Supplementary Information

Supplementary Information relating to Figure 2.

Equations (1) and (2) are integrated numerically, starting from a motionless bridge at equilibrium and $N = 50$ pedestrians with random phases Θ_i, then incrementing N by 10 at each step. Parameters: $M = 1.13\times 10^5$ kg, $B = 1.10\times 10^4$ kg/s, $K = 4.73\times 10^6\,\mathrm{kg/s^2}$, $G = 30\,\mathrm{kg\,m/s^2}$, $C = 16\,\mathrm{m^{-1}s^{-1}}$, $\alpha = \pi/2$, $\Omega_0 = 6.47\,\mathrm{rad/s}$ corresponding to $1.03\mathrm{Hz}$, and $\varsigma = 0.0075$. Normally distributed Ω_i, with mean Ω_0 and standard deviation 0.63 rad/s, corresponding to $0.1\,\mathrm{Hz}$.

Supplementary Methods

We begin by describing the methods used to derive the formula for the critical crowd size:

$$N_c = \frac{4\varsigma}{\pi}\left[\frac{K}{GC\,P(\Omega_0)}\right].$$

The calculation involves three main steps. First, we non-dimensionalize the governing differential equations in a way that brings out the appropriate length and time scales. This scaling also reveals a small parameter in the system, which allows us to make progress by using standard asymptotic methods.

In the second step, these asymptotic methods are applied to the scaled equations to obtain the so-called "averaged equations" that govern the evolution of slowly-varying amplitudes and phases in the problem.

The third step is to calculate all the steady-state solutions of the averaged equations. There are two such solutions: one corresponds to a motionless bridge with desynchronized walkers on it, and the other corresponds to a vibrating bridge with many (but not all) of the pedestrians phase-locked to the bridge's motion. By studying where the wobbling state bifurcates from the motionless one, we obtain the desired formula. The approach used in the third step is a direct extension of a method first developed[1,2] for the analysis of large systems of coupled biological oscillators.

Before proceeding, we wish to clarify that two teams working independently (S.H.S./D.M.A./A.McR. and B.E./E.O.) developed equivalent versions of the model in this paper. The formulation discussed here is that of S.H.S./D.M.A./A.McR.

Scaling. The governing equations, as presented in the main text, are

$$M\frac{d^2X}{dt^2}+B\frac{dX}{dt}+KX=G\sum_{i=1}^{N}\sin\Theta_i$$

$$\frac{d\Theta_i}{dt}=\Omega_i+CA\sin(\Psi-\Theta_i+\alpha),\qquad i=1,\ldots,N\,.$$

To scale the system, we introduce the following parameters

$$\begin{aligned}
\Omega_0 &= \sqrt{K/M}\\
2\varsigma &= B\,\Omega_0/K\\
L_1 &= NG/K\\
L_2 &= \Omega_0/C\\
L &= \sqrt{L_1 L_2}\\
\varepsilon &= \sqrt{\frac{L_1}{L_2}}=\sqrt{\frac{NGC}{K\Omega_0}}
\end{aligned}\tag{1}$$

and dimensionless variables

$$\begin{aligned}
\tau &= \Omega_0 t\\
x &= X/L\\
a &= A/L\,.
\end{aligned}\tag{2}$$

Then the model equations can be rewritten in dimensionless form as

$$\frac{d^2x}{d\tau^2} + 2\varsigma \frac{dx}{d\tau} + x = \varepsilon \langle \sin \Theta_i \rangle \qquad (3)$$

$$\frac{d\Theta_i}{d\tau} = \frac{\Omega_i}{\Omega_0} + \varepsilon a \sin(\Psi - \Theta_i + \alpha), \qquad i = 1, \ldots, N, \qquad (4)$$

where the angular brackets denote an average over all the pedestrians:

$$\langle \sin \Theta_i \rangle = \frac{1}{N} \sum_{i=1}^{N} \sin \Theta_i \, . \qquad (5)$$

The advantage of the scaling used here is that a parameter ε appears in front of *both* the coupling terms on the right hand side of equations (3), (4). Hence, in the limit $\varepsilon \to 0$ the system decouples completely and thereby reduces to a much more tractable form.

To render the system suitable for averaging theory, we simplify it further by supposing that

$$\varsigma = \varepsilon b \qquad (6)$$

and

$$\frac{\Omega_i}{\Omega_0} = 1 + \varepsilon \omega_i \, . \qquad (7)$$

These assumptions are valid if the damping is weak and the pedestrians have nearly identical walking frequencies that also happen to be close to the bridge's resonant frequency. (All of these conditions hold for the Millennium Bridge.) The detunings ω_i are distributed with a probability density $p(\omega)$, obtained from the original distribution of walking frequencies via the transformations $p(\omega)d\omega = P(\Omega)d\Omega$ and $\Omega = \Omega_0(1 + \varepsilon\omega)$, the latter coming from equation (7).

