Modeling and analysis of systems with nonlinear functional dependence on random quantities

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Many real-world systems exhibit noisy evolution; interpreting their finite-time behavior as arising from continuous-time processes (in the Itô or Stratonovich sense) has led to significant success in modeling and analysis in a variety of fields. Here we argue that a class of differential equations where evolution depends nonlinearly on a random or effectively-random quantity may exhibit finite-time stochastic behavior in line with an equivalent Itô process, which is of great utility for their numerical simulation and theoretical analysis. We put forward a method for this conversion, develop an equilibrium-moment relation for Itô attractors, and show that this relation holds for our example system. This work enables the theoretical and numerical examination of a wide class of mathematical models which might otherwise be oversimplified due to a lack of appropriate tools.

I. GENERALIZING LANGEVIN EQUATIONS

Langevin equations are often used to represent theoretical differential behavior for systems exhibiting stochastic dynamics (see, e.g., [1, 2]). These equations have a standard form, which we will aim to generalize:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t) + g(x,t)\eta_t\,,$$

where η_t represents the "Gaussian white noise" term, δ-correlated in continuous time. If $g(x,t)$ exhibits x dependence, such Langevin equations are ill-defined, necessitating either the Itô or Stratonovich interpretation, which will differ in their "drift" behavior [2, 3].

Here, we seek to generalize to systems of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = R(x,t,\eta_t). \tag{1}$$

We argue that, with the proper conversion procedure based on the central limit theorem [4], these Langevin-type systems may be reduced to equivalent Itô behavior, allowing for consistent simulation and theoretical analysis.

As a motivating example, we start by highlighting the difference between two similar-looking Langevin-type equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x^3 + \eta_t, \tag{2}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x+\eta_t)^3. \tag{3}$$

Equation (2) is a classic Langevin equation with cubic attraction towards zero and diffusive noise—easily interpreted (in either the Itô or Stratonovich sense) as the stochastic differential equation (SDE) $\mathrm{d}x = -x^3\mathrm{d}t + \mathrm{d}W$ (where $\mathrm{d}W$ represents the usual derivative of a Wiener process), enabling all the analytical and numerical options that entails.

Equation (3), however, is notably different in that the nonlinear cubing operation happens to a fundamentally random quantity, linking the deterministic and random parts of the equation. Naïve numerical simulation simply converges to deterministic behavior as the time-step shrinks, since the fluctuations average out before x changes considerably. If timestep-independent stochastic behavior is desired, we must develop a new consistent and coherent interpretation of this equation.

"Baked-in" stochasticity of this type might arise in a variety of physical modeling scenarios. For example, nonlinear drag forces acting on a macroscopic object in a turbulent flow would cause velocity to evolve according to this type of Langevin equation, with "noise" coming from rapidly fluctuating relative fluid velocity—including, e.g., viscous drag on a cylinder in a turbulent wake [5]. We compute results for this velocity distribution, and its stark difference from a naïve approach, at the end of this section. Physical systems with nonlinear feedback based on rapidly fluctuating quantities or quantities subject

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to random measurement error would also be of this type. Inasmuch as measurement error acts as independent random variation of a quantity, the behavior of simulated or artificially forced dynamical systems would also benefit from this analysis. Our interest was motivated by an earlier model for individuals reacting to a stochastic political environment [6]. A variety of other physics-inspired nonlinear models of complex real-world phenomena may also share this form.

We note that the systems we are concerned with differ from other ways in which nonlinearity can arise in stochastic systems, for example in the deterministic part (e.g. [7]) or when x-dependence appears multiplied by the stochastic quantity (e.g., [8, 9]), or when functions are applied to a continuous random-walking quantity rather than the uncertain/noisy quantity itself (as Itô's lemma would handle [2]). Certain specific problems exhibiting nonlinear dependence on stochastic quantities have been examined [10], but a general theory of this class of Langevin equations has not been developed.

Our argument is based on the consideration that over any finite time-scale, a theoretical system such as Eq. (3) will have experienced a large enough number of nearly-independent increments that the generalized central limit theorem should apply [11]. That is, the net increment over any finite time must be drawn from the family of *stable distributions*, or—if the intrinsic noise has finite variance—a Gaussian distribution in particular [11]. This intuitively dovetails with the more practically-motivated necessary condition that, in the numerical simulation of any continuous-time system, its behavior must not depend sensitively on the simulated timestep; that is, one relatively large step must result in the same distribution (in an ensemble average sense) as the commensurate number of arbitrarily small steps.

We proceed henceforth with the assumption of finite underlying variance. This means that the increment over any small but finite time must be drawn from a Gaussian distribution with mean equal to the mean of the underlying process. We may also choose this distribution's variance per unit time to likewise match the underlying process, maintaining consistency with the classic Langevin-Itô conversion and agreement in standard cases.

By this reasoning, we argue that every such stochastic process with finite variance is in fact *equivalent* to an Itô SDE over any finite time-scale: in particular, the SDE with deterministic part matching the underlying mean behavior and random part matching its standard deviation. We note that this is not a one-to-one mapping, but rather many-to-one: any stochastic process with the same mean and standard deviation would behave identically, and thus be represented by the same Itô SDE.

That is, for a general stochastic system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = R(x,t,\eta_t) \sim P(r|x,t),$$

where R is some finite-variance stochastic quantity dependent on x and δ-correlated in time, with distribution P, one should simulate the Itô SDE

$$\mathrm{d}x = F(x,t)\mathrm{d}t + G(x,t)\mathrm{d}W, \quad \text{where}$$

$$F(x,t) = \text{mean}\left[R(x,t)\right] = \int_{-\infty}^{\infty} rP(r|x,t)\mathrm{d}r,$$

$$G(x,t) = \text{std}\left[R(x,t)\right] = \sqrt{\int_{-\infty}^{\infty} \left[r - F(x,t)\right]^2 P(r|x,t)\mathrm{d}r},$$

if these quantities exist. We will limit ourselves to stationary and autonomous processes (i.e., $F(x,t) = F(x)$ and $G(x,t) = G(x)$) from this point forward, but the theory should extend to non-stationary processes.

