Midwest Dynamics and Group Actions Seminar 13/04

Extremal behaviour and spectral radius of random matrix products

1- Joint spectrum 2- Extremal behaviour 3. Spectral rad.

G = reductive lin Lie gp

$S \subseteq G$ → finite / → infinite, bounded

| $G = SL_d(\mathbb{R})$, $d = 2$ or 3. |

* $S^n = \{ g_1 \cdots g_n \mid g_i \in S \} \subseteq G$

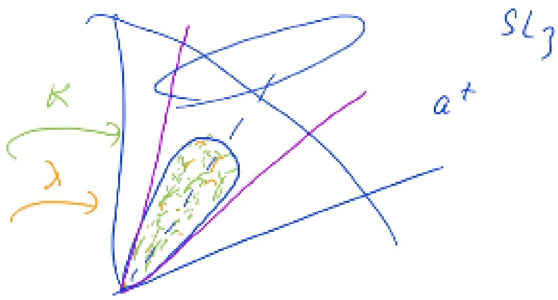
$G = KA^+K$ $\quad G = SL_d(\mathbb{R})$ $\quad K = SO_d(\mathbb{R})$

* $\kappa(g) = (\log \alpha_1, \dots, \log \alpha_d) \in \mathbb{R}^{d-1}$ $\quad A^+ = \left\{ \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_d \end{pmatrix} \;\middle|\; \begin{array}{l} \prod \alpha_i = 1 \\ \alpha_i > 0 \\ \alpha_1 \geq \dots \geq \alpha_d \end{array} \right\}$

$\alpha_1 = \|g\|$ $\qquad \mathfrak{a}^+$

* $\lambda(g) = (\log \lambda_1, \dots, \log \lambda_d) \in \mathbb{R}^{d-1}$ λ_i's are moduli of eigenvalues of g $\quad \lambda_1 \geq \dots \geq \lambda_d$

$\dfrac{\kappa(S^n)}{n}$ $\qquad \dfrac{\lambda(S^n)}{n}$



__Q__: Do these sequences converge for Hausdorff metric? Limit? Same?

__Thm__: (Breuillard - S 18') $\quad \left(\bigcup_{i \geq 1} S^i \right)$

Suppose S generates a semigroup that is Zariski in G. Then, ① $\frac{1}{n}\kappa(S^n)$, $\frac{1}{n}\lambda(S^n)$ converge in Hausdorff metric to the same compact set, called joint spectrum $J(S)$ of S.

② $J(S)$ is compact, convex set of __non-empty interior__ $\subset \mathbb{R}^{d-1}$

__Rk__: Zariski density is necessary for non-empty interior and convexity.

proofs : * make use of Abels-Margulis-Soifer and Benoist

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part 2 : Random matrix products

G ss. Let μ be a proba on G and let $X_1, X_2 \dots$ be iid random variables $\sim \mu$.

$\left(\mu = \frac{1}{2}\delta_{\begin{pmatrix}1&1\\&1\end{pmatrix}} + \frac{1}{2}\delta_{\begin{pmatrix}1&\\1&1\end{pmatrix}}\right)$ $\quad L_n = X_n \dots X_1$.

Furstenberg - Kesten 60s : Under a finite moment assumption

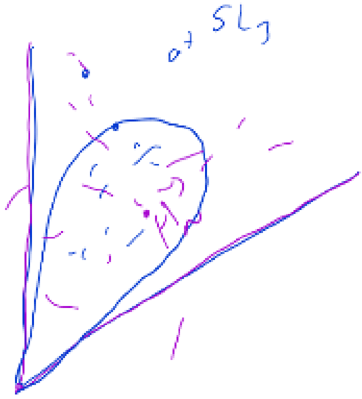
$$\frac{1}{n} K(L_n) \xrightarrow{a.s.} \vec{\lambda}_\mu \in \mathfrak{a}^+$$

Furstenberg (60s) if $\overset{\text{semigroup}}{\text{Supp}(\mu)}$ is Zariski dense then $\vec{\lambda}_\mu \neq 0$. $\quad \lambda_\mu = (\lambda_1, \dots, \lambda_d)$

Guivarc'h-Raugi + Goldsheid-Margulis (80s)

(supp μ) Zariski dense, then $\vec{\lambda}_\mu \notin$ walls.