Next, define new phase variables, θ_i and ψ, by viewing the dynamics in a frame rotating at the resonant frequency:

$$\theta_i = \Theta_i - \tau \tag{8}$$

$$\psi = \Psi - \tau. \tag{9}$$

Then the model's governing equations (3), (4) become

$$\frac{d^2x}{d\tau^2} + x = \varepsilon\left[\left\langle \sin(\tau + \theta_i)\right\rangle - 2b\frac{dx}{d\tau}\right] \tag{10}$$

$$\frac{d\theta_i}{d\tau} = \varepsilon\left[\omega_i + a\sin(\psi - \theta_i + \alpha)\right], \qquad i = 1, \ldots, N. \tag{11}$$

Averaging theory. If $\varepsilon = 0$, the solution of equation (10) would be $x(\tau) = a\sin(\tau + \psi)$, where a and ψ are constants. Likewise, the solution of equation (11) would be $\theta_i(\tau) = \text{const}$. For the case of interest, where $0 < \varepsilon << 1$, the amplitude and phase variables are not quite constant—they have time derivatives of order ε. But they are *almost* constant in the sense that they vary slowly, on a time scale $\tau = O(1/\varepsilon)$. The method of averaging[3] is a standard procedure for deriving the equations governing these slow variations. Following the usual approach, we find that

$$\dot{a} = -ba - \tfrac{1}{2}\left\langle \sin(\psi - \theta_i)\right\rangle \tag{12}$$

$$a\dot{\psi} = -\tfrac{1}{2}\left\langle \cos(\psi - \theta_i)\right\rangle \tag{13}$$

$$\dot{\theta}_i = \omega_i + a\sin(\psi - \theta_i + \alpha), \qquad i = 1, \ldots, N, \tag{14}$$

where the overdot denotes differentiation with respect to slow time $T = \varepsilon\tau$. (All of this presupposes that ε truly is small for the parameter values appropriate to the Millennium Bridge; we'll see below that this is the case.)

Speaking in more physical terms, the averaged equations describe the gradual evolution of the bridge vibrations and the crowd synchronization process. Both of these are intuitively expected to be slow, in the sense that they would seem to require tens or possibly hundreds of footsteps by the walkers before they could grow significantly. Our analysis will show that these processes are indeed slower than a single footstep, but not very much slower; the fast and slow time scales are separated by only about one order of magnitude.

Stationary solutions. The averaged equations (12)–(14) can be viewed as a nonlinear dynamical system with a state space of dimension $N+2$, and parametrized by the random detunings ω_i. Given all this complexity, there is no hope of solving these equations in general. Nevertheless, a great deal can be learned by adopting a statistical approach in the spirit of kinetic theory, fluid mechanics, or traffic flow. This is the method that has often been used in the study of coupled biological oscillators, and in fact, equations (12)–(14) are very similar to a classic model in that field, known as the Kuramoto model[1,2].

It turns out, remarkably, that in the special case where $\alpha = \pi/2$ and $p(\omega)$ is even and unimodal, the system has steady states precisely isomorphic to those in the Kuramoto model. In other words, our problem can be mapped onto that one, at least as far as steady states are concerned (the transient behaviour is somewhat different, however). For simplicity, we'll restrict attention to this special case from now on, although we have found that the method can be readily extended to arbitrary α and $p(\omega)$.

To uncover the connection to the Kuramoto model, let us look for statistically steady solutions of equations (12)–(14), for $N >> 1$. These will have a time-independent value of a (to be determined self-consistently, as part of the solution). Ordinarily, the corresponding condition for ψ would be that $\dot{\psi} = q$, where q is some unknown constant that also has to be found self-consistently. But here, because of the particularly felicitous special case we are considering, it turns out that one can insist on a self-consistent solution with constant ψ; that is, we can impose $\dot{\psi}=0$ a priori. Then, because the averaged equations possess a global rotational symmetry (they are left unchanged by $\psi \to \psi + \psi_0$, $\theta_i \to \theta_i + \psi_0$ for any constant ψ_0), we can fix ψ equal to any value we like, without loss of generality. We choose

$$\psi(T) \equiv -\pi/2 \tag{15}$$

so that $\psi + \alpha = 0$ and hence equation (14) conveniently reduces to

$$\dot{\theta}_i = \omega_i - a\sin\theta_i, \qquad i = 1,\ldots,N. \tag{16}$$

Also, by setting $\dot{a}=0$ in equation (12) and recalling $\alpha = \pi/2$, we find

$$\begin{aligned} ba &= \tfrac{1}{2}\left\langle \sin(\theta_i + \tfrac{\pi}{2})\right\rangle \\ &= \tfrac{1}{2}\left\langle \cos\theta_i \right\rangle. \end{aligned} \tag{17}$$

