

# Supplemental Material: Symmetry breaking in optimal timing of traffic signals on an idealized two-way street

Mark J. Panaggio,<sup>1,\*</sup> Bertand J. Ottino-Löffler,<sup>2</sup> Peiguang Hu,<sup>3</sup> and Daniel M. Abrams<sup>1,4</sup>

<sup>1</sup>*Department of Engineering Sciences and Applied Mathematics,  
Northwestern University, Evanston, IL 60208, USA*

<sup>2</sup>*Department of Mathematics, Caltech, Pasadena, CA 91126, USA*

<sup>3</sup>*Engineering Science Program, National University of Singapore, Singapore*

<sup>4</sup>*Northwestern Institute on Complex Systems, Northwestern University, Evanston, IL 60208, USA*

(Dated: August 12, 2013)

## S1. DERIVATION OF EASTBOUND EFFICIENCY

Recall that the length of a trip is determined by  $N_L$  (a trip represents the periodic trajectory in which a vehicle passes through  $N_L$  lights before stopping and waiting for a time  $W$ ) which satisfies the following.

$$(N - 1/2) \leq N_L (r_C - r_\Delta) < N . \quad (\text{S1})$$

For simplicity, we define  $M = r_C - r_\Delta$ . We can eliminate  $N$  from the above expression by noting that it is the smallest integer that satisfies

$$N_L M < N \leq N_L M + \frac{1}{2}$$

for a positive integer  $N_L$ . Clearly this is satisfied if  $N = \lceil N_L M \rceil$  where  $N_L$  satisfies the equation

$$\left\lceil N_L M + \frac{1}{2} \right\rceil = \lceil N_L M \rceil$$

This equation is periodic in  $M$ . This can be verified by noting that we can write  $M = \lfloor M \rfloor + \{M\}$ . Expanding the arguments of both sides of the equation and simplifying integer terms, we note that the resulting equation depends only on the fractional part  $\{M\}$

$$\left\lceil N_L \{M\} + \frac{1}{2} \right\rceil = \lceil N_L \{M\} \rceil \quad (\text{S2})$$

It is straightforward to show that the solution to this equation is  $N_L = \lceil \frac{1}{2\{M\}} \rceil$ . Suppose  $N_L < \frac{1}{2\{M\}}$ . Then

$$\begin{aligned} \frac{1}{2} \leq N_L \{M\} + \frac{1}{2} < \frac{1}{2\{M\}} \{M\} + \frac{1}{2} = 1 &\implies \left\lceil N_L \{M\} + \frac{1}{2} \right\rceil = 0 \text{ and} \\ 0 \leq N_L \{M\} < \frac{1}{2\{M\}} \{M\} = 1/2 &\implies \lceil N_L \{M\} \rceil = 1. \end{aligned}$$

So  $N_L < \frac{1}{2\{M\}}$  does not satisfy equation S2. Now suppose  $N_L = \lceil \frac{1}{2\{M\}} \rceil$  (the smallest integer greater than or equal to  $\frac{1}{2\{M\}}$ ). If  $\{M\} > 1/2$ , then  $N_L = 1$  satisfies the equation, so the assumption is satisfied. If  $\{M\} \leq 1/2$ , then

$$\begin{aligned} 1 = \frac{1}{2\{M\}} \{M\} + \frac{1}{2} \leq N_L \{M\} + \frac{1}{2} < \left( \frac{1}{2\{M\}} + 1 \right) \{M\} + \frac{1}{2} < 1 + \{M\} &\implies \left\lceil N_L \{M\} + \frac{1}{2} \right\rceil = 1 \text{ and} \\ 1/2 = \left( \frac{1}{2\{M\}} \right) \{M\} \leq N_L \{M\} < \left( \frac{1}{2\{M\}} + 1 \right) \{M\} < \frac{1}{2} + \{M\} < 1 &\implies \lceil N_L \{M\} \rceil = 1. \end{aligned}$$

So,  $N_L = \lceil \frac{1}{2\{M\}} \rceil$  must be the smallest integer solution to Eq. (S2).

