Measure rigidity and orbit closure classification of random walks on surfaces

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Setting

Given a manifold M, a point $x \in M$ and a semigroup Γ acting on M,

what can we say about:

- the orbit of x under Γ,

$$\mathrm{Orbit}(x, \Gamma) := \{\varphi(x) \mid \varphi \in \Gamma\}?$$

- the Γ-invariant probability measures ν on M?

When can we classify all of them?

Circle

Say $M = S^1 = [0,1]/\sim$, $\quad f(x) = 3x \mod 1$, $\quad \Gamma = \langle f \rangle$ is cyclic,

- If $x = p/q$ is rational, $\mathrm{Orbit}(x, \Gamma) \subset \{0, 1/q, \dots, (q-1)/q\}$ is finite.
- By the pointwise ergodic theorem, we know that for almost every point $x \in S^1$, $\mathrm{Orbit}(x, \Gamma)$ is dense.
- But there are points $x \in S^1$ where $\mathrm{Orbit}(x, \Gamma)$ is neither finite nor dense, for instance for certain $x \in S^1$, the closure of its orbit

$$\overline{\mathrm{Orbit}(x, \Gamma)} = \text{ middle third Cantor set.}$$

(And many orbit closures of Hausdorff dimension between 0 and 1!)

Furstenberg's $\times 2 \times 3$ problem

Nonetheless, if we take $M = S^1$ and $\Gamma = \langle f, g \rangle$, where

$$f(x) = 2x \bmod 1, \qquad g(x) = 3x \bmod 1,$$

we have the following theorem of Furstenberg:

Theorem (Furstenberg, 1967)

For all $x \in S^1$, $\mathrm{Orbit}(x, \Gamma)$ is either finite or dense.

For invariant measures...

Conjecture (Furstenberg, 1967)

Every ergodic Γ-invariant probability measure ν on S^1 is either finitely supported or the Lebesgue measure.

Free group action on 2-torus

For dim $M = 2$, one observes similar phenomenon. Say $M = \mathbb{T}^2$, and $\Gamma = \langle f, g \rangle$ with

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \in SL_2(\mathbb{Z})$$

which acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by left multiplication.
Then $\mathrm{Orbit}(x, \langle f \rangle)$ can be neither finite nor dense. Nonetheless it follows from a theorem of Bourgain-Furman-Lindenstrauss-Mozes that

Theorem (Bourgain-Furman-Lindenstrauss-Mozes, 2007)

- *For all $x \in \mathbb{T}^2$, $\mathrm{Orbit}(x, \langle f, g \rangle)$ is either finite or dense.*
- *Every ergodic Γ-invariant probability measure ν on $\mathbb{T}^2$ is either finitely supported or the Lebesgue measure.*

Stationary measure

In fact, the theorem of BFLM classifies stationary measures on $\mathbb{T}^d$.

Let X be a metric space, G be a group acting continuously on X. Let μ be a probability measure on G.

Definition

A measure ν on X is μ-stationary if

$$\nu = \mu * \nu := \int_G g_* \nu \, d\mu(g).$$

In other words, ν is "invariant on average" under the random walk driven by μ.

Stationary measure

Definition

A measure ν on X is μ-stationary if

$$\nu = \mu * \nu := \int_G g_* \nu \, d\mu(g).$$

Basic facts: Let $\Gamma = \langle \mathrm{supp}\ \mu \rangle \subset G$.

- Every Γ-invariant measure is μ-stationary.
- Every finitely supported μ-stationary measure is Γ-invariant.
- (Choquet-Deny) If Γ is abelian, every μ-stationary measure is Γ-invariant (stiffness).
- (Kakutani) If X is compact, there exists a μ-stationary measure on X. (Even though Γ-invariant measure may not exist for non-amenable Γ!)

Zariski dense toral automorphism

Theorem (Bourgain-Furman-Lindenstrauss-Mozes, Benoist-Quint)

Let μ be a compactly supported probability measure on $SL_d(\mathbb{Z})$. If $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is a Zariski dense subsemigroup of $SL_d(\mathbb{R})$, then

- *For all $x \in \mathbb{T}^d$, $\mathrm{Orbit}(x, \Gamma)$ is either finite or dense.*
- *Every ergodic μ-stationary probability measure ν on $\mathbb{T}^d$ is either finitely supported or the Lebesgue measure.*
- *Every infinite orbit "equidistributes" on $\mathbb{T}^d$.*

- The Zariski density assumption is necessary since the theorem is false for say cyclic Γ generated by a hyperbolic element in $SL_d(\mathbb{Z})$.
- The second conclusion implies that under the given assumptions, every μ-stationary measure is Γ-invariant (i.e. stiffness).