Likewise, because ψ is constant, equation (13) yields

$$\left\langle \sin\theta_i \right\rangle = 0. \tag{18}$$

Equations (16)–(18) are identical, up to change of notation, with equations found in the solution of the Kuramoto model[1,2]. Borrowing results known for that model, one can prove that there are two stationary states for our system.

One of them has $a=0$ (the bridge is completely motionless) and the phases θ_i are uniformly dispersed over the interval $0 \le \theta \le 2\pi$ (the pedestrians are totally desynchronized from one another). This state exists for all values of the damping parameter b, and no matter how broad or narrow the distribution $p(\omega)$ is. It is not necessarily stable, of course—in fact, it is stable only if the dimensionless damping is large enough, or the dimensionless detuning distribution is broad enough.

The other stationary state has $a>0$ (the bridge is wobbling). Now the pedestrians split into two groups, corresponding to a state of partial synchronization within the population. Walkers with detunings $|\omega_i| \le a$ (that is, with intrinsic walking frequencies sufficiently close to the resonant frequency of the bridge) are the "sensitive" people who lock in to the wobbles of the bridge, as can be seen from equation (16). They lock at final phases θ_i^* given by

$$\sin\theta_i^* = \omega_i / a \tag{19}$$

and therefore drive the bridge with an oscillating force $G\sin(\Omega_0 t + \theta_i^*)$.

To understand the physical significance of this result, recall that the bridge displacement is given by $X = A\sin\Psi = A\sin(\Omega_0 t + \psi) = -A\cos\Omega_0 t$, where the last equality follows from equation (15); hence the bridge velocity is $dX/dt = \Omega_0 A\sin\Omega_0 t$. Comparing this to the excitation force $G\sin(\Omega_0 t + \theta_i^*)$ generated by a single phase-locked pedestrian, we see that the excitation force leads or lags the bridge velocity by an amount given by equation (19). For example, the perfectly resonant pedestrians (whose detunings are exactly zero) exert a force that is precisely in phase with the bridge velocity, and therefore pump energy into the wobbles with maximum efficiency. The other phase-locked pedestrians lead or lag the bridge velocity by an amount that increases with their detuning but still results in a net transfer of energy into the vibrations of the bridge.

Now we turn to the desynchronized pedestrians, whose detuning falls in the range $|\omega_i| > a$. These people never fall into synchrony with the bridge's vibrations. Instead, they drift monotonically through all phases relative to the bridge, though they hesitate longest near a relative phase relationship of $(\pi/2)\,\mathrm{sgn}(\omega_i)$ (the phase where their speed $|\dot{\theta}_i|$ is minimized in equation (16)). The incessant phase drift of these pedestrians would seem to violate the original assumption that the system as a whole is statistically steady, but the stationarity condition can be enforced nonetheless. The key is to require that these pedestrians distribute their phases according to a stationary density $\rho(\theta,\omega)$.

The condition for stationarity (just as in traffic flow or fluid mechanics) is that the density at a given θ must be inversely proportional to the speed $|\dot{\theta}|$ there, as prescribed by equation (16). Thus

$$\rho(\theta,\omega)=\frac{1}{2\pi}\frac{\sqrt{\omega^2-a^2}}{|\omega-a\sin\theta|}, \qquad \text{for } |\omega|>a \tag{20}$$

where the normalization constant has been chosen to ensure that $\int_0^{2\pi}\rho(\theta,\omega)\,d\theta=1$ for all $|\omega|>a$.

Having characterized the phase distributions of both the locked and desynchronized parts of the crowd, it remains only to make sure that equations (17) and (18) are satisfied. Both of those equations involve averages over the entire population, and hence will involve two contributions—one from pedestrians who are locked to the bridge's vibrations, and another from those who are desynchronized.

It is easy to check that equation (18) is satisfied automatically if we assume (as we have been) that $p(\omega)=p(-\omega)$ for all ω; this is where the symmetry assumption on the detunings, and hence on the original walking frequencies, comes into play.