Once we have this Itô equation, we may use standard numerical integration techniques for individual trajectories, or convert the system to a Fokker-Planck form and evolve the solution's probability distribution $\rho(x)$ directly, with

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F(x)\rho(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[G(x)^2\rho(x,t)\right] .$$

As an example, we will now examine a slightly generalized version of Eq. (3) to determine the effect of noise with arbitrary constant amplitude σ:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x + \sigma\eta_t)^3. \tag{4}$$

In section S1 of the Supplemental Material (SM), we examine a yet more general version of this attractor with arbitrary positive-integer exponent, but for illustration and concreteness henceforth focus on this cubic nonlinear-stochastic attractor. Using the shorthand notation $N(r|\mu,\sigma) = e^{\frac{-(r-\mu)^2}{2\sigma^2}}/(\sigma\sqrt{2\pi})$, we have:

$$F(x|\sigma) = \int_{-\infty}^{\infty} -r^3 N(r|x,\sigma)\mathrm{d}r$$
$$= -x^3 - 3\sigma^2 x$$

and

$$G(x|\sigma) = \sqrt{\int_{-\infty}^{\infty} [-r^3 - F(x)]^2 N(r|x,\sigma)\mathrm{d}r}$$
$$= \sqrt{9\sigma^2 x^4 + 36\sigma^4 x^2 + 15\sigma^6}.$$

So we argue that the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x+\sigma\eta_t)^3$$

is equivalent to the Itô SDE

$$\mathrm{d}x = (-x^3 - 3\sigma^2 x)\ \mathrm{d}t + \sqrt{15\sigma^6 + 36\sigma^4 x^2 + 9\sigma^2 x^4}\ \mathrm{d}W, \qquad (5)$$

which is amenable to various methods of simulation and analysis like any other Itô equation. We note that this Itô equation is significantly different from anything one might obtain from the similar-looking but simply additive Langevin form in Eq. (2).

To reiterate: a naïve interpretation of Eq. (4) would lead to the Itô SDE

$$\mathrm{d}x = -x^3\mathrm{d}t + \sigma^3\mathrm{d}W\ , \qquad (6)$$

which has completely different physical behavior than our proposed interpretation in Eq. (5)[12]. Basic properties like the variance of the equilibrium distribution differ, with divergence possible in Eq. (5) but not in Eq. (6). This has significant implications for all types of stochastic models used throughout physics.

As an illustrative physical example, we consider the regime of quadratic drag with rapidly varying relative fluid velocity—of relevance to the behavior of particles in well-developed turbulence. In the one-dimensional case without stochasticity, relative velocity v would vary as $\mathrm{d}v/\mathrm{d}t = -cv|v|$ (here the constant c sets the time scale, and we set it to 1 henceforth). When random velocity fluctuations are included, we have:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -(v+\sigma\eta_t)|v+\sigma\eta_t|. \qquad (7)$$

This might naïvely be modeled by the Itô equation

$$\mathrm{d}v = -v|v|\mathrm{d}t + \sigma^2\mathrm{d}W\ , \qquad (8)$$

which has an exact solution for its steady-state probability distribution

$$p(v) = \frac{C}{\sigma^{2/3}}\exp\left(\frac{-2|v^3|}{3\sigma^2}\right)\ , \qquad (9)$$

where C is a normalization constant, namely $3^{7/6}\Gamma(2/3)/(2^{5/3}\pi)$.

But this system is more faithfully modeled by using our proposed conversion, which yields

$$\mathrm{d}v = F_2(v|\sigma)\mathrm{d}t + G_2(v|\sigma)\mathrm{d}W\ , \qquad (10)$$

where

$$F_2(v|\sigma) = -\left(\sigma^2+v^2\right)\mathrm{Erf}\left(\frac{v}{\sigma\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}x\sigma e^{\frac{-v^2}{2\sigma^2}},$$
$$G_2(v|\sigma) = \sqrt{v^4 + 6v^2\sigma^2 + 3\sigma^4 + 3[F_2(v|\sigma)]^2}$$

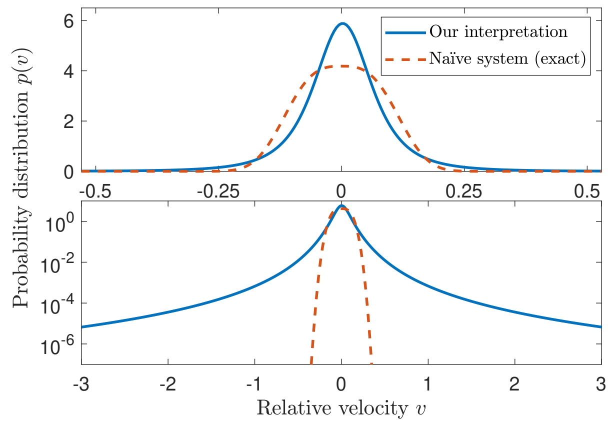

FIG. 1: **Equilibrium velocity distributions.** Comparison of equilibrium distributions for the drag system in Eq. (7) with $\sigma = 0.2$, computed by Fokker-Planck integration of our proposed behavior (10) and compared to the exact solution (9) for a naïve interpretation of the system's behavior. **Top:** Linear scale. **Bottom:** Zoomed-out log-scale view, emphasizing clear differences in implied behaviors.

(computation details in section S1 of the SM). The significant difference in behavior between these systems is illustrated in in Fig. 1.