## S2. EXPLORATION OF DISCONTINUITIES

In our model, an isolated vehicle always travels an integer number of blocks before stopping. This results in a discontinuous eastbound efficiency. For  $r_C = r_\Delta$ ,  $N_L$  is infinite. For  $r_\Delta$  slightly greater than  $r_C$ ,  $N_L$  drops to 1 and then increases for increasing  $r_\Delta \pmod{1}$ .  $N_L$  is always an integer, so it increases in integer increments, and each jump produces a jump in efficiency. Immediately after the jump, the efficiency reaches a local maximum, before decreasing again. It can be shown that these peaks in efficiency are located at the following locations:

$$r_\Delta - r_C = \left\{ 0, 1/2, 3/4, \dots, \frac{2N_L - 1}{2N_L} \right\}. \quad (\text{S3})$$

The first peak  $r_\Delta = r_C$  represents a green wave, the ideal case in which all vehicles travelling in a single direction are able to proceed indefinitely without stopping. The second peak corresponds to timings in which a vehicle leaving a green light at the instant it turns green arrives at the next light immediately before it turns red. Thus the car travels two blocks and then stops for an instant at a red light before it turns green. We call this a ‘‘red wave’’ peak because the absolute minimum efficiency (where cars stop at every red light and wait for the entire red cycle) lies immediately to the left of this peak. The third peak occurs when the car makes it through the first two lights (arriving at the second at the instant before it turns red), but then stops at the third and waits the remainder of the red cycle. Subsequent peaks are analogous.

Between these peaks, the efficiency decays like

$$E = \begin{cases} \frac{N_L r_C}{N_L r_\Delta + 1} & \text{for } r_\Delta < r_C \\ \frac{N_L r_C}{N_L (r_\Delta - 1) + 1} & \text{for } r_\Delta > r_C \end{cases}$$

reaching minima of

$$E = \frac{N_L r_C}{N_L r_C + 1/2}$$

and maxima of

$$E = \frac{N_L r_C}{N_L r_C + 1 - \frac{N_L}{2(N_L - 1)}}.$$

This is illustrated in Fig. S1.

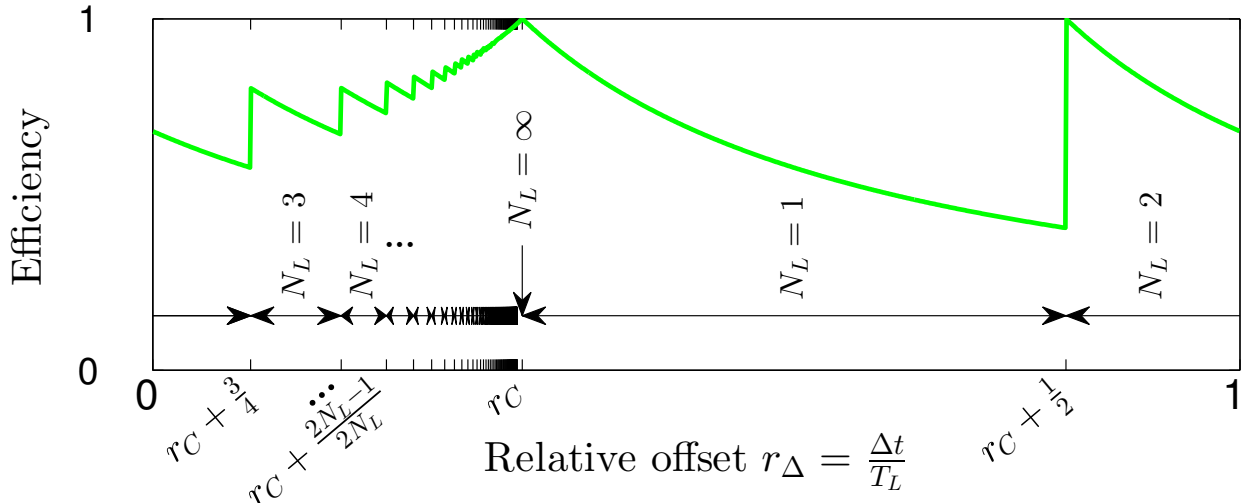


FIG. S1. Peaks of theoretical efficiency. The eastbound efficiency is plotted for  $r_C = 0.34$  with ticks marking the local efficiency maxima.

