Supplementary material: The dynamics of social group competition: modeling the decline of religious affiliation

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S1. GENERALITY OF MODEL

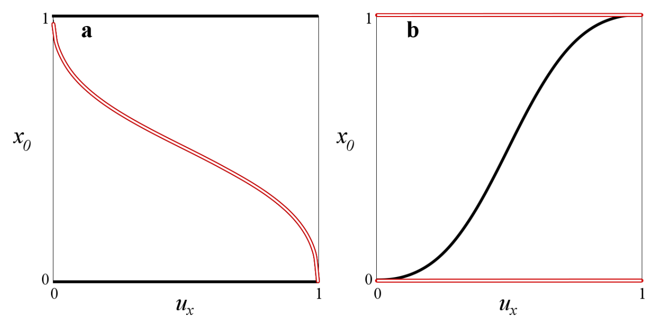
The model presented in this paper is applied to the widespread phenomenon of religious shift, but may be more generally applicable to a variety of competitive social systems. The model allows for either competitive exclusion ($a \geq 1$) or stable coexistence ($a < 1$) in systems composed of two social groups, and makes sense in the context of social networks. A similar model (reference 1) was applied to the phenomenon of language death. Some other competitive social systems in which identical or very similar models may apply include, for example, smoker vs. non-smoker, vegetarian vs. meat-eater, obese vs. non-obese, and Mac user vs. PC user.

S2. WHY THREE FIXED POINTS

In the main text of our paper, we state that there can be at most three fixed points for "generic" functions $P_{yx}(x;u)$ that satisfy our assumptions of symmetry, monotonicity, C^∞ continuity, and limiting properties. In this section we will clarify the meaning of "generic".

P_{yx} is a non-negative function of x parametrized by u (which will henceforth be abbreviated as simply u). The fixed points x^* can be written as solutions to the equation $0 = (1-x)P_{yx}(x;u) - xP_{yx}(1-x;1-u)$ for a given value of u. When $u = 0$ the limiting properties $P_{yx}(0;u) = 0$ and $P_{yx}(x;0) = 0$, along with monotonicity, imply that only $x^* = 0$ and $x^* = 1$ can be fixed points, with $x^* = 0$ the only stable fixed point. When $u = 1$, similarly, only $x^* = 0$ and $x^* = 1$ can be fixed points, with only $x^* = 1$ stable.

If there is a single intermediate fixed point $x^* \neq 0, x^* \neq 1$ for all values of $u \in (0,1)$, then it must limit to $x^* \to 0$ when $u \to 0$ and $x^* \to 1$ when $u \to 1$ (assuming it's stable—the opposite will be true if it is unstable). In order for other fixed points to appear, the continuous curve connecting $(x;u) = (0;0)$ to $(x;u) = (1;1)$ would have to have zero slope at some value of u (see Supporting Figure S1). Thus the condition for a single intermediate fixed point is that $dx/du > 0$ for all u (stable), or $dx/du < 0$ for all u (unstable).



SUPPORTING FIG. S1. Typical fixed points for Eq. (1). Here $P_{yx}(x;u) = cx^a u$, with (a) $a = 3$ and (b) $a = \frac{1}{2}$. Red open lines indicate unstable branches, black solid lines indicate stable branches of fixed points. Panel (a) demonstrates that the intermediate unstable branch of fixed points $x_u^*(u)$ serves as a separatrix, with all other initial conditions leading to $x = 0$ or $x = 1$. Panel (b) demonstrates how the stable fixed point $x_s^*(u)$ typically varies with u. If the intermediate fixed point is unstable, it must limit to $x_u^* \to 1$ when $u \to 0$ and $x_u^* \to 0$ when $u \to 1$.

For a separable function $P_{yx}(x;u) = X(x)U(u)$, $X(x) > 0$, $U(u) > 0$, the implications of this condition are as follows. The fixed point equation $(1-x)P_{yx}(x;u) = xP_{yx}(1-x;1-u)$ becomes $(1-x)X(x)U(u) = xX(1-x)U(1-u)$, and, assuming $x \neq 0$ and $x \neq 1$,

$$\frac{U(u)}{U(1-u)} = \frac{x}{1-x}\frac{X(1-x)}{X(x)} \ . \tag{S1}$$

Thus

$$\left[\frac{U(u)}{U(1-u)}\right]' \frac{du}{dx} = \left[\frac{x}{1-x}\frac{X(1-x)}{X(x)}\right]' \ . \tag{S2}$$

Since $\left[\frac{U(u)}{U(1-u)}\right]' = [U'(u)U(1-u) + U(u)U'(1-u)]/U^2(1-u) > 0 \,\forall\, u$ due to assumptions (monotonicity implies that U and U' must both be positive for all nonzero arguments), the condition $dx/du < 0$ becomes

$$\left[\frac{x}{1-x}\frac{X(1-x)}{X(x)}\right]' < 0 \ . \tag{S3}$$

This can be simplified to

$$\frac{X'(x)}{X(x)} + \frac{X'(1-x)}{X(1-x)} > \frac{1}{x} + \frac{1}{1-x} \ . \qquad \text{(S4)}$$