Thus equation (17) is thus the only remaining equation to be satisfied. It determines the self-consistent amplitude a of the bridge's steady-state vibrations, a quantity that until now has been fixed but unknown. Writing the average in equation (17) as a sum of contributions from the locked and desynchronized pedestrians, we obtain the self-consistency equation

$$2ba=\int_{-a}^{a}\cos\theta\,p(\omega)d\omega\;+\;\frac{1}{2\pi}\int_{|\omega|>a}\sqrt{\omega^2-a^2}\left[\int_0^{2\pi}\frac{\cos\theta\,d\theta}{|\omega-a\sin\theta|}\right]p(\omega)\,d\omega\;, \tag{21}$$

where, in the first integral, ω and θ are invertibly related by $\omega=a\sin\theta$ (because of equation (19)). One can check that the term in brackets in the second integral vanishes—this is equivalent to the physically plausible statement that the desynchronized pedestrians do no net work on the bridge. Therefore, after substituting $\omega = a\sin\theta$ in the first integral, we find

$$2ba = \int_{-\pi/2}^{\pi/2} \cos\theta \, p(a\sin\theta) a\cos\theta \, d\theta \ , \tag{22}$$

where we notice that both sides contain a constant multiplicative factor of a. Thus, as claimed, there are two types of steady solutions:

- $a = 0$: The bridge is motionless, in which case equation (20) yields $\rho(\theta,\omega) \equiv 1/(2\pi)$, meaning that all the pedestrians are desynchronized and uniformly scattered in phase;
- $a > 0$: The bridge is wobbling with dimensionless amplitude a given implicitly by the solution of

$$2b = \int_{-\pi/2}^{\pi/2} \cos^2\theta \, p(a\sin\theta) \, d\theta \, . \tag{23}$$

This branch of solutions splits off the desynchronized branch at a critical value of b, which can be found by taking the limit $a \to 0^+$ in equation (23) and evaluating the integral. We find

$$b_c = \frac{\pi}{4} p(0). \tag{24}$$

Under our assumption that $p(\omega)$ is symmetrical and bell-shaped with a maximum at $\omega = 0$, one can check that the bifurcation is supercritical. That is, solutions of equation (23) exist only for $b \le b_c$, indicating that the motionless state loses stability once the dimensionless damping is too small, after which it gives rise to a stable wobbling state.

The bifurcation condition (24) can be transformed back into the original dimensional variables, using the relationships $p(\omega)d\omega = P(\Omega)d\Omega$ and $\Omega = \Omega_0(1+\varepsilon\omega)$, along with the definitions of ε and b in equations (1) and (6). The result is the desired formula for the critical crowd size,

$$N_c = \frac{4\varsigma}{\pi}\left[\frac{K}{GC\,P(\Omega_0)}\right]. \tag{25}$$

Interpretation and testable aspects of the model. There are various assumptions and testable features of the model that warrant further comment.

For example, the equation for the bridge dynamics (Eq.(1) in the main text) assumes that the pedestrians drive the bridge by imparting sideways forces on it. This force is known[4-7] to be quite small compared with the front-to-back force generated during walking, which in turn is quite small compared with the up-and-down force generated during walking. So one could reasonably ask whether these other loads helped to drive the bridge, perhaps through interesting modal interactions with the bridge. This possibility was considered during the early investigation on the bridge, back in 2000. However, it was generally concluded that the modal interactions did not play a crucial role in the opening day events. For example, on p.26 in Dallard et al.[4], the authors estimate the effect of vertical and torsional forces and show that "the lateral component of load was by far the most significant." Furthermore, experiments by McRobie et al.[6] on a large rig suspended from vertical cables (which had only one degree of freedom – a purely lateral motion) were able to replicate the synchronization and instability phenomenon – thereby demonstrating that lateral forces, rather than vertical or longitudinal, were the important effect.

The hypothetical equation for the walker dynamics (Eq.(2) in the main text) is much more tentative. It should be viewed as a simple, qualitative attempt to describe how people might alter their gait when they are walking on a platform swaying from side to side. The form of the equation is motivated by an analogy with phase-locked loops, lasers, and other nonlinear systems that are able to synchronize to an external periodic drive. Equations of this form have been shown to give a reasonable approximation to the dynamics of various biological rhythms, such as the flash rhythm of southeast Asian fireflies when driven by an oscillating light stimulus[8]. The parameter C in the equation quantifies the impact of a stimulus (here, the bridge vibration) of amplitude A and phase Ψ. Specifically, C controls how fast a pedestrian unconsciously shifts the phase of his walking cycle in response to the sideways oscillations of the platform on which he is walking.

Notice that Eq. (2) also assumes that the rate of phase shifting is directly proportional to the drive strength A. This seems plausible—as the bridge vibrations tend to zero, their effect on the walker should vanish as well. A linear dependence on A is both the simplest and most generic way to incorporate this effect.