### S3. COMPUTATION OF BANDWIDTH

In this system, each intersection has a maximum flow rate that is determined by the vehicle speed and the green time. This flow rate limits the length of a platoon that can clear the light in a single cycle. We refer to this as  $L_0$ .

$$L_0 = v \frac{T_L}{2} = \frac{\Delta x T_L}{2T_C} = \frac{\Delta x}{2r_C}$$

Although a platoon of length  $L_0$  is able to clear a single light during one cycle, the platoon may be unable to clear subsequent lights. The theory in the manuscript was derived for a single vehicle, but it is possible to show that there is a platoon length  $L$  (often greater than the length of a single vehicle) for which this theory is valid. In practice, the platoon length can only take on discrete values due to the fact that a platoon must consist of an integer number of vehicles. In this analysis, for simplicity, we allow the platoon length to take on any value (this is equivalent to using a vehicle length of 0). This assumption means that the platoon length used in subsequent calculations can deviate from the actual platoon length by up to one car length. This deviation is negligible as long as the length of a vehicle is small relative to the block length.

We define bandwidth  $B = \frac{L}{L_0}$  as the fraction of the maximum allowable platoon length that can achieve the efficiency from theory. This can also be interpreted as the fraction of vehicles departing during the first green cycle that will be able to achieve the theoretical efficiency. This is simply the standard definition of bandwidth in seconds normalized by the green time  $T_L/2$ . Thus a bandwidth of 1 means that a platoon of length  $L_0$  is able to achieve the efficiency of a single vehicle. A bandwidth of 0 means that only the single car can achieve this efficiency, and that no other vehicles can perform as well.

There are two factors that restrict the bandwidth: the phases of downstream lights upon the arrival of the platoon and the initial phases of upstream lights.

*Downstream bandwidth*—To compute the downstream bandwidth, we note that the first vehicle in a platoon arrives at light  $n$  at time  $nT_C$ . For lights  $1, 2, \dots, N_L - 1$ , the light is green on arrival and the remaining green time restricts the bandwidth. The phase of light  $n$  upon arrival is  $n\omega(T_C - \Delta t)$  which satisfies

$$k_n 2\pi < n\omega(T_C - \Delta t) < (k_n + 1/2)2\pi$$

for integer  $k_n = \lfloor n(r_C - r_\Delta) \rfloor$  where the leftmost expression represents the phase when the light turns green and the rightmost expression represents the phase when the light turns red. The remaining green time is

$$\left(k_n + \frac{1}{2}\right) T_L - n(T_C - \Delta t).$$

We divide this by the total green time  $T_L/2$  for proper normalization. Taking the minimum over the first  $N_L - 1$  lights yields the downstream bandwidth

$$B_{\text{down}} = 2 \min\{ \lfloor n(r_C - r_\Delta) \rfloor + 1/2 - n(r_C - r_\Delta) \}_{n=1}^{N_L-1}.$$

Note that for  $r_C < r_\Delta < r_C + 1/2$ , the downstream bandwidth is 1. This is due to the fact that vehicles encounter a red after traveling a single block. In this region, the upstream lights limit the bandwidth.

*Upstream bandwidth*—The upstream bandwidth is necessary because  $L_0$  may exceed the length of a single block allowing an upstream light to segment the platoon. Given a platoon of length  $L_0$  waiting at light 0, we now determine which of those vehicles will be able to reach light 0 before it turns red. Upstream lights  $-1$  to  $-n$  are green if  $n\Delta t < T_L/2$  or  $n < \frac{1}{2r_\Delta}$ . We choose  $n$  as the total number of consecutive upstream green lights

$$n = \left\lfloor \frac{1}{2r_\Delta} \right\rfloor.$$

This limits the length of platoon that can clear light 0 to at most  $n + 1$  blocks. The remaining green time for light  $-n$  further restricts the length of the last fragment. It remains green for time  $\frac{T_L}{2} - n\Delta t$ , so only vehicles within  $v(\frac{T_L}{2} - n\Delta t)$  of light  $-n$  will be able to advance before it turns red. As a result, the platoon fragment beyond light  $-n$  is limited to the lesser of  $v(\frac{T_L}{2} - n\Delta t)$  and  $\Delta x$ . Hence the upstream bandwidth is