II. EQUILIBRIUM MOMENT ANALYSIS OF ITÔ SDES

We now shift our focus from Itô interpretation of generalized Langevin equations to a technique of equilibrium analysis for Itô SDEs themselves. Equilibrium distributions are of considerable interest in any system where they exist. However, in some cases, direct analytical calculation of the steady-state distribution (as described in, e.g., [2]) requires integrals that fail to converge. Our interpretation of the stochastic cubic attractor (5) is of this type, and we will show that this technique yields insight into its structure.

Suppose we seek to examine the equilibrium distribution (if it exists) of the autonomous Itô SDE

$$\mathrm{d}x = F(x)\mathrm{d}t + G(x)\mathrm{d}W. \qquad (11)$$

We will use Euler-Maruyama numerical integration [13] as a guide: in discrete time, we have

$$\Delta x = F(x)\Delta t + G(x)\sqrt{\Delta t}\ \eta, \qquad (12)$$

where $\eta \sim N(0,1)$. We may write the expression for the distribution of the new value $\xi = x + \Delta x$ from any previous position x:

$$\xi \sim N\left(x + F(x)\Delta t\ ,\ G(x)\sqrt{\Delta t}\right)\ ,$$
$$P(\xi|x) = \frac{1}{G(x)\sqrt{2\pi\Delta t}} e^{\frac{-[\xi - x - F(x)\Delta t]^2}{2G(x)^2\Delta t}}\ . \quad (13)$$

Given this probability density function (PDF) for the outcome of a single step from any initial position x, we may write an expression for the evolution of the solution PDF from initial state $\rho_k(x)$ to subsequent state $\rho_{k+1}(x)$ a short time Δt later:

$$\rho_{k+1}(\xi) = \int_{-\infty}^{\infty} P(\xi|x)\rho_k(x)\mathrm{d}x\ .$$

At equilibrium, this operation leaves the distribution $\rho_k = \rho_{k+1} = \rho^*$ unchanged, i.e.,

$$\rho^*(\xi) = \int_{-\infty}^{\infty} P(\xi|x)\rho^*(x)\mathrm{d}x\ . \quad (14)$$

Rather than attempt to solve this implicit integral equation for ρ^* directly, we instead examine the second (raw) moment of the distribution μ_2 by multiplying both sides of Eq. (14) by ξ^2 and integrating over all ξ:

$$\mu_2 = \int_{-\infty}^{\infty} \xi^2\rho^*(\xi)\mathrm{d}\xi = \int_{-\infty}^{\infty} \xi^2 \left[\int_{-\infty}^{\infty} \rho^*(x)P(\xi|x)\mathrm{d}x\right]\mathrm{d}\xi$$
$$= \int_{-\infty}^{\infty} \rho^*(x) \int_{-\infty}^{\infty} \xi^2 \frac{1}{G(x)\sqrt{2\pi\Delta t}} e^{\frac{-[\xi - x - F(x)\Delta t]^2}{2G(x)^2\Delta t}} \mathrm{d}\xi\ \mathrm{d}x\ .$$

After swapping the order of integration[14], we observe that the inner integral over ξ is of the form

$$\frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{\frac{-(u-a)^2}{2s^2}}\mathrm{d}u = a^2 + s^2$$

with $u = \xi$, $a = x + F(x)\Delta t$, and $s = G(x)\sqrt{\Delta t}$. So we find

$$\mu_2 = \int_{-\infty}^{\infty} \rho^*(x)\Big[x^2 + 2xF(x)\Delta t + F(x)^2\Delta t^2 + G(x)^2\Delta t\Big]\mathrm{d}x\ .$$

Distributing the integral and subtracting μ_2 from both sides (note that the integral of x^2 against ρ^* is simply the definition of μ_2), we find

$$0 = \Delta t \int_{-\infty}^{\infty} \rho^*(x)\left[2xF(x) + G(x)^2\right]\mathrm{d}x + \Delta t^2 \int_{-\infty}^{\infty} \rho^*(x)F(x)^2\mathrm{d}x\ , \quad (15)$$

which should hold exactly for any such Itô system. Enforcing this to leading order in Δt for our cubic stochastic attractor gives

$$\begin{aligned} 0 = &\int_{-\infty}^{\infty} \rho^*(x)\big[2x(-x^3 - 3\sigma^2 x) + (9\sigma^2x^4 + 36\sigma^4x^2 + 15\sigma^6)\big]\mathrm{d}x \\ =&15\sigma^6 \int_{-\infty}^{\infty} \rho^*(x)\mathrm{d}x + (36\sigma^4 - 6\sigma^2)\int_{-\infty}^{\infty} x^2\rho^*(x)\mathrm{d}x + (9\sigma^2 - 2)\int_{-\infty}^{\infty} x^4\rho^*(x)\mathrm{d}x\ . \\ =&15\sigma^6 + 6\sigma^2(6\sigma^2 - 1)\mu_2 + (9\sigma^2 - 2)\mu_4 \end{aligned} \quad (16)$$

So we obtain a relationship between moments of the equilibrium ρ^*.

However we notice a problem: if σ is large enough that $9\sigma^2 - 2 > 0$ and $6\sigma^2 - 1 > 0$ (i.e., $\sigma > \sqrt{2}/3$), all terms on the right hand side are positive and there is no way for the equality to hold.