The direction of the inequality would be reversed for a stable intermediate fixed point. Note that a *sufficient* condition would be $X'/X > 1/x$. This, or the same condition with the inequality reversed, is clearly satisfied for any power law form $P_{yx}(x;u) \sim x^a$, $a \neq 1$. It is also satisfied for any function with a monotonic first derivative $X'(x)$ (Sketch of proof: Let $X'(x) = X'_0 + f(x)$, where $f(x) \equiv \int_0^x X''(\xi)d\xi$ is a monotonically increasing function. Then $xX'(x) = xX'_0 + xf(x)$ and $X(x) = \int_0^x X'(\xi)d\xi = xX'_0 + \int_0^x f(\xi)d\xi$. Thus $xX'(x) - X(x) = xf(x) - \int_0^x f(\xi)d\xi$. That last quantity is necessarily greater than zero for any monotonically increasing $f(x)$, and therefore $xX'(x) > X(x)$, or $X'/X > 1/x$.)

An analogous result holds for inseparable functions $P_{yx}(x;u)$. Using the same approach, the condition is:

$$\frac{1}{x}P'_{yx}(x;u) + \frac{1}{1-x}P'_{yx}(1-x;1-u) > \frac{1}{x^2}P_{yx}(x;u) + \frac{1}{(1-x)^2}P_{yx}(1-x;1-u) \ , \qquad \text{(S5)}$$

where prime notation represents a derivative with respect to the argument, not the parameter. Thus a sufficient condition is $P'_{yx}(x;u)/P_{yx}(x;u) > 1/x$ for all u. The full condition is satisfied by any function for which curvature doesn't change sign.

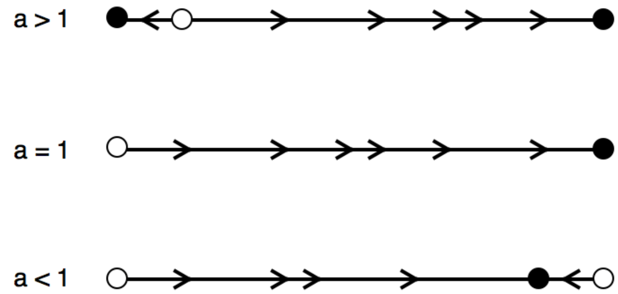
S3. STABILITY OF FIXED POINTS

Examine the stability of the fixed point at $x = 0$ (and note that the same argument will work for the stability of the fixed point at $x = 1$). Set $x = \eta$, a small perturbation from $x = 0$. Then

$$\begin{aligned} \dot{\eta} &= (1-\eta)P_{yx}(\eta;u) - \eta P_{yx}(1-\eta;1-u) \\ &\approx P_{yx}(0;u) + \eta\left[P'_{yx}(0;u) - P_{yx}(0;u) - P_{yx}(1;1-u)\right] \\ &= -\eta\left[P_{yx}(1;1-u) - P'_{yx}(0;u)\right] \qquad \text{(S6)} \end{aligned}$$

to $\mathcal{O}(\eta^2)$. So the fixed point $x^* = 0$ is stable to small perturbations if $P_{yx}(1;1-u) > P'_{yx}(0;u)$. For a power law $P_{yx} \sim x^a$, this will be true only when $a > 1$. The fixed point $x^* = 0$ will be unstable for $a < 0$, and its stability will depend on the sign of $u - \frac{1}{2}$ when $a = 1$.

The stability of the intermediate fixed point is fully determined once the stabilities of the two endpoints $x^* = 0$ and $x^* = 1$ are known. Because it is a one-dimensional flow, the intermediate fixed point must be stable when the endpoints are unstable, and vice-versa when the endpoints are stable (see Figure S2).



SUPPORTING FIG. S2. The flow in x for various values of the constant a. Filled circles indicate stable fixed points, while open circles indicate unstable fixed points. The leftmost fixed points correspond to $x = 0$, while the rightmost fixed points correspond to $x = 1$.

S4. DATA SETS AND MODEL FITS

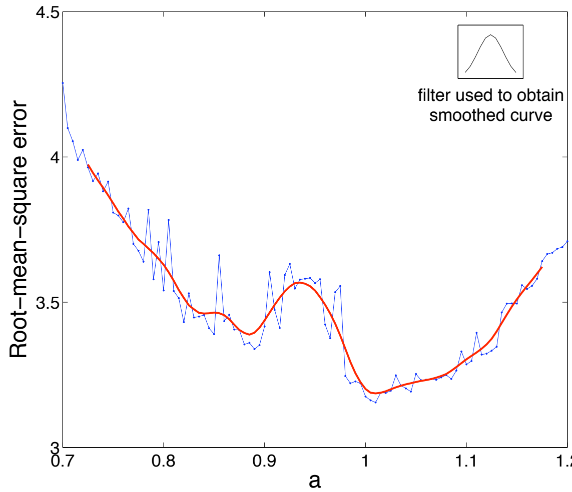
Data used in validating this model originated in census surveys from a range of countries worldwide. A total of 85 data sets had 5 or more independent data points. These came from various regions of 9 different countries: Australia, Austria, Canada, the Czech Republic, Finland, Ireland, the Netherlands, New Zealand, and Switzerland. Table I tabulates the names of the relevant publications and internet URLs (active as of May 2011) from which data sets and census information were obtained.