By the same token, the parameters C and α should depend on the drive frequency Ω_0. Since the phase-shifting influence of a moving platform should vanish in the limit of imperceptibly slow oscillations, we expect that $C \to 0$ as $\Omega_0 \to 0$. On purely mathematical grounds, one would expect that $C \sim \Omega_0$ for small Ω_0, since this is the generic case if the dependence is smooth. On physical grounds, however, one might expect a quadratic dependence $C \sim (\Omega_0)^2$. This would be the case if the phase shift were controlled by the inertial force experienced by the walker as he is being shaken from side to side.

Finally, on biological grounds, one would expect the sensitivity C to vary across the population, depending on a person's age, size, health, and so on. At the moment we have no way to estimate any of these inter-individual differences and therefore have to resort to estimating a population average for C.

Although Eq.(2) has a black-box character and cannot account for neural properties of the human locomotor system in any detail, it does have the virtue that it is straightforwardly testable. One could conduct biomechanical experiments on individual subjects, similar to those reported in ref. [6], or like those conducted at Imperial College and cited in Dallard et al.[5] (but whose results have not yet been fully published, to the best of our knowledge). The idea is to put subjects, one at a time, on platforms vibrating with a fixed, controlled amplitude A and prescribed driving frequency (so $\Psi = \Omega_0 t$ in Eq. (2) of the main text). One then records the footfalls and phases of the subjects during the transient period as they adjust their gaits to the vibrations of the platform. From that data one could extract a fitted C and α for each person, at a given amplitude A and driving frequency Ω_0. We believe that the prefactor of A in Eq.(2) should account for the main dependence on amplitude; this could be tested directly by checking whether C and α are roughly constant as the amplitude of excitation varies. (The algebraic dependence of C and α on Ω_0 could also be extracted in a similar way.) The subject's natural footfall frequency Ω_i could be measured directly by observing his normal gait on a motionless walkway. If the sinusoidal form of Eq.(2) turns out to be too crude, one could concoct more elaborate functional forms of the coupling term to see which best fits the recorded data.

Testable consequences. In the remainder of this Supplementary Information, we summarize the calculations used to derive various testable consequences of the model, three of which were mentioned in the main text (characteristic amplitude of the bridge's vibrations; synchronization time scale; and the empirical law that the correlated excitation force produced by the pedestrians is proportional to the magnitude of the bridge's velocity).

All the parameters in the model are known, in some cases only roughly, except for the biomechanical parameters C and α. They could be determined by measuring the changes in gait of individual subjects as they adjust to walking on vibrating platforms driven at a specified amplitude and frequency. Lacking such data (for now, at least), we have chosen to fix $\alpha = \pi/2$ for simplicity, and to estimate C from an experiment conducted on the Millennium Bridge. Our estimates could well need modification when the relevant data become available, but in the meantime, they seem to account reasonably well for diverse observations on opening day and in subsequent controlled experiments.

Fitting the formula (25) to the observed value of N_c, assuming that all other parameters have the values measured for the north span of the Millennium Bridge (before it was retrofitted with dampers) yields, as a first estimate, that $C \approx 15\,\mathrm{m}^{-1}\mathrm{s}^{-1}$. This estimate needs to be refined slightly, however, to account for an aspect of the procedure used in the experiments. Recall that the number of the people on the bridge was progressively incremented in a staircase fashion, as shown in Fig. 2a in the main text. This approach tends to overestimate N_c because it increases N faster than the system's transient dynamics can keep up with. The difficulty becomes especially acute where it matters the most—at the critical crowd size, where the eigenvalue governing transient growth approaches zero and the synchronization time scale tends to infinity (see below). In our simulations, we find that a value of C closer to 16 or 17 gives results that more faithfully reproduce the experimental data on the transient growth of the vibrations. Therefore, in what follows, we will assume $C \approx 16\,\mathrm{m}^{-1}\mathrm{s}^{-1}$.

Characteristic amplitude of vibrations. With all the parameters now fixed, the model makes a number of firm predictions. The characteristic length scale is given by

$$L = \sqrt{\frac{NG\Omega_0}{KC}} \approx 20\text{mm}\,, \tag{26}$$

where we are assuming $N = 160$, $G = 30\,\text{kg}\,\text{m/s}^2$, $\Omega_0 = 2\pi(1.03)\text{rad/s}$, $K = M\Omega_0^2$ with $M = 113{,}000\,\text{kg}$, and $C = 16\,\text{m}^{-1}\text{s}^{-1}$, all of which are appropriate for the experiment conducted on the north span.