If we had preserved all terms from Eq. (15), rather than truncating at leading order, we would have obtained the full, exact relation

$$\begin{aligned} 0 = 15\sigma^6 &+ (36\sigma^4 - 6\sigma^2 + 9\sigma^4\Delta t)\mu_2 \\ &+ (9\sigma^2 - 2 + 6\sigma^2\Delta t)\mu_4 + \Delta t\ \mu_6\ . \end{aligned} \quad (17)$$

This still does not avoid the problematic implication at large σ—in fact, it makes the situation slightly "worse" by adding more positive terms. This contradiction implies that we were wrong to treat μ_2 as finite: the equilibria for these values of σ must have infinite second moments.[15]

If we repeat our above analysis, but with the $2k^{\text{th}}$ moment of ρ^* instead of the second[16], we have

$$\begin{aligned} \mu_{2k} &= \int_{-\infty}^{\infty} \xi^{2k}\rho^*(\xi)\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \rho^*(x) \int_{-\infty}^{\infty} \frac{\xi^{2k}}{G(x)\sqrt{2\pi\Delta t}} e^{\frac{-[\xi - x - F(x)\Delta t]^2}{2G(x)^2\Delta t}} \mathrm{d}\xi\ \mathrm{d}x\ . \end{aligned} \quad (18)$$

Integrals of the following form arise:

$$I_{2k} := \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} u^{2k} e^{\frac{-(u-a)^2}{2\sigma^2}} \mathrm{d}u$$
$$= (2k)! \sum_{i=0}^{k} \frac{\sigma^{2i} a^{2k-2i}}{(2i)!!(2k-2i)!} \, .$$

So with any Itô SDE we have

$$\mu_{2k} = \int_{-\infty}^{\infty} \rho^*(x)$$
$$\times \left[(2k)! \sum_{i=0}^{k} \frac{(G(x)\sqrt{\Delta t})^{2i}(x + F(x)\Delta t)^{2k-2i}}{(2i)!!(2k-2i)!} \right] \mathrm{d}x$$
$$= \sum_{i=0}^{k} \frac{(2k)!}{(2i)!!(2k-2i)!} \Delta t^i \int_{-\infty}^{\infty} \rho^*(x) G(x)^{2i}$$
$$\times \sum_{j=0}^{2k-2i} \binom{2k-2i}{j} x^j [F(x)\Delta t]^{2k-2i-j} \mathrm{d}x \, .$$

Regrouping by powers of Δt and retaining only leading order behavior, we find that the constant term $(i = 0, j = 2k)$ cancels from the left hand side, leaving

$$0 = \Delta t \int_{-\infty}^{\infty} \rho^*(x) \left[2kx^{2k-1} F(x) + G(x)^2 \right] \mathrm{d}x \, .$$

This relation should hold for any equilibrium of an Itô SDE for which the $2k^{\text{th}}$ raw moment is finite. If $F(x)$ and $G(x)^2$ are polynomials, this may be used to obtain a recursion relation for all moments of the equilibrium ρ^*.

For example, in the case of our cubic nonlinear-stochastic attractor from Eq. (4),

$$0 = \int_{-\infty}^{\infty} \rho^*(x) \Big[2kx^{2k-1}(-x^3 - 3\sigma^2 x)$$
$$+ (9\sigma^2 x^4 + 36\sigma^4 x^2 + 15\sigma^6) \Big] \mathrm{d}x$$
$$= 15\sigma^6 + 36\sigma^4 \mu_2 + 9\sigma^2 \mu_4 - 6k\sigma^2 \mu_{2k} - 2k\mu_{2k+2}$$

for integers $k \geq 1$.

While this slightly under-specified system of equations doesn't yield exact moments, it implies that those moments should lie on a surface, which we confirm by numerical simulation (see section S2 of the SM). When the typical magnitude of x is small compared to σ (i.e., $\mu_2 \ll \sigma^2$), however, Eq. (5) is well approximated by an SDE with constant noise and linear drift: an Ornstein–Uhlenbeck process [17, 18] (see also, e.g., [19] or [20]). This implies a normal distribution at equilibrium, with moment relationship

$$\mu_4 = 3\mu_2^2. \tag{19}$$

Plugging this additional constraint into our lowest-order relation Eq. (16) yields

$$0 = 15\sigma^6 + 6\sigma^2(6\sigma^2 - 1)\mu_2 + 3(9\sigma^2 - 2)\mu_2^2 \, ,$$

which agrees well with simulation in the relevant parameter region: see Fig. 2. For a direct look at the Gaussian nature of equilibria across this transition, see section S3 of the SM.

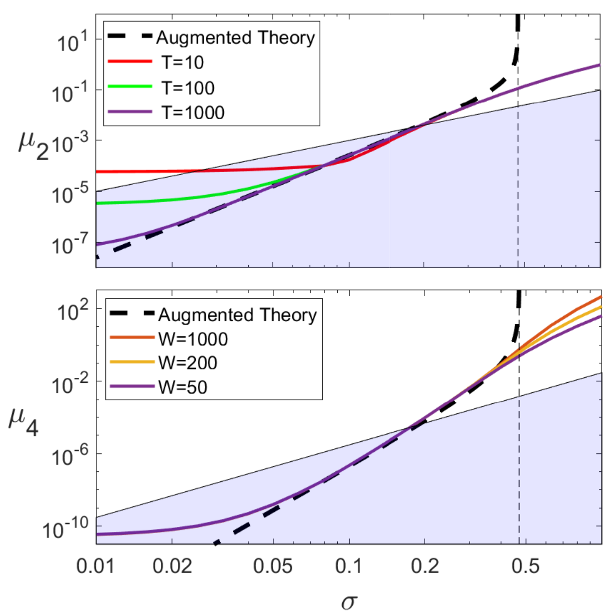

FIG. 2: **Numerical validation.** Comparison of numerical results (via Fokker-Planck evolution) to the theoretical relation, augmented with the extra Gaussian condition $\mu_4 = 3\mu_2^2$. The shaded region indicates $\mu_2 < 0.1\sigma^2$ (top) and correspondingly $\mu_4 < 0.03\sigma^4$ (bottom), where the Gaussian approximation (from $\mu_2 \ll \sigma^2$) should be most valid. **Top:** Smaller σ values take longer simulated time T to equilibrate, but do approach the theorized line. For high noise amplitudes, the relation need not hold, and indeed theory suggests that μ_2 and μ_4 should diverge for $\sigma > \sqrt{2}/3$ (indicated by the vertical dashed line). **Bottom:** As predicted by theory, the fourth moment μ_4 does indeed appear to diverge for $\sigma > \sqrt{2}/3$, though simulation with ever wider domain width W (measured in number of standard deviations of the equilibrium solution) is needed capture more of the distribution's tails (all curves shown for $T = 100$).