Homogeneous Setting

The theorem of Benoist-Quint works more generally for homogeneous spaces G/Λ.

Theorem (Benoist-Quint, 2011)

Let G be a connected simple real Lie group, Λ be a lattice in G, μ be a compactly supported probability measure on G. If $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is a Zariski dense subsemigroup of G, then

- *For all $x \in G/\Lambda$, $\mathrm{Orbit}(x, \Gamma)$ is either finite or dense.*
- *Every ergodic μ-stationary probability measure ν on G/Λ is either finitely supported or the Haar measure.*
- *Every infinite orbit "equidistributes" on G/Λ.*

Non-homogeneous setting

Let M be a closed manifold with (normalized) volume measure vol,
μ be a probability measure on $\mathrm{Diff}^2(M)$, $\Gamma = \langle \mathrm{supp}\ \mu \rangle$.

Under what condition on μ and/or Γ do we have that

- For all $x \in M$, $\mathrm{Orbit}(x, \Gamma)$ is either finite or dense.
- Every ergodic μ-stationary probability measure ν on M is either finitely supported or vol.
- Every infinite orbit "equidistributes" on M?

Uniform expansion

Definition

Let M be a Riemannian manifold, μ be a probability measure on $\mathrm{Diff}^2(M)$. We say that μ is uniformly expanding if there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $x \in M$ and $v \in T_x M$,

$$\int_{\mathrm{Diff}^2(M)} \log \frac{\|D_x f(v)\|}{\|v\|} d\mu^{(N)}(f) > C > 0.$$

Here $\mu^{(N)} := \mu * \mu * \cdots * \mu$ is the N-th convolution power of μ.

In other words, the random walk w.r.t. μ expands every vector $v \in T_x M$ at every point $x \in M$ on average.

Remark

Uniform expansion is an open condition.

Main result

Theorem (C.)

Let M be a closed 2-manifold with volume measure vol*. Let* μ *be a compactly supported probability measure on* $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ *that is uniformly expanding, and* $\Gamma := \langle \mathrm{supp}\ \mu \rangle$*. Then*

- *For all* $x \in M$, $\mathrm{Orbit}(x, \Gamma)$ *is either finite or dense.*
- *Every ergodic* μ*-stationary probability measure* ν *on* M *is either finitely supported or* vol*.*

Remark

- *For* $M = \mathbb{T}^2$ *and* μ *supported on* $SL_2(\mathbb{Z})$*, if* $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ *is Zariski dense in* $SL_2(\mathbb{R})$*, then* μ *is uniformly expanding.*
- *Since uniform expansion is an open condition, so the conclusion holds for small perturbations of Zariski dense toral automorphisms in* $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ *too.*

Result of Brown and Rodriguez Hertz

Theorem (Brown-Rodriguez Hertz, 2017)

Let M be a closed 2-manifold. Let μ be a measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$, and $\Gamma := \langle \mathrm{supp}\ \mu \rangle$. Let ν be an ergodic hyperbolic μ-stationary measure on M. Then at least one of the following three possibilities holds:

1. *ν is finitely supported.*
2. *$\nu = \mathrm{vol}|_A$ for some positive volume subset $A \subset M$ (local ergodicity).*
3. *For ν-a.e. $x \in M$, there exists $v \in \mathbb{P}(T_x M)$ that is contracted by $\mu^{\mathbb{N}}$-almost every word ω ("Stable distribution is non-random" in ν).*

1. Uniform expansion (UE) implies hyperbolicity and rules out (3).
2. UE and some version of the Hopf argument (related to ideas of Dolgopyat-Krikorian) show that $\nu = \mathrm{vol}$ in (2) (global ergodicity).
3. UE together with techniques (Margulis function) originated from Eskin-Margulis show that the classification of stationary measures implies equidistribution and orbit closure classification.

Result of Brown and Rodriguez Hertz

Thus uniform expansion is stronger than the assumptions of Brown-Rodriguez Hertz. But in some sense this is best possible.

Proposition (C.)