$$\begin{aligned} B_{\text{up}} &= \frac{1}{L_0} \min \left( L_0, n\Delta x + \min \left[ \Delta x, v \left( \frac{T_L}{2} - n\Delta t \right) \right] \right) \\ &= \min \left( 1, \left\lfloor \frac{1}{2r_\Delta} \right\rfloor 2r_C + \min \left[ 2r_C, \left( 1 - \left\lfloor \frac{1}{2r_\Delta} \right\rfloor 2r_\Delta \right) \right] \right). \end{aligned}$$

Note that for slow vehicles ( $r_C \geq 1/2$ ) the upstream bandwidth is 1; this occurs because  $L_0$  is less than one block making the phases of upstream lights irrelevant. For very fast cars ( $r_C \ll 1/2$ ),  $L_0$  is large, and the upstream lights tend to play the dominant role in determining the bandwidth.

*Composite bandwidth*— The true bandwidth is the minimum of the upstream and downstream bandwidths

$$B = \min(B_{\text{down}}, B_{\text{up}}), \quad (\text{S4})$$

and it is bounded  $0 \leq B \leq 1$ . Figure S2 overlays the bandwidth on the plot of the eastbound efficiency. It is clear that all peaks with the exception of the green wave peak have zero bandwidth.

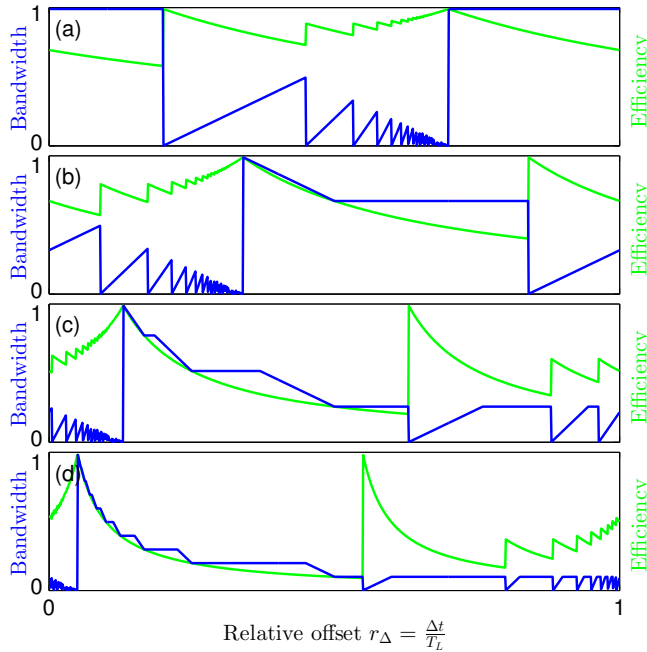


FIG. S2. Bandwidth of theoretical efficiency. The blue (dark gray) curve representing the bandwidth as a function of  $r_\Delta$  is displayed along with the green (light gray) curve representing the eastbound efficiency for (a)  $r_C = 0.7$ , (b)  $r_C = 0.34$ , (c)  $r_C = 0.13$ , and (d)  $r_C = 0.05$ .

One shortcoming of this approach is that it does not provide information about how the other vehicles perform relative to the first car in the platoon. Thus small bandwidth cannot necessarily be equated with poor efficiency or even large deviation from the theoretical efficiency. It simply indicates that deviations from the theory are possible. Large bandwidth, on the other hand, indicates that the theory is likely valid even for moderate traffic densities. Simulations reveal that despite the low bandwidth, timings near secondary peaks still perform well at low densities.

#### S4. JAMMING THRESHOLD

In our simulations, the cars were randomly placed in the system at according to an initial density  $\rho$ . We observed that, below a critical density  $\rho_{\text{crit}}$ , the efficiency is independent of density and is accurately predicted by the theory. Above that density, the efficiency decreases as vehicle density increases. In the theory, we assume vehicles are non-interacting. In simulations with large numbers of vehicles, this assumption no longer holds. Thus the degradation of efficiency can be attributed to the increasing frequency of interactions between vehicles. We now compute the “Jamming threshold”, the critical vehicle density for which vehicle interactions can no longer be ignored.

