exist a subsequence $N'_k \to \infty$, a mps $(X, \mu, T)$, and a function $f \in L^\infty(\mu)$ such that

$$\int T^{n_1} f \cdots T^{n_\ell} f \, d\mu = \lim_{k\to\infty} \mathbb{E}^{\log}_{n\in[N'_k]} a(n+n_1)\cdots a(n+n_\ell),$$

for all $\ell \in \mathbb{N}$ and $n_1, \dots, n_\ell \in \mathbb{Z}$.

- $X = \mathbb{U}^{\mathbb{Z}}$, $(Tx)(k) = x(k+1)$, $f(x) = x(0)$, only $\mu$ varies.
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## Definition (Furstenberg systems)

- Any such system is called a **Furstenberg system of** $a\colon \mathbb{N} \to \mathbb{U}$. If $a = \lambda$ we call it a **Liouville system.**
- A Furstenberg system of a strictly increasing $a\colon \mathbb{N} \to \mathbb{N}$ with range a set $S$ of positive density, is any Furstenberg system of $\mathbf{1}_S$.
- A sequence $a\colon \mathbb{N} \to \mathbb{U}$ (or $a\colon \mathbb{N} \to \mathbb{N}$) is ergodic, totally ergodic, or 0-entropy (deterministic) if all its Furstenberg systems are.

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# The Chowla and Sarnak conjecture in ergodic terms

## Chowla conjecture (Ergodic reformulation)

Logarithmic Chowla conjecture $\Leftrightarrow$ All Liouville systems are Bernoulli systems.

- But it is not even known if any Liouville system is ergodic.
- (F., 2016): If a Liouville system is ergodic iff it is Bernoulli.

## Definition (Furstenberg 1967)

Two mps $(X, \mu, T)$, $(Y, \nu, S)$ are disjoint if the only $(T \times S)$-invariant measure on $X \times Y$ with marginals $\mu$ and $\nu$ is $\mu \times \nu$.

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The logarithmic Sarnak conjecture holds if all Liouville systems are disjoint from all 0-entropy mps.

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# A disjointess property and applications

**Theorem (F., Host 2018)**

All Liouville systems are disjoint from all **totally ergodic systems of 0-entropy**.

**Ergodic Sarnak conjecture:** If $a\colon \mathbb{N} \to \mathbb{U}$ is a totally ergodic sequence, using disjointness we get

$$\mathbb{E}^{\log}_{n\in\mathbb{N}}\, \lambda(n)\, a(n) = \mathbb{E}^{\log}_{n\in\mathbb{N}}\, \lambda(n) \cdot \mathbb{E}^{\log}_{n\in\mathbb{N}}\, a(n) = 0.$$

In order to deal with more general ergodic sequences we also have to use:

$$\mathbb{E}^{\log}_{n\in\mathbb{N}}\, \lambda(n)\, \lambda(n+h) = 0$$

for every $h \in \mathbb{N}$ (Tao 2015).

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**Chowla for totally ergodic weights:** If $a\colon \mathbb{N} \to \mathbb{U}$ is totally ergodic, has 0-entropy and mean 0 (for ex. $a(n) = e(n\alpha)$ with $\alpha$ irrational), then using disjointness we get

$$\mathbb{E}^{\log}_{n\in\mathbb{N}}\, a(n)\, \lambda(n+n_1)\cdots\lambda(n+n_\ell) = \mathbb{E}^{\log}_{n\in\mathbb{N}}\, a(n) \cdot \mathbb{E}^{\log}_{n\in\mathbb{N}}\, \lambda(n+n_1)\cdots\lambda(n+n_\ell) = 0.$$

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**Chowla along deterministic sequences:** If $a\colon \mathbb{N} \to \mathbb{N}$ is a totally ergodic sequence of 0-entropy, because of disjointness we get for every $\mathbf{M} = ([M_k])_{k\in\mathbb{N}}$ (assuming all limits exist)

$$\mathbb{E}^{\log}_{m\in\mathbf{M}} \prod_{j=1}^{\ell} \lambda(a(m+n_j)) = \mathbb{E}^{\log}_{n\in\mathbf{M}} \Big(\mathbb{E}^{\log}_{m\in\mathbf{M}} \prod_{j=1}^{\ell} \lambda(m+a(n+n_j))\Big).$$

For $\ell = 2$ ($n_1 \neq n_2$) and $\ell$ odd, the last averages are 0 by the results of Tao and Tao-Teräväinen.

# Disjointness property from a structural result

## Theorem (Structural result)

1. *A Liouville system cannot have irrational eigenvalues. ($Tf = e^{2\pi i\alpha} f$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, implies $f = 0$).*
2. *The "building blocks" of a Liouville system are Bernoulli systems and systems of algebraic structure (nilsystems).*

The first property is equivalent to showing that for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

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If a system satisfies the previous two structural properties, then it is disjoint from all totally ergodic systems of 0-entropy.

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# Tao's identity

Starting point in the proof of the two structural properties is:

**Theorem (Tao's identity 2015)**

If $a \in \ell^\infty(\mathbb{N})$, $\mathbf{N} = ([N_k])_{k\in\mathbb{N}}$, and all limits below exist, then

$$\mathbb{E}_{p\in\mathbb{P}}\, \mathbb{E}^{\log}_{n\in\mathbf{N}}\, a(pn) = \mathbb{E}^{\log}_{n\in\mathbf{N}}\, a(n).$$

More generally, for all $\ell, n_1, \ldots, n_\ell \in \mathbb{N}$ we have

$$\mathbb{E}_{p\in\mathbb{P}}\, \mathbb{E}^{\log}_{n\in\mathbf{N}} \prod_{j=1}^{\ell} a(pn + pn_j) = \mathbb{E}_{p\in\mathbb{P}}\, \mathbb{E}^{\log}_{n\in\mathbf{N}} \prod_{j=1}^{\ell} a(n + pn_j).$$

The identity is false for Cesàro averages (take $a(n) = n^i$).

