Left-orderable lattices in semi-simple Lie groups

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May 10, 2020

Joint project with

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A group Γ is *left-orderable* if it admits a total order which is invariant by left multiplications.

$$\forall f, g, h \in \Gamma : \qquad \text{If } f < g \text{ then } hf < hg$$

A folklore result

A countable group Γ is left-orderable iff it acts faithfully on the real line by orientation preserving homeomorphisms.

$$\Gamma \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$$

If $p \in \mathbb{R}$ is a free orbit (i.e. $\forall g \in \Gamma, \quad g(p) \neq p$), then we can define:

$$h <_p g \qquad \text{if} \quad h(p) < g(p).$$

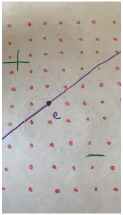
Left-orderable groups:

1. $\mathbb{Z}^n$, $\mathbb{F}_n$.
2. Braid groups. Some MCG's of surfaces. RAAG's.
3. Thompson's group F (consist of piecewise homeomorphisms of an interval)
4. Many more...

Non left-orderable groups:

1. Groups with torsion.
2. $\Gamma = \langle a, b | \;\; ab^7ab^{13}ab = e, \;\; ab^{-3}a^{-3}b = e, \;\; a^{-7}ba^{-2}b^3 = e, \;\; a^{-5}b^{-7}a^{-3}b^{-4} = e \rangle$.
3. Random groups. (Orlef, 2014) (Unknown for actions in the circle)
4. $SL_n(\mathbb{Z})$, when $n \geq 3$. (Witte-Morris, 1994)
5. It is unknown whether there exists an orderable group with property T.

Orders in $\mathbb{Z}^2$:



Orders in $\mathbb{F}_2$: There are many more orders (Super-exponentially many when looking at balls in the Cayley graph).

I will discuss the left-orderability of irreducible lattices in semi-simple Lie groups.

Notation: G is a Lie group, $G = \mathrm{Isom}(X)$, where X is the associated symmetric space. Γ is a lattice if $\mathrm{vol}(G/\Gamma) < \infty$.

Hyperbolic spaces, $G = \mathbf{SO}(n, 1)$:

Fundamental groups of hyperbolic surfaces are left-orderable.

A conjecture of Boyer-Gordon-Watson, relates left-orderability of fundamental groups of 3-manifolds with taut foliations and Floer homology. See a lecture of Nathan Dunfield on his webpage.

The fundamental group of a hyperbolic 3-manifold is virtually left orderable. No examples known in dimension higher than 3.

Other rank one symmetric spaces

It seems, no lattice in other rank one symmetric spaces (complex hyperbolic, quaternionic hyperbolic, Cayley plane) are known to be left-orderable.

Higher rank symmetric spaces

Zimmer program: Every smooth action on a manifold of an irreducible lattice in higher rank comes from a nice algebraic construction.

Our main result concerns irreducible lattices in higher rank:

An irreducible lattice Γ *in a connected semi-simple Lie group* G *of rank at least two is left-orderable iff* Γ *is torsion free and there exists a surjective morphism* $G \to PSL(2, \mathbb{R})$.

- Dave Witte-Morris proved this theorem for many lattices.

Example 1: The rank of $SL(3, \mathbb{R})$ is 3 - 1. $SL_3(\mathbb{Z})$ is not left-orderable.

Example 2: $SL(2, \mathbb{Z}(\sqrt{2}))$ embeds as a lattice in $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ via

$$A \to (A, \sigma(A)),$$

where $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$. $SL(2, \mathbb{Z}(\sqrt{2}))$ is not left-orderable. Passing to universal covering one gets a left-orderable lattice of higher rank.

Remark: Margulis showed all lattices in higher rank are arithmetic. So our theorem is mainly about groups similar to example 2.

A theorem of Ghys (1999):

If Γ is a lattice in a connected semi-simple Lie group G of rank at least two and $\Gamma \to \text{Homeo}^+(\mathbb{S}^1)$ is an action, then:

1. Either Γ has a finite orbit on $\mathbb{S}^1$.
2. Or there exists a surjective morphism $G \to PSL(2, \mathbb{R})$.

- This result was also proven by Burger-Monod around the same time for many lattices.Navas and Rezhnikov proved that any group with property T do not act smoothly in $\mathbb{S}^1$. Ghys Theorem was generalized by Bader-Furman for some non-linear groups.

Strategy of proof: Assume action minimal. Assume G simple.
Idea: Γ preserves a measure on $\mathbb{R}$. This implies $\Gamma \to \mathbb{Z}$, contradiction.
Suspension space:

$$Y := (G \times \mathbb{R})/(g, t) \sim (g\gamma^{-1}, \gamma(t))$$

- Y is an $\mathbb{R}$-bundle over G/Γ. G acts on Y.
- Γ preserves a measure in $\mathbb{R}$ iff G preserves a measure on Y.

Stiffness 1: Construct a G-stationary measure on Y and show is G-invariant.

Stiffness 2: Construct a P-invariant measure on Y and show is G-invariant.

Philosophy: Higher rank abelian (hyperbolic) actions have rigidity. Understand dynamics of A-action in Y and show G-invariance.