III. DISCUSSION AND LIMITATIONS

The first proposition of this paper—the argument for Ito-equivalency of nonlinear Langevin-type systems—is really a proposed definition rather than a theoretical result. Like Langevin equations themselves, the notation is simple and intuitive, but solid mathematical interpretation requires the use of the more rigorous notation, and we propose that interpretation in terms of Itô calculus.

We apply logic based on the central limit theorem for finite-variance random variables, but the Langevin noise terms are not regular random variables and their variance may not be well-defined or finite. If variance is treated as well-defined but not finite, other (non-Gaussian) stable distributions per time-step may arise, rather than normally distributed Itô time-steps.

We also note the perhaps-undesirable sensitivity to the assumption of Gaussian underlying noise in Eq. (1). In particular, the assumption that η_t is normally distributed may be incorrect for some systems with biased or irregularly shaped noise, and if the noise shape is known it should be used.

The Itô equilibrium analysis ending with Eq. (15) applies to any Itô system with an equilibrium where the second raw moment of that equilibrium is finite, but it is of particular use when the functions F and G^2 are polynomial in nature, since this allows the analysis to culminate with a relation between even moments rather than merely integrals against an unknown distribution.

Finally, in the analysis of our particular cubic-attractor equilibrium, we have employed an approximation (valid only for $\sigma^2 \gg \mu_2$) which allowed us to fully prescribe the moments of the equilibrium when they are finite. It remains unclear whether a more general constraint valid for arbitrary σ can be found.

IV. CONCLUSIONS

We have shown that a class of “nonlinear-stochastic” Langevin equations may be interpreted such that they have well-defined behavior after conversion to an equivalent Itô system. We have applied this theory to a class of nonlinear attracting fixed points to analyze their equilibria via moment relations, and showed that simulations bear out this analysis. This type of equilibrium may be more general than initially apparent, since nearly any isolated attracting fixed point is locally well-approximated by equations of this form.

This conversion technique should lead to more faithful physical modeling, yielding qualitatively different behavior when compared to simplifications which transform a deterministic quantity and add noise afterward. In particular, we have shown that there exists a critical noise level in one such system which leads to divergent moments of its equilibrium, something that cannot occur if x-independent noise is simply added after the nonlinear operation. Conversely, our reasoning also leads to the implication that apparent Itô behavior might be driven by any number of nonlinear Langevin processes.

Independent but complementary to these considerations, the equilibrium-analysis section of this work applies to all Itô systems, and can lead to recursive moment relations or other insights in their study when exact equilibrium solutions aren’t attainable. In particular, we note that the divergent moments this technique exposes for the example system would be difficult to deduce using numerical solutions of the Itô (or corresponding Fokker-Planck) equation in question.

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Supplemental Material: Modeling and analysis of systems with nonlinear functional dependence on random quantities

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S1. GENERALIZATION TO POSITIVE-INTEGER ATTRACTORS

Here we examine a generalization of the nonlinear attracting system from the main text: the n^{th}-order attracting fixed point. As in the main text, the nonlinear attracting function is applied to a Gaussian random variable (in the Langevin sense) centered on the current value (using $N(r;x,\sigma)$ as shorthand for the Gaussian pdf with mean x and standard deviation σ):

$$r := x + \sigma\eta, \text{ i.e. } r \sim N(r|x,\sigma)$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\operatorname{sgn}(r)|r|^n, \; n \in \mathbf{Z}^+ \; . \tag{1}$$

Using the proposed Itô conversion from the main text, this should be equivalent to the system

$$\mathrm{d}x = F(x|\sigma,n)\mathrm{d}t + G(x|\sigma,n)\mathrm{d}W, \text{ where} \tag{2}$$
$$F(x|\sigma,n) = \left\langle \frac{\mathrm{d}x}{\mathrm{d}t} \right\rangle = \int_{-\infty}^{\infty} \Big[-\operatorname{sgn}(r)|r|^n \Big] N(r|x,\sigma)\mathrm{d}r \tag{3}$$
$$G(x|\sigma,n) = \operatorname{std}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \sqrt{\int_{-\infty}^{\infty} [-\operatorname{sgn}(r)|r|^n - F(x|\sigma,n)]^2 N(r|x,\sigma)\mathrm{d}r}. \tag{4}$$

We first note that G can be easily computed in terms of F:

$$\begin{aligned}
G(x|\sigma,n) &= \sqrt{\int_{-\infty}^{\infty} \Big[-\operatorname{sgn}(r)|r|^n - F(x|\sigma,n) \Big]^2 N(r|x,\sigma)\mathrm{d}r} \\
&= \sqrt{\int_{-\infty}^{\infty} \Big[r^{2n} + 2\operatorname{sgn}(r)|r|^n F(x|\sigma,n) + F^2(x|\sigma,n) \Big] N(r|x,\sigma)\mathrm{d}r} \\
&= \sqrt{\int_{-\infty}^{\infty} r^{2n} N(r|x,\sigma)\mathrm{d}r + 2F(x|\sigma,n)\int_{-\infty}^{\infty} \operatorname{sgn}(r)|r|^n N(r|x,\sigma) + F^2(x|\sigma,n)\int_{-\infty}^{\infty} N(r|x,\sigma)\mathrm{d}r} \\
&= \sqrt{\mu_{2n,x,\sigma} + 3F^2(x|\sigma,n)},
\end{aligned}$$