The length scale L sets the typical size of the bridge's vibrations, and compares well with the data shown in Fig.4 of Dallard et al.[4], where the maximum acceleration seen was 80 milli-g, equivalent to an amplitude of 20 mm. Note that this experiment was stopped (for safety concerns) before the acceleration had reached a steady state; otherwise even larger vibrations and accelerations would have occurred. Also, for comparison, the maximum displacements observed on opening day were about 70 mm on the center span, and 50 mm on the south span, though with considerably larger numbers of pedestrians.

Synchronization time scale. The model predicts that the system has three important time scales.

The fast time scale is set by the frequency of a pedestrian's walking cycle and by the resonant frequency of the bridge. Since the dimensionless fast time variable is $\tau = \Omega_0 t$, the fast time scale can be taken as $t_{fast} = 2\pi/\Omega_0$. Thus the fast time scale is on the order of one second.

The slow time scale controls how long it takes for the crowd and the bridge to interact substantially. Since the slow time variable is $T = \varepsilon\tau = \varepsilon\Omega_0 t$, the slow time scale is $t_{slow} = 2\pi/\varepsilon\Omega_0$ and hence

$$t_{slow}/t_{fast} = 1/\varepsilon\ .$$

Substituting the parameter values from above, we find

$$\varepsilon = \sqrt{NGC/K\Omega_0} \approx 0.05\,, \tag{27}$$

which is indeed small compared to unity, as the theory supposed from the start. Thus, we see that the slow time scale is about 20 times longer than the fast one, and hence is on the order of tens of seconds. This time scale is comparable to that observed for the growth of bridge oscillations in the crowd tests reported by Dallard et al. (for example, see Figures 15 and 17 in ref. [5].)

The third time scale, which we call the onset time scale, is super-slow. A limiting case of the slow time scale, it operates only very close to the bifurcation point where the bridge switches from stable to unstable. As can be shown by asymptotic analysis of equation (43) below, at the onset of instability the crowd synchronization and bridge vibrations grow exponentially, but with a divergent time constant given asymptotically by

$$t_{onset} \sim \frac{1+(4\sigma_\omega^2)^{-1}}{\varsigma\Omega_0}\left(\frac{N_c}{N-N_c}\right) \tag{28}$$

as $N \to N_c$ from above. Here σ_ω is the (dimensionless) standard deviation of the detunings; for pedestrians with Gaussian intrinsic frequencies of mean 1.0 Hz and standard deviation $\sigma_f = 0.1$ Hz, we find that $\sigma_\omega = (2\pi/\varepsilon\Omega_0)\sigma_f \approx 2$. In particular, when $C = 16$, we find that $N_c = 148$; hence, for $\varsigma = 0.0075$ and $\Omega_0 = (2\pi)(1.03)$ we obtain

$$t_{onset} \approx \frac{3200}{N-N_c}\ \text{sec}\ . \tag{29}$$

Thus, for $N = 164$, say, we expect that the synchronization order parameter $R(t)$ will require a super-slow time scale on the order of $3200/(164-148) = 200$ sec to grow by a factor of e. Note that this time is longer than the length of one of the staircase plateaus in Fig. 2a. In other words, near the critical crowd size, people were being added to the bridge faster than the system could respond. This explains why the staircase procedure tends to overestimate N_c.

In reality, the super-slow time scale may not be quite as long as our estimate suggests. In Fig.4 of Dallard et al.[4], it took about 60 seconds for the vibration amplitude to reach a level of 20 mm after the critical crowd size had been exceeded, whereas our simulation (Fig.2 of the main text) predicts a time scale 2-3 times longer than that. Perhaps it would be too optimistic to hope for better agreement, given the many simplifications and uncertainties in the model. For example, we have assumed that pedestrians exert a maximum lateral force G on the bridge, independent of how much the bridge is wobbling. In fact, G is known to increase with the vibration amplitude A, as people widen their stance and change their gait to keep their balance[5,6]. Incorporating this effect would speed up the super-slow time scale, as desired, by strengthening the positive feedback loop between synchrony and wobbling, thereby making the dynamics even more unstable. Another source for the discrepancy could be uncertainties in the model parameters, especially C and α. Finally, our simulations show that the critical crowd size and super-slow time scale vary sensitively from one run to the next, depending on the order in which pedestrians are added to the bridge (bear in mind that different pedestrians have different intrinsic frequencies Ω_i, sampled at random from the underlying distribution).