Remark 1: This strategy was used in work of Brown,Rodriguez-Hertz,Wang (2014) about stiffness of actions of lattices. This work was applied by Brown,Fisher,Hurtado in the solution of Zimmer's conjecture (2016).

Remark 2: Our method follows same philosophy but avoids use of entropy

Big problems: $\mathbb{R}$ is not compact. Action is not smooth.

Theorem (Deroin's space of quasi-periodic actions (2011))

Assume Γ^* *acts on* $\mathbb{R}$ *without a discrete orbit. There exists one-dimensional laminated compact space* D *such that:*

1. Γ *acts on* D *and preserve each leaf.*
2. *The action is Lipschitz in each one dimensional leaf.*
3. *The original action is conjugate to the action in a leaf of* D*.*

Warning: D is in general infinite dimensional and its size is related to the possible left-orders of Γ.

Remark: D is related to space of orders constructed* by Witte-Morris.

Random walks by homeomorphisms of $\mathbb{R}$:

Suppose μ is a finitely supported, symmetric measure on Γ. Assume Γ fixed point free. Fix $p \in \mathbb{R}$. Consider the random walk:

$$X_n(p) = g(X_{n-1}(p))$$

g is chosen as determined by μ.

What happen as $n \to \infty$?

Theorem (Deroin-Kleptsyn-Navas-Parwani (2012))

1. *For all* $p \in \mathbb{R}$, $\limsup X_n(p) = \infty$ *and* $\liminf X_n(p) = -\infty$ *almost surely.*
2. *There exists a stationary Radon measure in* $\mathbb{R}$. *(unique* for minimal action).*
3. *Under necessary assumptions**: For all* $p, q \in \mathbb{R}$ $\lim X_n(p) - X_n(q) = 0$.

DNKP Theorem implies that up to conjugation, Lebesgue is stationary: For all $x, y \in \mathbb{R}$, $x - y = \sum \mu(\gamma)(\gamma(x) - \gamma(y))$, moreover:

1. **Lipschitz:** $|\gamma(x) - \gamma(y)| \leq \frac{1}{\mu(\gamma)}|x - y|$,
2. **Bounded displacement and non-triviality**:

$$\forall x, \quad \frac{1}{C_\mu} \leq \sum \mu(\gamma)|\gamma(x) - x| \leq C_\mu$$

3. **Harmonicity:** $\forall x, \; x = \sum \mu(\gamma)\gamma(x)$.

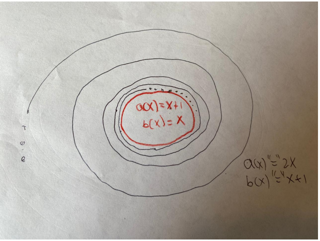
$D := \{(\Phi, p) | p \in \mathbb{R}, \;\; \Phi : \Gamma \to \text{Homeo}^+(\mathbb{R}) \text{ satisfying } 1), 2) \text{ and } 3)\}/\sim$

The equivalence relation $\sim$ is defined by translations:
$(\Phi, p) \sim (T^t \Phi T^{-t}, p + t)$.
There is an $\mathbb{R}$-flow in D sending (Φ, p) to $(\Phi, p + t)$.

Example 1: For $\Gamma = \mathbb{Z}^2$, D consist of actions by translations. D is topologically $\mathbb{S}^1$.

Example 2: For Γ lift of action by homeomorphisms of $\mathbb{S}^1$, $D = \mathbb{S}^1$.

Example 3: For $\Gamma = \{a, b | aba^{-1} = b^2\}$.



Some other applications of D:

1. A left orderable, amenable group has surjection to $\mathbb{Z}$. (Witte-Morris).
2. Understanding of Hyde-Lodha 's example of f.g. simple left orderable group. (Triestino-Matte Bon)
3. Rigidity of actions of Thompson's groups and other related work. (Rivas, Matte Bon, Lodha, Triestino).

Some open questions related to D and Harmonic actions:

1. Is there a CLT for harmonic actions?, large deviations?, LLT?
2. What are the groups with the most dense orbits in 1-dimensions.

Thank you and have a nice week.

Ideas of proof of main theorem Let $X = (G \times D)/\Gamma$ be the suspension space for the Γ action on D. X is a G-space.
Fix a maximal compact subgroup $K \subset G$, and let m_G be a probability measure on G which is

- absolutely continuous wrt Haar.
- invariant by left and right multiplications by K, and
- symmetric.

A general machinery shows that there exists on X a measure m_X which is m_G-stationary, namely which satisfies the convolution equation

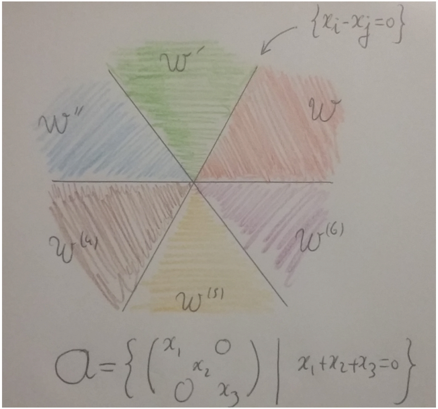
$$m_G \star m_X = \int g_* m_X \; m_G(dg) = m_X.$$

Our goal is to establish that m_X is indeed G-invariant; we construct D, X and m_X are constructed in such a way that m_X is ergodic and conditionals measures along leafs of D are abs. continuous with respect to Lebesgue. For constructing D, we choose μ in Γ a dicretization probability measure for the Brownian motion in the symmetric space $K \backslash G$. (G/P is the poisson boundary of (Γ, μ)).