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where the first term is not quite $F(x|\sigma, 2n)$—due to the sign difference—but rather the (much simpler) $2n^{\text{th}}$ non-central moment of the normal distribution $N(r|x, \sigma)$:

$$\begin{aligned}
\mu_{2n,x,\sigma} &:= \int_{-\infty}^{\infty} r^{2n} N(r|x,\sigma)\mathrm{d}r \\
&= \int_{-\infty}^{\infty} (x+z)^{2n} N(z|0,\sigma)\mathrm{d}z \\
&= \sum_{i=0}^{2n} \binom{2n}{i} x^{2n-i} \int_{-\infty}^{\infty} z^i N(z|0,\sigma)\mathrm{d}z \\
&= \sum_{j=0}^{n} \binom{2n}{2j} (2j-1)!! x^{2n-2j} \sigma^{2j} \ .
\end{aligned}$$

We now seek F. If n is odd, this calculation is simply the non-central moment again:

$$\begin{aligned}
F(x|\sigma, n) &= \int_{-\infty}^{\infty} -r^n \left[\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(r-x)^2}{2\sigma^2}} \right] \mathrm{d}r \quad \text{if } n \text{ odd} \\
&= \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \binom{n}{i} (i-1)!! x^{n-i} \sigma^i && (5) \\
&= \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} A_{n,i}\ x^{n-i} \sigma^i, && (6) \\
\text{where } A_{n,i} &: = \binom{n}{i} (i-1)!! = \frac{n!}{i!!(n-i)!} \ . && (7)
\end{aligned}$$

However if n is even, we must split the integral and the boundary terms no longer cancel due to the sign difference:

$$\begin{aligned}
F(x|\sigma, n) &= \int_{-\infty}^{\infty} \Big[-\operatorname{sgn}(r)|r|^n \Big] N(r; x, \sigma)\mathrm{d}r \\
&= \int_{-\infty}^{0} r^n N(r; x, \sigma)\mathrm{d}r - \int_{0}^{\infty} r^n N(r; x, \sigma)\mathrm{d}r \\
&= \int_{-\infty}^{\infty} r^n N(r; x, \sigma)\mathrm{d}r - 2\int_{0}^{\infty} r^n N(r; x, \sigma)\mathrm{d}r \\
&= \sum_{\substack{i=0 \\ i \text{ even}}}^{n} A_{n,i}\ x^{n-i}\sigma^i - 2\int_{-x}^{\infty} (x+w)^n N(w; 0, \sigma)\mathrm{d}w \\
&= \sum_{\substack{i=0 \\ i \text{ even}}}^{n} A_{n,i}\ x^{n-i}\sigma^i - 2\sum_{i=0}^{n} \binom{n}{i} x^i \underbrace{\int_{-x}^{\infty} w^{n-i} N(w; 0, \sigma)\mathrm{d}w}_{J_{-x}(n-i)} \ . && (8)
\end{aligned}$$

We then need to process the expression marked J using IBP, and unlike before we have boundary terms:

$$\begin{aligned}
J_{-x}(p) &: = \int_{-x}^{\infty} w^p N(w; 0, \sigma)\mathrm{d}w \\
&= \left[(-1)^{p-1} \frac{(p-1)!!}{\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} \sum_{\substack{j=1 \\ j \text{ odd}}}^{p} \frac{x^{p-j}\sigma^j}{(p-j)!!} \right] + \begin{cases} 0, & \text{if p odd} \\ \frac{(p-1)!!}{2}\sigma^p \left[1 + \operatorname{Erf}\left(\frac{x}{\sigma\sqrt{2}} \right) \right], & \text{if p even} \ . \end{cases}
\end{aligned}$$

We can now expand the relevant sum from (8), separating out the Erf parts of J from the sums:

$$
\begin{aligned}
-2\sum_{i=0}^{n}\binom{n}{i}x^i J_{-x}(n-i) = -2 &\sum_{\substack{i=0\\ n-i \text{ even}}}^{n}\binom{n}{i}x^i\frac{(n-i-1)!!}{2}\sigma^{n-i}\left[1+\text{Erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right] \\
&-2\sum_{i=0}^{n}\binom{n}{i}x^i(-1)^{n-i-1}\frac{(n-i-1)!!}{\sqrt{2\pi}}e^{\frac{-x^2}{2\sigma^2}}\sum_{\substack{j=1\\ j \text{ odd}}}^{n-i}\frac{x^{n-i-j}\sigma^j}{(n-i-j)!!} \\
= -&\left[1+\text{Erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\sum_{\substack{i=0\\ i \text{ even}}}^{n}A_{n,n-i}\ x^i\sigma^{n-i} \\
&+\sqrt{\frac{2}{\pi}}e^{\frac{-x^2}{2\sigma^2}}\sum_{i=0}^{n}(-1)^i A_{n,n-i}\sum_{\substack{j=1\\ j \text{ odd}}}^{n-i}\frac{x^{n-j}\sigma^j}{(n-i-j)!!}\ ,
\end{aligned}
$$

using the fact that n is even to recondition the sums and simplify the alternating negative sign. We now notice that we can combine all terms of constant j in the second line, rearranging the order of the sums:

$$
\sqrt{\frac{2}{\pi}}e^{\frac{-x^2}{2\sigma^2}}\sum_{i=0}^{n}(-1)^i A_{n,n-i}\sum_{\substack{j=1\\ j \text{ odd}}}^{n-i}\frac{x^{n-j}\sigma^j}{(n-i-j)!!} = \sqrt{\frac{2}{\pi}}e^{\frac{-x^2}{2\sigma^2}}\sum_{\substack{j=1\\ j \text{ odd}}}^{n-1}x^{n-j}\sigma^j\sum_{i=0}^{n-j}(-1)^i\frac{A_{n,n-i}}{(n-i-j)!!}\ .
$$