Because questions on census surveys generally ask individuals to choose a particular "religious affiliation," we can only use our model to attempt to explain their choices: while it is not possible to know from census data what portion of religiously affiliated individuals truly believe in God (or agree with any particular aspect of religious theology), it *is* possible to know an individual's declared affiliation. Thus if an individual declares he is religious, even for purely utilitarian non-theological reasons, we consider him to be part of the religiously affiliated group. Our hope is that the model will capture the dynamics of this *declared* affiliation.

The census reports that we used each had distinct methods for categorizing religious affiliation as "none." Usually this included "atheist," "agnostic," "secular deist," "humanist," "spiritual but not religious," and other similar answers.

Fitting was done by minimizing root-mean-square errors. Using a functional form $P_{yx} = cx^a u$, the parameters c and a were taken to be universal while the parameter u was allowed to vary with each data set. This was accomplished by simultaneously optimizing c, a, and $u_1 ... u_N$ such that the RMS error summed over all N data sets was minimized.

Supporting Figure S3 shows how that summed error varied with the parameter a. We chose $a \approx 1$ for the fits discussed in this paper both for simplicity and because of the broad minimum visible around $a \approx 1$. The parameter



SUPPORTING FIG. S3. Summed root-mean-square error over all data sets versus parameter a in $P_{yx} = cx^a u$. The error was calculated by finding the combination of parameters c, x_{0i} and u_i (where i varies over all data sets) that minimized the root mean square error between the model predictions and the data. Blue curve indicates exact error calculations, red indicates smoothed error after convolution with a Gaussian (inset). Note that there appears to be a broad minimum near $a = 1$.

c, which simply sets a time scale, was approximately 0.2.

S5. PERTURBATION OF NETWORK STRUCTURE

In this section we examine in greater depth the implications of Eq. (3), a continuous deterministic system with arbitrary coupling.

All-to-all coupling

If $G(\xi, \xi') = 1/2$ then there is uniform all-to-all coupling, and we see that $x(\xi, t) = \frac{1}{2}\int_{-1}^{1} R(\xi', t) d\xi' = \bar{R}(t)$, independent of space, where $\bar{R}$ is the spatially averaged value of R.

Then (3) becomes

$$\frac{\partial R}{\partial t} = (1 - R) P_{yx}(\bar{R}; u) - R P_{yx}(1 - \bar{R}; 1 - u) \qquad \text{(S7)}$$

If at some time t^* $R(\xi, t^*) = R_0(t^*)$ is independent of space, then $\bar{R}(t^*) = R_0(t^*)$ and Eq. (S7) becomes

$$\frac{\partial R_0}{\partial t} = (1 - R_0) P_{yx}(R_0; u) - R_0 P_{yx}(1 - R_0; 1 - u) \ , \qquad \text{(S8)}$$

which follows dynamics identical to the original two-group discrete system.

Country	Source(s)
Australia	2008 Year Book Australia, Table 14.38 www.abs.gov.au
Austria	“Die Habsburgermonarchie 1848-1918 Band IV: Die Konfessionen,” Statistik Austria (1995) and private communication with Statistik Austria. www.statistik.at
Canada	Canada Statistical Yearbook (years 1891-1901, 1911, 1921, 1931, 1941, 1951, 1971, 1991) and Statistics Canada online database. www.statcan.gc.ca, www.statcan.gc.ca/pub/11-516-x/sectiona/4147436-eng.htm
The Czech Republic	Czech Demographic Handbook 2007, Table 1-19 www.czso.cz, www.czso.cz/csu/2008edicniplan.nsf/engp/4032-08, www.czso.cz/csu/2008edicniplan.nsf/engt/24003E05ED/$File/4032080119.pdf
Finland	The Finland Year Book (years 1943, 1945, 1962, 1963, 1964, 1965, 1968, 1969, 1970, 1971, 1972, 1976, 1981, 1986) and private communication with Statistics Finland www.stat.fi, www.stat.fi/tup/suoluk/suoluk_vaesto_en.html
Ireland	Central Statistics Office Ireland online database www.cso.ie, www.cso.ie/px/pxeirestat/Dialog/varval.asp?ma=CNA28&ti=Population+(Number)+by+County,+Year+and+Religious+Denomination
The Netherlands	Centraal Bureau voor de Statistiek (Statistics Netherlands) online database www.cbs.nl, statline.cbs.nl/StatWeb/publication/?VW=T&DM=SLNL&PA=37944&D1=0&D2=a&HD=091126-1210&HDR=T&STB=G1 (in Dutch)
New Zealand	Hoverd, W. J. “No Longer a Christian Country? - Religious Demographic Change in New Zealand 1966-2006,” New Zealand Sociology **23**(1) 2008, and private communication with Statistics New Zealand www.stats.govt.nz, www.stats.govt.nz/Census/2006CensusHomePage/classification-counts-tables/about-people/religious-affiliation.aspx
Switzerland	Private communication with Swiss Federal Statistical Office (Bundesamt für Statistik BFS) www.statistik.admin.ch