Weyl chambers Consider the case $G = \mathrm{SL}(3,\mathbb{R})$. We set $K = \mathrm{SO}(3,\mathbb{R})$, and let $A \subset G = \mathrm{SL}(3,\mathbb{R})$ be the subgroup of diagonal matrices with positive coefficients. Each $a \in \mathrm{lie}(A) \simeq \mathbb{R}^2$ determines a solvable subgroup $P^a = AN^a$, where N^a is the strong unstable foliation of a:

$$N^a := \{b \in G \mid e^{ta} b e^{-ta} \to_{t\to-\infty} e_G\}.$$

For generic a's, there are only six possibilities for the N^a's, which defines a decomposition of A into six Weyl chambers:



$P^{\mathcal{W}}$-invariant measures

For each Weyl chamber $\mathcal{W}$, we have the Iwasawa decomposition $G = KP^{\mathcal{W}}$. Applying Furstenberg's Poisson formula to the function $g \mapsto g_* m_X$, which is harmonic and bounded (since m_X is stationary), one proves that:

There exists a unique probability measure $m_X^{\mathcal{W}}$ on X which satisfies

- *$m_X^{\mathcal{W}}$ is $P^{\mathcal{W}}$-invariant and $P^{\mathcal{W}}$-ergodic,*
- *the K-average of $m_X^{\mathcal{W}}$ wrt the normalized Haar measure on K equals m_X.*

Global contraction property

The lamination defined by the flow T on the quasi-periodic space Z produces a one dimensional oriented lamination $\mathcal{L}$ on the suspension space X, which is invariant by the G-action.

We say that an element $a \in \mathrm{lie}(A)$ has the *global contraction property wrt some probability measure m on X* if for m-a.e. $x \in X$, the flow associated to a contracts globally the leaf $\mathcal{L}(x)$ in the sense that

$$d(e^{ta}(y), e^{ta}(z)) \rightarrow_{t \rightarrow +\infty} 0 \text{ for every } y, z \in \mathcal{L}(x).$$

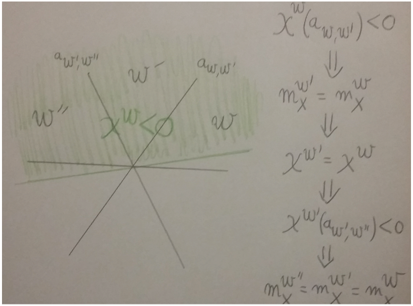
Lyapunov exponents

For each Weyl chamber $\mathcal{W}$, there exists an open half-space in lie(A) consisting of elements whose exponential have the global contraction property wrt to $m_X^{\mathcal{W}}$. Moreover, this half-space intersects the interior of $\mathcal{W}$.

This half-space is determined by a Lyapunov exponent functional being negative. The Lyapunov exponent is the exponential rate of the derivative in the direction of $\mathcal{L}$ of an element of A. It is linear functional in lie(A) and is denoted by $\chi^{\mathcal{W}}$: lie(A) $\to \mathbb{R}$.

Propagating invariance

Assume that $\mathcal{W}, \mathcal{W}'$ are two adjacent Weyl chambers, and denote by $a^{\mathcal{W},\mathcal{W}'}$ a non zero element in $\mathcal{W} \cap \mathcal{W}'$. Assume that the flow a has the global contraction property wrt $m_X^{\mathcal{W}}$. Then $m_X^{\mathcal{W}} = m_X^{\mathcal{W}'}$.



Idea of the proof main Lemma: We show there are two generic points x_1, x_2 for $m_X^{\mathcal{W}}$ and $m_X^{\mathcal{W}'}$ with the same ergodic averages. There is a nice relation between $m_X^{\mathcal{W}}$ and $m_X^{\mathcal{W}'}$, they are related via: $k^* m_X^{\mathcal{W}}$ and $m_X^{\mathcal{W}'}$ for $k \in K$. (k is an element of the Weyl group). So we can take x_1, x_2 generic in the same real leaf. As both measures are N_a-invariant, one can change the a-future of $\pi_{G/\Gamma}(x_1)$ to coincide** with the future of $\pi_{G/\Gamma} x_2$. More formally, there exists $n_1, n_2 \in N_a$ such that:

$$\lim_{t \to \infty} d_{G/\Gamma}(e^{ta}\pi_{G/\Gamma}(x_1), e^{ta}\pi_{G/\Gamma}(x_2)) = 0$$

Using the global contraction property we have $d_X(e^{ta}x_1, e^{ta}x_2) = 0$ and we are done.

Thank you!