So we may rename index variables $i \to k$ and $j \to i$ and define another constant:

$$
B_{n,i} := \sqrt{\frac{2}{\pi}}\sum_{k=0}^{n-i}(-1)^k\frac{A_{n,n-k}}{(n-k-i)!!} = \sqrt{\frac{2}{\pi}}\sum_{k=0}^{n-i}(-1)^k\binom{n}{k}\frac{(n-k-1)!!}{(n-k-i)!!}
$$

in order to reunify the sums for even and odd i:

$$
-2\sum_{i=0}^{n}\binom{n}{i}x^i J_{-x}(n-i) = \sum_{i=0}^{n}x^{n-i}\sigma^i\cdot\begin{cases}-A_{n,n-i}\left[1+\text{Erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right], & i \text{ even}\\ B_{n,i}\ e^{\frac{-x^2}{2\sigma^2}}, & i \text{ odd}\end{cases}.
$$

And finally recombine with the first term of (8) to get F itself:

$$
\begin{aligned}
F(x|\sigma,n)|_{n \text{ even}} &= \sum_{\substack{i=0\\ i \text{ even}}}^{n}A_{n,i}\ x^{n-i}\ \sigma^i + \sum_{i=0}^{n}x^{n-i}\sigma^i\cdot\begin{cases}-A_{n,n-i}\left[1+\text{Erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right], & i \text{ even}\\ B_{n,i}\ e^{\frac{-x^2}{2\sigma^2}}, & i \text{ odd}\end{cases} \\
&= \sum_{i=0}^{n}x^{n-i}\sigma^i\cdot\begin{cases}A_{n,i}-A_{n,n-i}\left[1+\text{Erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right], & i \text{ even}\\ B_{n,i}\ e^{\frac{-x^2}{2\sigma^2}}, & i \text{ odd}\end{cases}.
\end{aligned} \tag{9}
$$

In particular, for the $n = 2$ case which might have utility for modeling drag amidst turbulence, we have

$$
\begin{aligned}
F(x|\sigma,2) &= -\left(\sigma^2+x^2\right)\text{Erf}\left(\frac{x}{\sigma\sqrt{2}}\right) - \sqrt{\frac{2}{\pi}}x\sigma e^{\frac{-x^2}{2\sigma^2}}\ , \\
\implies G(x|\sigma,2) &= \sqrt{\mu_{4,x,\sigma}+3F^2(x|\sigma,2)} \\
&= \sqrt{x^4+6x^2\sigma^2+3\sigma^4+3\left[(\sigma^2+x^2)\,\text{Erf}\left(\frac{x}{\sigma\sqrt{2}}\right)+\sqrt{\frac{2}{\pi}}x\sigma e^{\frac{-x^2}{2\sigma^2}}\right]^2}\ .
\end{aligned}
$$

S2. ADDITIONAL FIGURES

Here we have provided some additional 3-dimensional figures elaborating on the 2-dimensional ones from the main text.

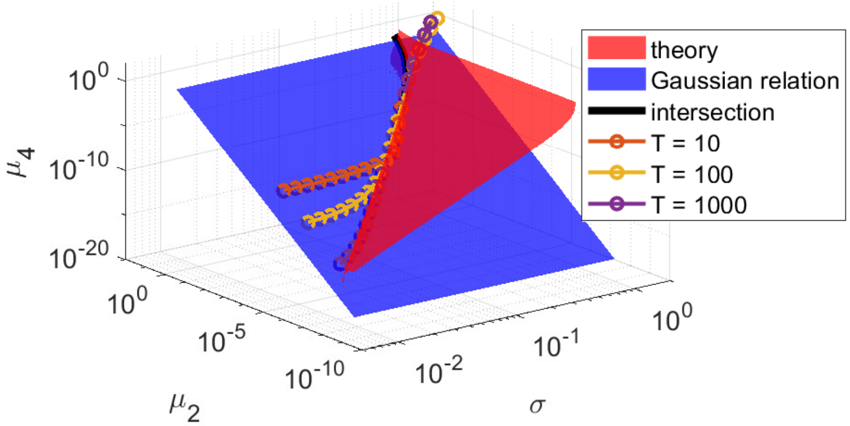

FIG. S1: **Small-noise convergence.** Three dimensional version of Fig. 1 (top panel) from the main paper; at small noise σ, solutions converge over increasing simulated time T to the intersection of the Gaussian condition and Eq. (11), our theorized surface from the main paper relating the equilibrium's second and fourth moments to the inherent noise σ: $0 = 15\sigma^6 + 6\sigma^2(6\sigma^2 - 1)\mu_2 + (9\sigma^2 - 2)\mu_4$.

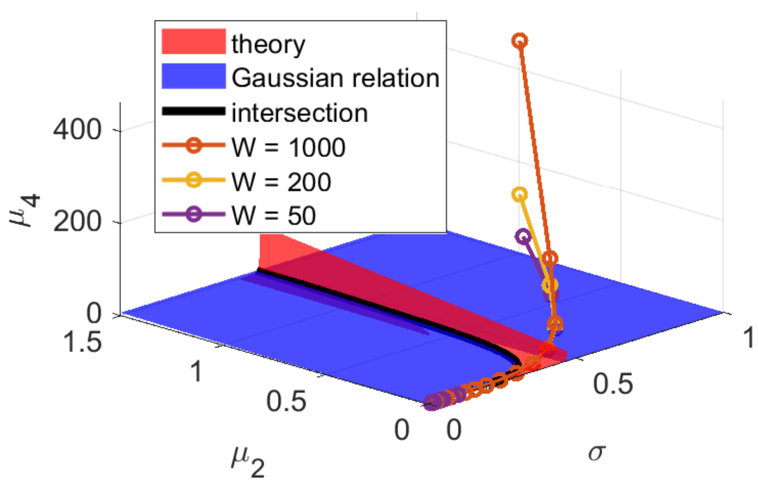

FIG. S2: **Large-noise divergence.** Three dimensional version of Fig. 1 (bottom panel) from main paper relating noise amplitude σ to the second and fourth moments of the equilibrium (μ_2 and μ_4, respectively), but on a linear scale. We can see the clear suggestion of divergence for (at least) μ_4; as our domain captures more standard deviations W of the solution, the measured μ_4 grows without bound. We propose that noise values σ beyond the asymptote "curtain" $\sigma^* = \sqrt{2}/3 \approx 0.47$ must have divergent second and fourth moments.