Vehicles interact when they stop behind other vehicles waiting at red lights. At low densities, this causes vehicles to form platoons which then behave as a single unit. These interactions cease once a steady state platoon distribution is reached, and consequently they do not cause significant reductions in efficiency. However, when these platoons exceed a critical length, either repeated segmentation by red lights or coalescence with other platoons becomes unavoidable. When these interactions cause vehicles to stop earlier than they would have in isolation, the efficiency decreases.

The two types of platoon interactions that cause a degradation of efficiency, platoon coalescence and platoon segmentation, become significant at different densities. So we compute two different thresholds that can explain the behavior visible in Fig. 5.

*Platoon coalescence*—The frequency of coalescence depends on the relative speeds of the green wave and the individual vehicles. When  $0 \leq \Delta t < \frac{T_L}{2}$  and  $\Delta t > T_C$ , cars travel faster than the green wave. As a result, vehicles accumulate at the front of each string of green lights. After the initial transients decay, these vehicles form platoons that behave as a unit provided the platoon does not fill up the series of consecutive green lights and is short enough to be able to make it through a light in a single cycle. Since half of the system is occupied by green lights and there are no vehicles between red lights at equilibrium, the critical density must be  $\rho_{\text{crit}} = \min[1/(2r_C), 1/2]$ . See Fig. S3 for an illustration of this scenario.

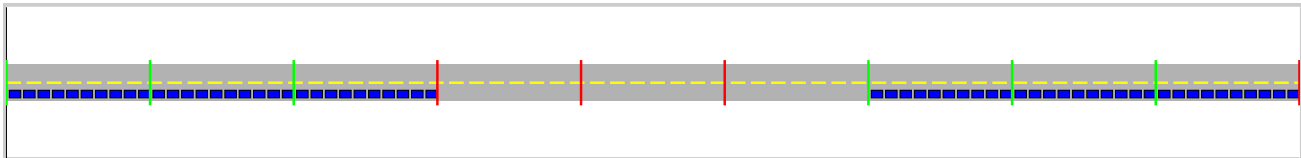


FIG. S3. Platoon distribution near the jamming threshold for  $r_C < r_\Delta$ . This displays a snapshot from a simulation for  $r_\Delta = 1/6$  where eastbound cars travel faster than the green wave. The green (light gray) and red (dark gray) vertical bars represent the traffic lights, and the blue rectangles represent individual vehicles. This snapshot is taken at the instant before the light on the far right turns green and the light at the trailing end of the platoon turns red. Note that for this particular value of  $r_\Delta$  the green string consists of  $n = \lfloor \frac{1}{2r_\Delta} \rfloor = 3$  lights. This string is filled with a platoon of vehicles that advances at the green wave speed. Adding additional vehicles would cause vehicles at the trailing end of the green string to stop prematurely due to the coalescence at the front of the platoon.

When  $0 \leq \Delta t < \frac{T_L}{2}$  and  $\Delta t \leq T_C$ , cars travel slower than the green wave. Thus, the front end of the red wave catches vehicles in the green section at a constant rate. This leaves queues of constant length in the red section which are then left behind at the trailing end of the red wave. However, if there are too many vehicles in the green section, then the cars at the front of the green section will be slowed down by the cars left behind by the red wave. This occurs when the green section is full of cars (density = 1), and the platoons in the red section surpass a critical length determined by the “catch up rate” (density =  $1 - \frac{r_\Delta}{r_C}$ ). See Fig. S4 for an illustration of this scenario. The critical density is a weighted average of the densities in the green segments and red segments:  $\rho_{\text{crit}} = 1/2 + 1/2(1 - r_\Delta/r_C)$ .