TABLE I. Sources of census data.

Perturbation off of uniform R with all-to-all coupling

We impose a destabilizing perturbation such that the portion of the population with $\xi < 0$ has lower R and the portion with $\xi > 0$ has higher R, i.e.,

$$R(\xi, t_0) = R_0 + \epsilon \ \mathrm{sgn}\, \xi \ , \qquad \text{(S9)}$$

where ϵ is a small parameter. Then $x(\xi) = \frac{1}{2}\int_{-1}^{1} R(\xi')d\xi' = R_0$, and from (3) we get

$$\frac{\partial R}{\partial t} =(1 - R_0 - \epsilon \operatorname{sgn}\xi)P_{yx}(R_0; u) - (R_0 + \epsilon \operatorname{sgn}\xi)P_{yx}(1 - R_0; 1 - u) \tag{S10}$$

We can also look at the dynamics of the mean religious affiliation $\bar{R}$,

$$\frac{\partial \bar{R}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\int_{-1}^{1} R(\xi', t)d\xi'\right) = \frac{1}{2}\int_{-1}^{1} \frac{\partial R(\xi', t)}{\partial t} d\xi' \ . \tag{S11}$$

Plugging Eq. (S10) into Eq. (S11) and simplifying gives

$$\frac{\partial \bar{R}}{\partial t} = \frac{\partial R_0}{\partial t} = (1-R_0)P_{yx}(R_0; u) - R_0 P_{yx}(1-R_0; 1-u) \ , \tag{S12}$$

so the mean religious affiliation $\bar{R}(t)$ continues to follow the dynamics of the original system despite the perturbation.

Rearranging Eq. (S10), we see

$$\frac{\partial R}{\partial t} = \frac{\partial R_0}{\partial t} - \epsilon \operatorname{sgn}\xi(P_{yx}(\bar{R}; u) + P_{yx}(1 - \bar{R}; 1 - u)) \ , \tag{S13}$$

and direct differentiation of Eq. (S9) yields

$$\frac{\partial R}{\partial t} = \frac{\partial R_0}{\partial t} + \frac{\partial \epsilon}{\partial t} \operatorname{sgn}\xi \ . \tag{S14}$$

Equating these expressions yields a differential equation for $\epsilon(t)$:

$$\frac{\partial \epsilon}{\partial t} = -\epsilon(P_{yx}(\bar{R}; u) + P_{yx}(1 - \bar{R}; 1 - u)) \ . \tag{S15}$$

Note that ϵ remains independent of the spatial coordinate, and that $\epsilon \to 0$ as $t \to \infty$, for any initial condition (the time constant may vary with the parameter u and the state $\bar{R}$). So the initial perturbation must damp out, and the system must evolve to a single affiliation as $t \to \infty$, just as the original system (1) did.

Non-uniform coupling

We consider the case of non-uniform spatial coupling as the continuum limit of a discrete network where the links are nearly but not quite all-to-all. In that case, a very destabilizing perturbation would be one in which the network is segregated into two clusters, each one more strongly coupled internally than externally. As described in the main text, one kernel representing such coupling is

$$G(\xi, \xi') = \frac{1}{2} + \frac{1}{2}\delta(2H(\xi) - 1)(2H(\xi') - 1) \ , \tag{S16}$$

where δ is a small parameter ($\delta \ll 1$) that determines the amplitude of the perturbation and $H(\xi)$ represents the Heaviside step function.

If the initial state of the population is uniform such that $R(\xi, t_0) = R_0$ then $x(\xi, t_0) = \int_{-1}^{1} G(\xi, \xi')R_0 d\xi' = R_0$ and R will satisfy Eq. (3), giving

$$\frac{\partial R_0}{\partial t} = (1-R_0)P_{yx}(R_0; u) - R_0 P_{yx}(1-R_0; 1-u) \ . \tag{S17}$$

Thus R will remain uniform in space and will follow the same dynamics as the original system, *despite* a non-uniform coupling kernel of arbitrary amplitude.

Perturbation off of uniform R with non-uniform coupling

As before, we impose a destabilizing perturbation such that the portion of the population with $\xi < 0$ has lower R and the portion with $\xi > 0$ has higher R, i.e., $R(\xi, t_0) = R_0 + \epsilon \operatorname{sgn}\xi$, where ϵ is again a small parameter. This should conspire with the perturbed coupling kernel to maximally destabilize the uniform state.