Empirical law F = kV. Dallard et al. [4,5] conducted a series of controlled crowd tests on the Millennium Bridge. A key result of these investigations was the discovery that the crowd exerts a "correlated excitation force" on the bridge that increases in proportion to the bridge's velocity. Thus, the more the bridge moves, the more the crowd pushes it to move further. We find that an effect of this sort occurs in our model as well, as a consequence of the crowd's increasing synchronization.

First we need to find our model's versions of the relevant quantities. The bridge velocity is given by

$$dX/dt = \Omega_0 A\cos(\Omega_0 t + \psi)\,, \tag{30}$$

to leading order in ε; that is, we are neglecting the slow variations in amplitude and phase that contribute an $O(\varepsilon)$ correction to this dominant term. Hence the magnitude of the velocity (itself a slowly-varying function of time) is given by

$$V(T) = \Omega_0 A(T) = \Omega_0 La(T)\,, \tag{31}$$

where $a(T)$ is the dimensionless amplitude variable that appears in the averaged equations (12)–(14).

The correlated excitation force F (per person) is found in three steps: first we write the excitation force per person, given by $G\langle\sin\Theta_i\rangle$; then we take the component of this force that is in phase with the bridge's velocity; and finally we take the magnitude of that in-phase (or "correlated") component. To extract the component in phase with the velocity in equation (30), we write

$$\begin{aligned}
G\langle\sin\Theta_i\rangle &= G\langle\sin(\Omega_0 t + \theta_i)\rangle \\
&= G\langle\sin(\Omega_0 t + \psi + (\theta_i - \psi))\rangle \\
&= G\sin(\Omega_0 t + \psi)\langle\cos(\theta_i - \psi)\rangle + G\cos(\Omega_0 t + \psi)\langle\sin(\theta_i - \psi)\rangle
\end{aligned}$$

which shows that the second term in the last line is the in-phase component. Its magnitude is therefore

$$F = G\langle \sin(\theta_i - \psi)\rangle . \tag{32}$$

This can be conveniently expressed in terms of the complex order parameter[1,2]

$$Re^{i\phi} = \langle \exp(i\theta_j)\rangle \tag{33}$$

as

$$\begin{aligned} F &= G\langle \sin(\theta_i - \psi)\rangle \\ &= GR\sin(\phi - \psi) \end{aligned} \tag{34}$$

where F, R, ψ, ϕ and θ_j are all functions of slow-time T.

Equation (34) highlights the crucial role that synchronization plays in generating the correlated force: it shows that F is proportional to R, the amount of phase coherence among the pedestrians. Thus, the more synchronized the crowd becomes, the more force they collectively impart to the bridge.

The empirical law found by Dallard et al.[5] states that $F \approx kV$, meaning that the ratio F/V is approximately constant during the transient when the wobbles are building up and the crowd is becoming synchronized. From equations (31) and (34), we see that the model's counterpart of this ratio is

$$\frac{F}{V} = \left[\frac{G}{\Omega_0 L}\right]\frac{R(T)}{a(T)}\sin\big(\phi(T) - \psi(T)\big), \tag{35}$$

which is *not* constant with respect to T, but which is *almost* constant, as we'll show next. To do so, we need to look more closely into the dynamics implied by the averaged equations.

An important feature of dynamics is that the averaged equations (12)–(14) possess an antisymmetric invariant manifold[3] in the continuum limit $N \to \infty$. This follows from our earlier assumptions that $\alpha = \pi/2$ and that $p(\omega)$ is even. Here's an intuitive explanation: suppose the detunings come in antisymmetric pairs, $\omega_i = -\omega_{-i}$, consistent with the evenness of $p(\omega)$, and suppose we start the system with $\psi(0) = -\pi/2$, $\phi(0) = 0$ and with all the corresponding phases also arranged antisymmetrically: $\theta_i(0) = -\theta_{-i}(0)$. Then equations (12)–(14) imply that these conditions will continue to hold for all time. In other words, a solution to the system that starts on this manifold will stay on it forever. Therefore, for initial conditions restricted to this manifold, we can set $\psi(T) \equiv -\pi/2$ and $\phi(T) \equiv 0$ for all T. Hence the dynamics on the invariant manifold reduce to

$$\dot{a} = \tfrac{1}{2} R - ba \tag{36}$$

$$\dot{\theta}_i = \omega_i - a \sin\theta_i \ , \quad i = 1, \ldots, N, \tag{37}$$

where

$$R = \langle \cos\theta_i \rangle, \tag{38}$$

and equation (35) simplifies to

$$\frac{F}{V} = \left[\frac{G}{\Omega_0 L} \right] \frac{R(T)}{a(T)} \quad . \tag{39}$$

The constancy of the ratio F/V thus boils down to the constancy of the ratio $R(T)/a(T)$, or in physical terms, the ratio between the amount of phase synchronization and the amplitude of wobbling, at any given time. Figures 2b and 2c in the main text already illustrated that these two functions follow nearly parallel time courses, as if one were just a constant multiple of the other. Thus, on numerical grounds, we have seen evidence for the approximate constancy of the ratio in equation (39), and hence for the empirical law in question. What remains is to understand why this near-constancy is logically implied by the governing equations.