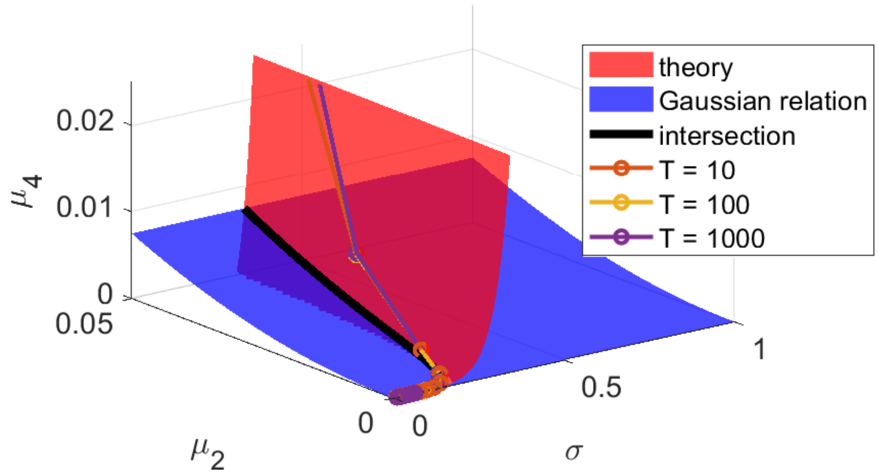

FIG. S3: **Gaussian validity boundary.** Three dimensional, linear-scale zoom of the region where the Gaussian assumption fails to hold. Our solutions fall off the intersection line due to their non-Gaussian nature.

S3. GAUSSIAN/NON-GAUSSIAN TRANSITION

In the main text we argue that the small-noise limit of the stochastic-cubic-attractor system leads to a Gaussian equilibrium. We can see this transition clearly in Fig. S4.

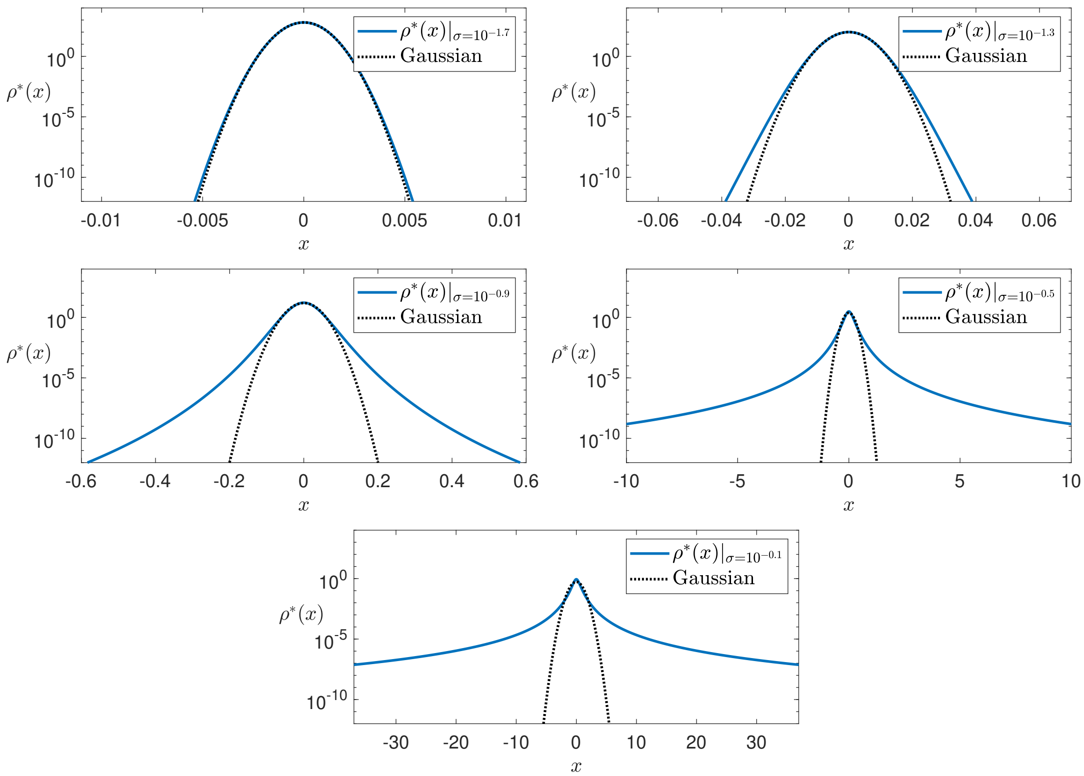

FIG. S4: **Equilibrium shape transition.** Comparison of equilibrium distributions for different noise values to Gaussian distributions with the same standard deviation. **Top Left:** For small σ, equilibria exhibit the signature parabolic shape (on semi-log axes) indicating a near-Gaussian distribution. **Other panels:** For larger σ, equilibria exhibit increasingly "fat tails" differentiating them from Gaussians.