The definition of x gives

$$\begin{aligned} x(\xi, t) =&(R_0 - \epsilon)\int_{-1}^{0} G(\xi, \xi')d\xi' + \\ &(R_0 + \epsilon)\int_{0}^{1} G(\xi, \xi')d\xi' \\ =&R_0 + \epsilon\delta \operatorname{sgn}\xi \ , \end{aligned} \tag{S18}$$

and from (3) we get

$$\frac{\partial R}{\partial t} = (1 - R_0 - \epsilon \operatorname{sgn}\xi)P_{yx}(R_0 + \epsilon\delta \operatorname{sgn}\xi; u) - (R_0 + \epsilon \operatorname{sgn}\xi)P_{yx}(1 - R_0 - \epsilon\delta \operatorname{sgn}\xi; 1 - u) \ . \tag{S19}$$

Directly differentiating $R(\xi, t_0) = R_0 + \epsilon \operatorname{sgn}\xi$, then rearranging terms on the right-hand-side of Eq. (S19) gives

$$\begin{aligned} \frac{\partial R_0}{\partial t} + \operatorname{sgn}\xi\frac{\partial \epsilon}{\partial t} =& (1 - R_0)P_{yx}(R_0 + \epsilon\delta \operatorname{sgn}\xi; u) \\ &- R_0 P_{yx}(1 - R_0 - \epsilon\delta \operatorname{sgn}\xi; 1 - u) - \epsilon \operatorname{sgn}\xi\,[P_{yx}(R_0 \\ &+ \epsilon\delta \operatorname{sgn}\xi; u) + P_{yx}(1 - R_0 - \epsilon\delta \operatorname{sgn}\xi; 1 - u)] \ . \end{aligned} \tag{S20}$$

Now calculate $\frac{\partial R_0}{\partial t}$ directly. Note that $R_0 = \bar{R} = \frac{1}{2}\int_{-1}^{1} R(\xi', t)d\xi'$, so

$$\begin{aligned}\frac{\partial R_0}{\partial t} =& \frac{\partial}{\partial t}\left(\frac{1}{2}\int_{-1}^{1} R(\xi',t)d\xi'\right) = \frac{1}{2}\int_{-1}^{1}\frac{\partial R(\xi',t)}{\partial t}d\xi' \\ =& \frac{1}{2}\int_{-1}^{1}\Big[(1-R_0-\epsilon\ \mathrm{sgn}\,\xi')P_{yx}(R_0+\epsilon\delta\ \mathrm{sgn}\,\xi';u)-(R_0+\epsilon\ \mathrm{sgn}\,\xi')P_{yx}(1-R_0-\epsilon\delta\ \mathrm{sgn}\,\xi';1-u)\Big]d\xi' \\ =& \frac{1}{2}(1-R_0)\big[P_{yx}(R_0-\epsilon\delta;u)+P_{yx}(R_0+\epsilon\delta;u)\big]-\frac{1}{2}R_0\big[P_{yx}(1-R_0+\epsilon\delta;1-u)+P_{yx}(1-R_0-\epsilon\delta;1-u)\big] \\ &+\frac{1}{2}\epsilon\big[P_{yx}(R_0-\epsilon\delta;u)-P_{yx}(R_0+\epsilon\delta;u)\big]+\frac{1}{2}\epsilon\big[P_{yx}(1-R_0+\epsilon\delta;1-u)-P_{yx}(1-R_0-\epsilon\delta;1-u)\big]\ . \qquad \text{(S21)}\end{aligned}$$

Taylor expanding in both ϵ and δ eliminates all first order terms in ϵ, leaving

$$\frac{\partial R_0}{\partial t} = (1-R_0)P_{yx}(R_0;u)-R_0P_{yx}(1-R_0;1-u)+\mathcal{O}(\epsilon^2\delta)\ , \qquad \text{(S22)}$$

so the assumption that the mean religious affiliation follows the dynamics of the original unperturbed system is well justified.

Similarly Taylor expanding Eq. (S20) to first order in ϵ and δ allows canceling of the $\partial R_0/\partial t$ terms on either side (using Eq. (S22)), leaving an equation in ϵ:

$$\begin{aligned}\frac{\partial \epsilon}{\partial t} =& -\epsilon\Big\{P_{yx}(R_0;u)+P_{yx}(1-R_0;1-u) \\ &-\delta\big[(1-R_0)P'_{yx}(R_0;u) \\ &+R_0P'_{yx}(1-R_0;1-u)\big]\Big\}\ . \qquad \text{(S23)}\end{aligned}$$