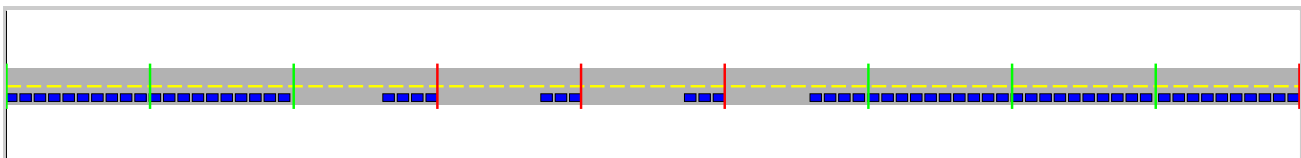


FIG. S4. Platoon distribution near the jamming threshold for  $r_C > r_\Delta$ . This displays a snapshot from a simulation for  $r_\Delta = 1/6$  where eastbound cars travel slower than the green wave. This snapshot is taken at the instant before the light on the far right turns green and the light at the trailing end of the long platoon in the green string turns red. Note that for this particular value of  $r_\Delta$  the green string is filled with vehicles while shorter platoons remain queued at red lights. Adding additional vehicles would cause the short platoons behind red lights to increase in length thereby causing vehicles at the end of the long platoon to stop prematurely when coalescence takes place.

For left moving waves ( $\Delta t > T_L/2$ ) the behavior is more complex and less intuitive. This case is less relevant for urban networks where  $r_C$  will typically be small (optimal  $r_\Delta$  are generally close to  $r_C$ ). Nonetheless, replacing  $r_\Delta$  with  $1 - r_\Delta$  in the solution for right-moving waves yields good agreement with simulations.

The predicted thresholds for platoon coalescence are marked by an open square and displayed in panel (a) of Fig. 5 in the main text.

*Platoon segmentation*—Platoon segmentation is related to the bandwidth. A platoon longer than the bandwidth will necessarily be segmented as it travels through the system. At the threshold, the system is filled with platoons of length  $L_P = L_0 B$ . These platoons are separated by gaps of length  $G$ . Simulations suggest that the variation in  $G$  is small near the threshold, so we assume  $G$  is a constant. Under this assumption, the critical density of vehicles in the

system is

$$\rho_{\text{crit}} = \frac{L_P}{L_P + G} \quad (\text{S5})$$

In order to determine this density, one additional equation is needed. A platoon of length  $L_0$  will be segmented into  $N_L$  pieces by downstream lights. Thus the region of length  $L_0$  will contain  $N_L$  platoons and  $N_L - 1$  gaps. The missing equation is

$$N_L L_P + (N_L - 1)G = L_0 \text{ for } N_L > 1. \quad (\text{S6})$$

If  $N_L = 1$ , this equation is redundant and instead

$$L_P + G = \Delta x / r_\Delta \text{ for } N_L = 1. \quad (\text{S7})$$

This is due to the fact that there is one platoon per “wavelength” in the green wave. Equations (S5), (S6) and (S7) can be solved for  $\rho_{\text{crit}}$  yielding the predictions marked by a solid circle and displayed in panel (a) of Fig. 5 in the main text. These predictions, which assume a vehicle length of 0, agree with the changes in slope of the efficiency curves displayed in Fig. 5, which were obtained from simulations with a vehicle length of 1/25th of the block length. Thus, the discretization of allowable platoon lengths can safely be ignored as long as the vehicle length is small relative to the block length, and the changes in slope can be attributed to platoon segmentation and coalescence.

## S5. SUPPLEMENTAL VIDEOS

Three video clips from simulations are included to demonstrate the platoon dynamics observed. These display eastbound and westbound traffic on an 8 block segment of the total 50 block long system. Traffic lights are indicated by red and green vertical lines, and individual vehicles are marked by blue rectangles. In these videos, the vehicles have a length equal to 1/10th of the length of a block. The videos correspond to  $r_C = 0.34$ ,  $\rho = 0.25$ , and three different values of  $r_\Delta$  corresponding to a non-green-wave optimum ( $r_\Delta = 0.15$ ), a green wave ( $r_\Delta = 0.34$ ), and a sub-optimal timing ( $r_\Delta = 0.44$ ). The total inflow and outflow for both eastbound and westbound traffic are also displayed. Due to the short timespan over which these totals are computed, they are highly dependent on the initial condition and as a result should not be viewed as a reflection of effectiveness of the timing scheme.