Let M be a closed 2-manifold. Let μ be a measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$. Then μ is uniformly expanding if and only if for every ergodic μ-stationary measure ν on M,

1. *ν is hyperbolic,*
2. *Stable distribution is not non-random in ν.*

Verify uniform expansion

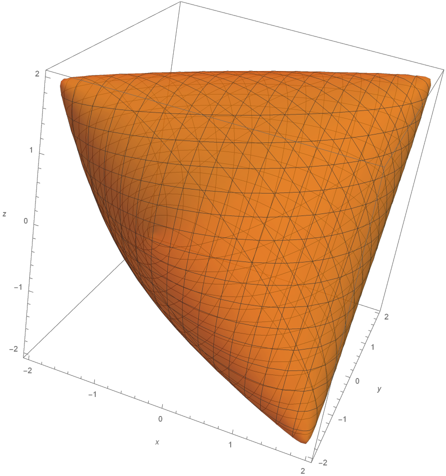
How hard is it to verify the uniform expansion condition? We checked it in two settings:

1. Discrete perturbation of the standard map (verified by hand)
2. $\mathrm{Out}(F_2)$-action on the character variety $\mathrm{Hom}(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$ (verified numerically).

Application: Out(F2)-action on character variety

The character variety $\mathrm{Hom}(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$ can be embedded in $\mathbb{R}^3$ via trace coordinates, with image given by

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - xyz - 2 \in [-2, 2]\} \subset \mathbb{R}^3.$$



Application: Out(F2)-action on character variety

Moreover, under the natural action of $\mathrm{Out}(F_2)$, the ergodic components are the compact surfaces

$$\{x^2 + y^2 + z^2 - xyz - 2 = k\} \subset \mathbb{R}^3$$

for $k \in [-2, 2]$, corresponding to relative character varieties $\mathrm{Hom}_k(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$. Under such identification, the action of $\mathrm{Out}(F_2)$ is generated by two Dehn twists

$$T_X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ xz - y \end{pmatrix}, \qquad T_Y \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ yz - x \end{pmatrix}.$$

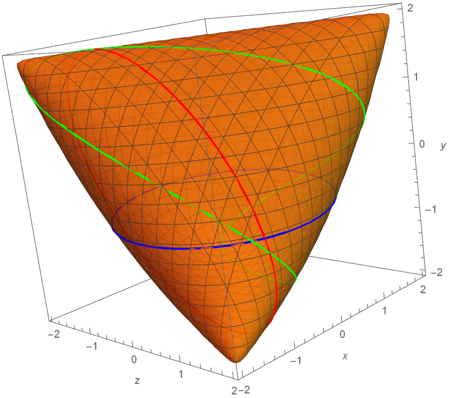
Application: Out(F2)-action on character variety

For $k = 1.99$, the relative character variety is

$$\{x^2 + y^2 + z^2 - xyz - 2 = k\} \subset \mathbb{R}^3$$

with maps

$$T_X(x, y, z) = (x, z, xz - y), \qquad T_Y(x, y, z) = (z, y, yz - x).$$



Application: Out(F2)-action on character variety

Recall that uniform expansion means that there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $P \in M$ and $v \in T_P M$,

$$\int_{\mathrm{Diff}^2(M)} \log \frac{\|D_P f(v)\|}{\|v\|} d\mu^{(N)}(f) > C > 0.$$

Given the explicit form of both the compact surface and the maps, one can verify uniform expansion numerically:

1. Check UE on a grid on the (compact) unit tangent bundle $T^1 M$ using a program,
2. Extend to nearby points by the smooth dependence of the left hand side on $(P, \theta) \in T^1 M$.

Time complexity: $O(\lambda^6 A^2)$, where λ, A are C^1 and C^2 norms of f.

Application: Out(F2)-action on character variety

Theorem (C.)

For k near 2, $\mu = \frac{1}{2}\delta_{T_X} + \frac{1}{2}\delta_{T_Y}$ is uniformly expanding on $Hom_k(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$.

Corollary

For k near 2, let $X = Hom_k(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$, then

- *every $\mathrm{Out}(F_2)$-orbit on X is either finite or dense.*
- *Every infinite orbit equidistribute on X.*
- *Every ergodic $\mathrm{Out}(F_2)$-invariant measure on X is either finitely supported or the natural volume measure.*

Application: Out(F2)-action on character variety

Remark:

1. The topological statement was obtained by Previte and Xia for all $k \in [-2, 2]$ with a completely different method, using crucially the fact that $\mathrm{Out}(F_2)$ is generated by Dehn twists.
2. Our method is readily applicable for proper subgroups Γ of $\mathrm{Out}(F_2)$, including those without any powers of Dehn twists. It is only limited by computational power.
3. Are there faster algorithms to verify uniform expansion? Likely.

Thank you!