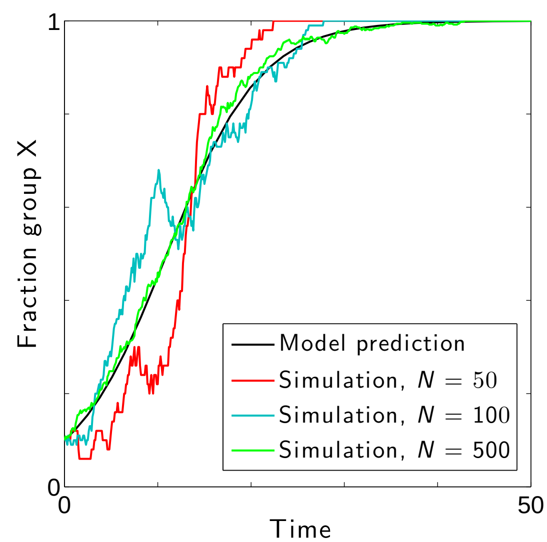
The sign of the quantity in braces in Eq. (S23) determines the stability of the uniform spatial state. It's clear that for sufficiently small δ, the uniform state will *always* be stable. However, in systems with an unstable intermediate fixed point (or no intermediate fixed point), the uniform state will remain stable even when the quantity in braces is initially positive! This is because Eq. (S22) will still hold for small ϵ, making R_0 approach a steady state value $R_0^*=0$ or $R_0^*=1$ from the original system. Since $\epsilon<\min(R_0,1-R_0)$ is required to maintain variables in the allowed domain, ϵ may initially grow, but it will have to eventually shrink as $R_0\to R_0^*$.

The above further shows that ϵ does not develop any additional spatial structure, so an initial state with $R=R_0+\epsilon\ \mathrm{sgn}\,\xi$ will maintain such a spatial structure as R_0 and ϵ evolve in time.

This calculation demonstrates that an understanding of the simple all-to-all discrete system gives insight into the more complex problem of religious shift on a social network. In numerical experiment, the results of the perturbation calculation described here remain valid even for very sparse networks quite different from all-to-all.

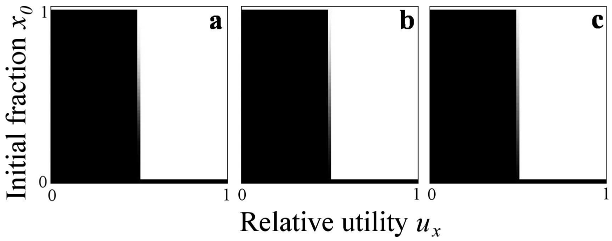
S6. NUMERICS

In the main text, we describe a numerical experiment that we performed on systems (2) and (3). Figure 3 of the main text shows the results of that experiment with a simulated size $N=500$, and in Figure S4, we show that the all-to-all system (1) becomes a good match to the discrete stochastic system (2) as the number of nodes increases (thus explaining why Figure 3 is well predicted by understanding the all-to-all system).



SUPPORTING FIG. S4. Comparison of simulation of discrete stochastic system (2) to model predictions for various system sizes (all-to-all coupling). Here $x_0 = 0.1$ and the total size of the network is 50 (red), 100 (blue), or 500 (green). The solid black line represents the solution to Eq. (1), the large N limit of this model.

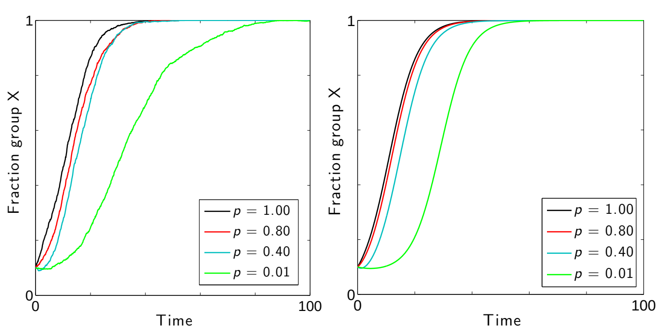
We also performed a our numerical experiment on (3), the continuous deterministic generalization of (1). Results are presented in Figure S5, where the steady states are indistinguishable from the all-to-all model.



SUPPORTING FIG. S5. Results of simulation of the continuous deterministic system (3) on a network with two initial clusters weakly coupled to one another. The ratio p of out-group coupling strength to in-group coupling strength is (a) $p = 0.01$; (b) $p = 0.40$; (c) $p = 0.80$ ($N = 500$). When $u = 1/2$, all points are fixed points, so the initial condition determines the final state. Steady states are indistinguishable from those of the all-to-all model (1) despite the non-uniform coupling and inhomogeneous initial conditions.