(simplicity of Lyapunov spectrum)



Breuillard - S 18' : $\quad S = \text{supp}\,\mu$

$\vec{\lambda}_\mu \in \overset{\circ}{J}(S)$ $\quad \frac{1}{n} K(S^n)$ $\quad \frac{1}{n} K(L_n)$

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the proof uses the non-degenerate CLT of Goldsheid-Guivarc'h / Benoist - Quint '16

Fix S and start varying μ

ex : $S = \left\{ \underset{a}{\begin{pmatrix}1&1\\&1\end{pmatrix}}, \underset{b}{\begin{pmatrix}1&\\1&1\end{pmatrix}} \right\}$ $\quad \mu_p = p\,\delta_a + (1-p)\,\delta_b$

$\rho(S)$ = joint spectral radius of S

$$= \lim_{n\to\infty} \sup_{g \in S^n} \|g\|^{1/n}$$

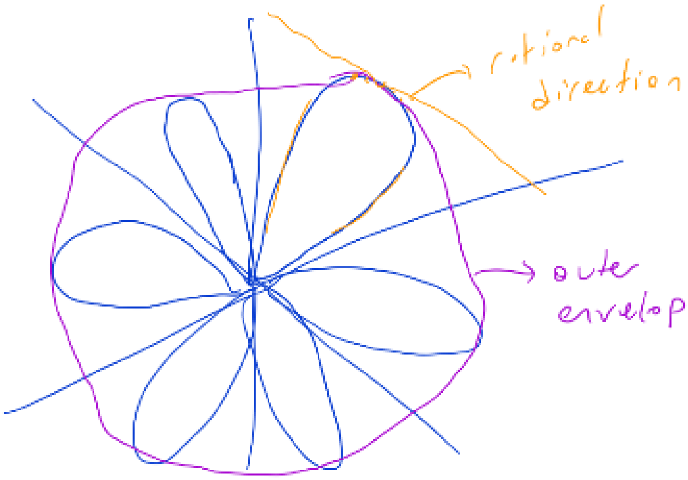
On the other hand, by subadditive ergodic theory, $\exists$ always an ergodic proba μ on $S^{\mathbb{N}}$ s.t. $\rho(S) = \lambda_1(\mu)$ (Morris 10)

Thm: (Breuillard-S '18) Suppose Z-dense

$\forall x \in \overset{\circ}{J}(S)$ $\exists \mu$ shift ergodic proba on $S^{\mathbb{N}}$ s.t. $\vec{\lambda}_\mu = x$.

Q: Gibbs?

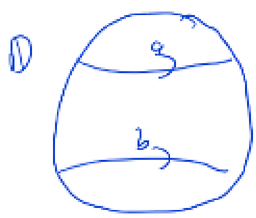
prop:



Bochi-Rams 13': S finite $\subseteq SL_2(\mathbb{R})$

S plays ping-pong, then $\forall \mu$ proba on $S^{\mathbb{N}}$ s.t. $\lambda_1(\mu) = \rho(S)$ has zero entropy.

Breuillard-S (ongoing): $S = \{a, b\} \subseteq SL_2(\mathbb{R})$ ping-pong.



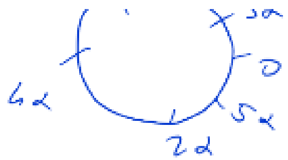
$\Rightarrow$ $\exists!$ proba measure μ s.t. $\lambda_1(\mu) = \rho(S)$ and this proba is a Sturmian proba of some parameter $\alpha \in [0,1)$.