**Corollary (Tao's identity for $\lambda$)**

For every $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$ we have

$$\mathbb{E}^{\log}_{n\in\mathbf{N}} \prod_{j=1}^{\ell} \lambda(n + n_j) = (-1)^\ell\, \mathbb{E}_{p\in\mathbb{P}}\, \mathbb{E}^{\log}_{n\in\mathbf{N}} \prod_{j=1}^{\ell} \lambda(n + pn_j).$$

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# Reduction to an ergodic statement

## An ergodic consequence of Tao's identity

Let $(X, \mu, T)$ be a Liouville system. Then for **some** $T$-generating $f\colon X \to \{-1, 1\}$ we have for every $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$

$$\int \prod_{j=1}^{\ell} T^{n_j} f \, d\mu = (-1)^\ell \, \mathbb{E}_{p \in \mathbb{P}} \int \prod_{j=1}^{\ell} T^{p n_j} f \, d\mu.$$

$T$-generating: The algebra generated by $T^n f$, $n \in \mathbb{Z}$, is dense in $L^2(\mu)$.

Hence, our task is reduced to showing that:

## Theorem (Ergodic structural result)

If $(X, \mu, T)$ satisfies the previous property, then

1. it has no irrational eigenvalues;
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# Systems of arithmetic progressions

It will be more convenient to work in an even more general setup:

## Definition

Let $(X, \mu, T)$ be a system. On $X^{\mathbb{Z}}$ we define the measure $\widetilde{\mu}$ as follows

$$\int \prod_{j=-m}^{m} f_j(x_j)\, d\widetilde{\mu} := \mathbb{E}_{p\in\mathbb{P}} \int \prod_{j=-m}^{m} T^{jp} f_j\, d\mu, \quad f_j \in L^\infty(\mu).$$

We call $(X^{\mathbb{Z}}, \widetilde{\mu}, S)$, where $S$ is the shift, the **system of arithmetic progressions (AP's)** with prime steps associated with $(X, \mu, T)$.

Relevance to our problem:

## Proposition (Factor property)

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# Two illuminating examples

## Example (Irrational rotations)

$Tt = t + \alpha \pmod 1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, acting on $\mathbb{T}$ with $m_{\mathbb{T}}$. Then

$$\int \prod_{j=-m}^{m} f_j(x_j)\, d\tilde{\mu} := \mathbb{E}_{p\in\mathbb{P}} \int \prod_{j=-m}^{m} f_j(t + jp\alpha)\, dt = \int\int \prod_{j=-m}^{m} f_j(t + js)\, dt ds.$$

System of AP's isomorphic to $(s, t) \mapsto (s, t + s)$ on $\mathbb{T}^2$ with $m_{\mathbb{T}^2}$.

## Example (Weak mixing systems)

$Tt = 2t \pmod 1$ acting on $\mathbb{T}$ with $m_{\mathbb{T}}$. Then

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# Systems of AP's: Structure of building blocks

**Theorem (Structure of building blocks)**

*The ergodic components of a system of AP's are direct products of Bernoulli systems and inverse limits of nilsystems.*

Proof uses:

1. Gowers uniformity of the modified von Mangoldt function (Green, Tao, and Ziegler, 2012).
2. A result about characteristic factors of Furstenberg averages (Host and Kra, 2005).
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# Systems of AP's: No irrational eigenvalues

**Theorem (No irrational eigenvalues)**

*A system of AP's has no irrational eigenvalues.*

Proof uses the following notion (Furstenberg and Katznelson 91):

**Definition (Partial strong stationarity)**

$(X^{\mathbb{Z}}, \nu, S)$ is partially strongly stationary if maps $\tau_r : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$, defined $(x(j))_{j\in\mathbb{Z}} \mapsto (x(rj))_{j\in\mathbb{Z}}$, are measure preserving for every $r \in d\mathbb{N} + 1$.

**Proposition**

A system of AP's is an inverse limit of partially sst systems.

Following an argument of Jenvey (1997) we show:

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**Definition (Partial strong stationarity)**

$(X^{\mathbb{Z}}, \nu, S)$ is partially strongly stationary if maps $\tau_r \colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$, defined $(x(j))_{j \in \mathbb{Z}} \mapsto (x(rj))_{j \in \mathbb{Z}}$, are measure preserving for every $r \in d\mathbb{N} + 1$.

**Proposition**

A system of AP's is an inverse limit of partially sst systems.

Following an argument of Jenvey (1997) we show:

**Proposition (No irrational eigenvalues)**

A partial sst system has no irrational eigenvalues.

# Some open problems

**Problem (No non-trivial rational spectrum)**

*A Liouville system has no rational eigenvalues different than* $1$*.*

On the other hand, Möbius system does have rational eigenvalues.

**Problem (Not a mixture of circle rotations)**

*A Liouville system is not isomorphic to the non-ergodic system* $(\mathbb{T}^2, m_{\mathbb{T}^2}, T)$*, where* $T(s,t) = (s, t+s), \quad s,t \in \mathbb{T}$*.*

**Problem (Dichotomy)**

*If* $f : \mathbb{N} \to \{-1, 1\}$ *is multiplicative, then it has a unique Furstenberg system that is isomorphic to a procyclic* **or** *a Bernoulli system.*

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