S7. TIME DELAY

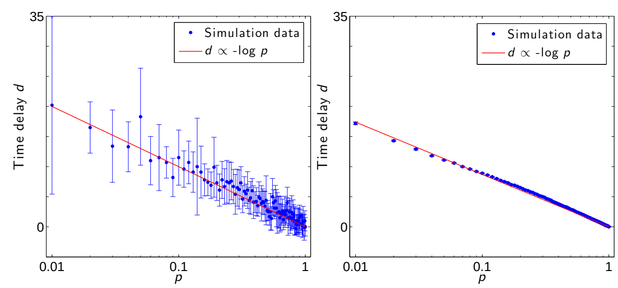
We define the effective time delay d to be the delay between the perturbed solution (not all-to-all) and the all-to-all solution (the logistic function, when $P_{yx} = cx^a u$ with $a \approx 1$), as measured when $\bar{R}$ has risen halfway to its asymptotic value of 1 (we assume a rising function with no loss of generality: the symmetric case of a decaying function can be examined under the change of variables $u \mapsto 1 - u, x_0 \mapsto 1 - x_0$). Thus d is the difference in the time t_c when a solution $\bar{R}(t_c) = \frac{1}{2}(1 + R_0)$ and $t_c^{\text{all-to-all}}$ when $\bar{R}_{\text{all-to-all}}(t_c^{\text{all-to-all}}) = \frac{1}{2}(1 + R_0)$. We observe this quantity to increase monotonically with the perturbation off of all-to-all δ—see Figure S6 for typical behavior at various values of δ. In the limit that $p \ll 1$ (δ near 1, nearly two separate clusters) we find that the curve is well approximated by $d \propto -\ln(p)/(2u - 1)$.



SUPPORTING FIG. S6. Variation in the behaviour of systems (2) and (3) with increasing perturbation off of all-to-all. This illustrates delay time as inter-cluster connection probability p varies. Equivalent values of the perturbation parameter δ in order of decreasing p are $\delta = 0$, $\delta = 0.11$, $\delta = 0.43$, and $\delta = 0.98$. Left panel: Discrete stochastic system (2) (ensemble averages of 10 realizations). Right panel: Continuous deterministic system (3). For all simulations $x(0) = 0.1$, $u = 0.6$ and $N = 500$.

We find this form by assuming $\bar{R}(t) \approx \bar{R}_0 + \eta y(t) + \mathcal{O}(\eta^2)$ for $\eta \ll 1$, then eliminating terms of order higher than linear in the equation governing $\bar{R}$ ((3) after simplifying for two cliques). We then expand this approximate equation for small p, retaining only lowest order terms. The resulting system is linear and can be solved exactly for the critical time t_c at which $\bar{R}$ rises to $(1 + x_0)/2$. The delay is simply the difference between that time and $t_c^{\text{all-to-all}} = -\ln(x_0/(1 + x_0))/(2u - 1)$, the critical time for the all-to-all $p = 1$ system.

Based on numerical work, the general behaviour of this approximation—$d \propto -\ln p$—seems to remain valid even for large p, although the additive constant seems to change (see Figure S7). That is to be expected, since $d \to 0$ is required for $p \to 1$.



SUPPORTING FIG. S7. Variation of time delay d with increasing inter-cluster connection probability p. Points and error bars indicate mean and standard deviation with 10 realizations. Lines show estimated logarithmic dependence. Left Panel: Discrete stochastic system (2). Right panel: Continuous deterministic system (3). For all simulations $N = 500$, $x(0) = 0.1$ and $u = 0.6$.

S8. LIMITATIONS OF MODEL

The models we have proposed have been greatly idealized, with the hope of capturing key aspects of the rapid growth of religious non-affiliation. We recognize that the simplification of this real-world social phenomenon limits our models applicability in several ways:

- We do not model birth, death, or migration effects. Differential rates for affiliated vs. unaffiliated groups could change the stability of fixed points, although we do not believe that these would affect the qualitative aspects of results.

- We assume constant u. Realistically, we believe that many factors affect this parameter, and that many of those factors change as societies evolve. Our hope is that the time scale for the rapid growth of non-affiliation differs from the time scale for drift in u, but we have no evidence that that is the case.

  We speculate that for most of human history, the perceived utility of religion was high and of non-affiliation low. Religiously non-affiliated people persisted but in small numbers. With the birth of modern secular societies, it appears that the perceived utility of adherence to religion versus non-affiliation has changed significantly in numerous countries, such as those with census data shown in Fig. 1, and the United States, where non-affiliation is growing rapidly. We would be interested in any direct or indirect evidence of changes in u, and believe this may be a fruitful topic of research for social scientists.

- We assume that individuals are free to choose their degree of religious affiliation. This may not be true in some societies, although it may still be possible to interpret such a restriction as a limiting case of our model where either $u \to 0$ or $u \to 1$. There may also be difficulties obtaining valid data in such societies.

- Each version of our model is analyzed in the limit of all-to-all social interaction, or nearly all-to-all. This limit is vindicated by the numerical work showing that similar dynamics occur even in very sparse networks.