Hedlund-Morse. Fix $\alpha \in [0,1)$ Consider

$T \longrightarrow T$

$u \longrightarrow u$

$x \mapsto x+\alpha$



Each time you cross 0 put 1, 0 otherwise

0 0 1 0 0 1 0 0 0

fact: The orbit closure of this sequence supports a unique shift-inv prob σ, called Sturmian measure of parameter α.

Lagarias-Wang finiteness "conjecture"

$S \subseteq Mat_d(\mathbb{R})$ bdd $\quad \rho(S) = \limsup_{n\to\infty} \left(\sup_{g\in S^n} \lambda_1(g)\right)^{1/n}$ (sup)

Berger-Wang equality

"Conjecture" '95: Given finite S, $\exists\, n_S \in \mathbb{N}$ and $g \in S^{n_S}$ s.t. $\lambda_1(g)^{1/n_S} = \rho(S)$.

Bousch-Mairesse 2002: $\to$ Sturmian measures.

Jenkinson-Pollicott 2016 $\to$ "a lot" of counterexamples.

Breuillard-S: ______.

Strategy: First show Sturmian rigidity $\|$

t ______________ , hit a parameter t

counterexamples Hausdorff dimension 0. for which Sturmian α is irrational.

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part 3: Spectral radius of RMP

$\|\cdot\|$ vs $\lambda_1(\cdot)$

F-K: $\frac{1}{n}\log\|L_n\| \xrightarrow{a.s.} \lambda_1 \in \mathbb{R}$. (L^1)

What happens for spectral radius?

Guivarc'h '90s, Benoist-Quint '16

μ has exponential moment and Γ_μ acts strongly irreducibly on $\mathbb{R}^d$ $\Rightarrow$ $\frac{1}{n}\rho(L_n) \xrightarrow{a.s.} \lambda_1$

Thm: (Aoun - S '19)
When μ has finite 2^{nd} order moment, then $\frac{1}{n}\log\rho(L_n) \xrightarrow{a.s.} \lambda_1 \in \mathbb{R}$.

$\frac{\lambda(S^n)}{n} \xrightarrow{?} J(s)$

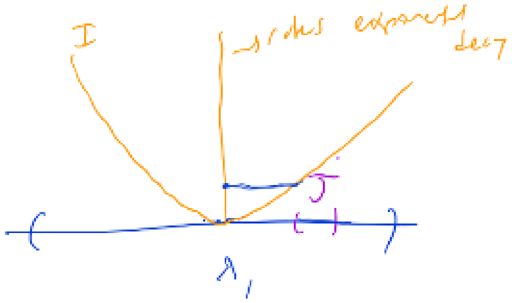
$a = \begin{pmatrix} 2 & \\ & 2^{-1} \end{pmatrix}$ $\sigma = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$S = \{a\sigma, \sigma\}$

Rk: this is for i.i.d products.
This fails for general stationary ergodic product. ||
$\to$ it's not hard to find Markovian counterexamples
(Avila - Bochi 2000)

- **Large deviations:**

$\frac{1}{n}\log\|L_n\| \to \lambda_1$

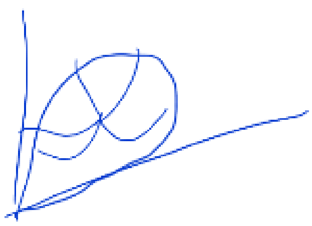


$$\mathbb{P}\left(\frac{1}{n}\log\|L_n\| \in J\right) \longrightarrow 0$$

Le Page, this is happening with exponential speed.

$$\mathbb{P}\left(\frac{1}{n}\log\|L_n\| \in J\right) \sim e^{-n \inf_{\alpha\in J} I(\alpha)}$$

S 16: $\frac{1}{n}\kappa(L_n)$ (λ)
satisfy an LDP with
a convex rate function.



conjecture: $\frac{1}{n}\lambda(L_n)$ satisfy LDP with the same function.

S - Sisto 20': we proved this for rk 1.

$*$ $\underset{a}{\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}}$ $\underset{b}{\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}}$ $0 \longrightarrow \lambda_1(ab)^{\frac{1}{2}}$