The remainder of the calculation will only be sketched, because the details are too lengthy to include here. Our numerics suggest that that the invariant manifold is locally attracting and hence provides a representative picture of the dynamics for a wide range of initial conditions. Next we take the continuum limit of the model, for which exact results can be obtained. As in the analysis of the Kuramoto model[1,2], the continuum limit is phrased in terms of a density $\rho(\theta,\omega,T)$, which describes how many people are at a particular phase θ in their walking cycle at time T, given that they have an intrinsic detuning ω. The evolution equation for this density, analogous to a continuity equation in fluid mechanics, is

$$\frac{\partial\rho}{\partial T} = -\frac{\partial}{\partial\theta}(\rho v) \tag{40}$$

where the velocity field in phase space is given by the Eulerian version of equation (37):

$$v(\theta,\omega,T) = \omega - a\sin\theta\,, \tag{41}$$

and the order parameter in equation (38) becomes

$$R = \int_0^{2\pi}\int_{-\infty}^{\infty} \cos\theta\,\rho(\theta,\omega,T)\,p(\omega)\,d\omega\,d\theta\,, \tag{42}$$

by the law of large numbers. Thus the flow on the invariant manifold is now given by the equations (36) and (40)–(42).

The fixed points of the continuum system are precisely the stationary densities discussed earlier, with the advantage that we are now in a position to analyze their linear stability.

In particular, consider the linearization about the state with the crowd completely desynchronized ($\rho = 1/2\pi$, $R = 0$) and the bridge motionless ($a = 0$). We find that this state loses stability when a single eigenvalue λ passes through zero onto the positive real axis. Then $R(T) = R_0 e^{\lambda T}$ and $a(T) = a_0 e^{\lambda T}$, where λ satisfies

$$\lambda + b = \tfrac{1}{4}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2+\omega^2}\, p(\omega)d\omega \quad . \tag{43}$$

The analysis shows further that along the unstable eigenvector, R and a are related by

$$R(T) = 2(b+\lambda)\, a(T) . \tag{44}$$

Therefore, for parameters just slightly above the instability, where λ is near 0, equation (44) implies that

$$R(T) \approx 2b\, a(T) \tag{45}$$

throughout the initial exponential growth away from the desynchronized state.

But that last result is remarkable, because the same proportionality holds at long time as well: equation (36) shows that as $T \to \infty$, the system settles to a fixed point (representing a steadily wobbling bridge and partially synchronized crowd) for which $R(\infty) = 2b\, a(\infty)$. In other words, the eventual steady state at long time lies on precisely the same diagonal line ($R = 2b\, a$) as the unstable eigendirection governing the initial exponential growth. Hence $R(T)$ and $a(T)$ stay close to this diagonal line, and therefore remain in nearly constant ratio, at both the beginning and end of their time evolution. In between, much the same is true: we find numerically that although $R(T) > 2b\, a(T)$, the two variables still hug the diagonal, hovering only slightly above it. So at all times, $R(T)$ and $a(T)$ are in nearly constant ratio. This explains why the right hand side of equation (39) stays nearly constant, and hence why the empirical law $F \approx kV$ holds in our model.

1. Kuramoto, Y. *Chemical Oscillations, Waves and Turbulence* (Springer-Verlag, Berlin, 1984).
2. Strogatz, S. H. *Physica D* **143,** 1– 20 (2000).

3. Guckenheimer, J. & Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983).

4. Dallard, P., et al. *J. Bridge Eng.* **6(6),** 412–417 (2001).

5. Dallard, P., et al. *Structural Engineer* **79(22),** 17–33 (2001).

6. McRobie, A., Morgenthal, G., Lasenby, J. & Ringer, M. *Proc. Inst. Civ. Eng. Bridge Eng.* **156(2),** 71–79 (2003).

7. Newland, D. E. *Proc. Instn Mech. Engrs* **218(5),** 477–492 (2004).

8. Ermentrout, G.B. & Rinzel, J. *Am. J. Physiol.* **246**, R102-R106 (1984).

Correspondence should be addressed to S.H.S. (strogatz@cornell.edu).