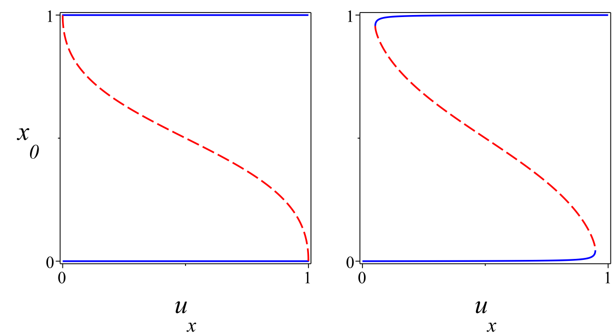
- All religions are treated as a single social group. This simplification allowed us to obtain analytical results, and analogous multi-group models may be examined in future work. We suspect that a system composed of many individual groups with distinct utilities could be aggregated into a two-group model where "effective utilities" are determined by weighted averaging of component group utilities.

- Our models do not allow for the emergence of a new social group if the population has reached consensus (i.e., everyone belongs to one group). Thus our models cannot describe the birth of a new religion to which no one previously belonged. As discussed in Supplementary Section S9, this phenomenon might be captured by relaxing our assumption that $P_{yx}(0;u) = 0$, an assumption that limits the applicability of the models at times when either group consists of an extremely small fraction of the population or vanishes altogether.

S9. BIRTH OF NEW SOCIAL GROUPS

The assumption that $P_{yx}(0;u) = 0$ indicates that no individual will ever switch to a social group with no members. That assumption prohibits the formation of a new social group. Here we discuss the impact of relaxing that assumption.

When $P_{yx}(0;u) \neq 0$, the argument in Supplementary Section 2 remains valid in a certain sense. The introduction of new fixed points along the intermediate branch in Supporting Figure S1 remains impossible for generic functions of the sort discussed there. However, the former linear branches of fixed points at $x^* = 0$ and $x^* = 1$ will no longer be lines, but rather curves, though each will still intersect the intermediate branch in a saddle-node bifurcation. The former positions of the saddle-node bifuractions—e.g., $(u^*, x^*) = (0, 1)$ and $(u^*, x^*) = (1, 0)$ when $a > 1$—will change, such that they will be shifted away from the corners of the graph. Figure S8 shows a typical example of how the position of the fixed points will change when $P_{yx}(0;u) \neq 0$.

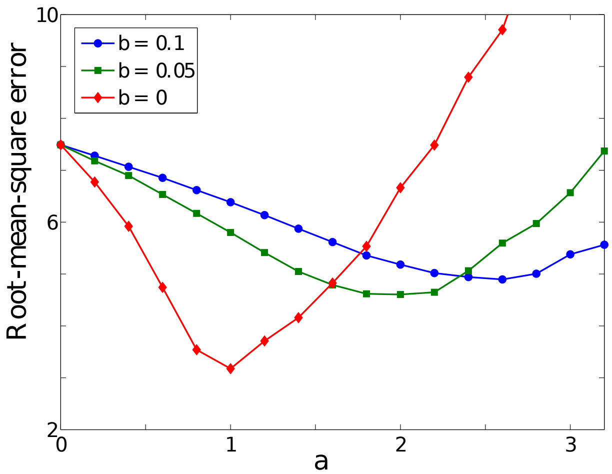


SUPPORTING FIG. S8. Typical example of fixed point positions for system where $P_{yx}(0;u) = 0$ (left panel) and $P_{yx}(0;u) \neq 0$ (right panel). Solid blue lines indicate stable fixed points, dashed red lines indicate unstable fixed points. Here we take $P_{yx}(x;u) = c(x+b)^a u$ with $a = 3, b = 0$ in the left panel and $a = 3, b = 0.1$ in the right panel.

The upshot of this change is that stable coexistence of two social groups *is* possible when $P_{yx}(0;u) \neq 0$ is allowed, but the qualitative change in the dynamics of the system is negligible for small $P_{yx}(0;u)$. Since we do not model lower-order effects such as birth, death, migration, or deliberate nonconformity, we feel that our model will not be accurate when either social group consists of a very small fraction of the population. Thus we believe that the gain in simplicity by setting $P_{yx}(0;u) = 0$ identically is justified.

Supporting Figure S9 shows the least-squares error in a fit to data with the form $P_{yx}(x;u) = (x+b)^a u$. It's clear that $b = 0$ does a good job of minimizing the error, further vindicating our simplifying assumption.

We also note that relaxing the assumption $P_{yx}(x;0) = 0$ has a similarly small effect on the dynamics for small nonzero values of $P_{yx}(x;0)$. Setting this to zero (besides seeming logically justified) not only simplifies the model by reducing free parameters, but also provides the best fit to data.



SUPPORTING FIG. S9. Summed root-mean-square error over all data sets versus parameter a in $P_{yx}(x;u) = c(x+b)^a u$. The error was calculated by finding the combination of parameters b, c, x_{0i} and u_i (where i varies over all data sets) that minimized the root mean square error between the model predictions and the data. Note that the best-fit value of b appears to be near $b = 0$ and $a = 1$, corresponding to the case shown in Supporting Figure S3.