Strong Disorder Renormalization Group

Strong Disorder Renormalization Group (SDRG) is a formal apparatus to extract global physics of strongly-disordered systems via successive decimation of highest energy scales. SDRG was developed by Dasgupta and Ma [MDH79] [DM80] to investigate the low-energy behavior of the random Heisenberg chain and later extended and formalized by Bhatt and Lee[BL82], and Daniel Fisher[Fis92], who showed that the technique gives asymptotically exact low energy results when the critical behavior is controlled by infinite randomness fixed points.

Disorder

Disordered systems (not to be confused with disordered, like the paramagnetic, phases) are useful models for impurities, vacancies, or dislocations in a perfect lattice. We distinguish between quenched, or frozen-in disorder, and annealed disorder, which fluctuates during the duration of an experiment. Annealed disorder is typically easier to deal with, since these impurities can be assumed to be in thermal equilibrium with the rest of the system, and the partition function can then be averaged out over those degrees of freedom for a statistical mechanical description. Quenched disorder, however, presents more difficulties and often needs to be handled using techniques from SDRG.

The prototypical example for a disordered lattice is the modified Ising chain Hamiltonian with a non-uniform exchange interaction, viz

$$\mathcal{H} = \sum_i B_i \hat{S}_i + \sum_{<ij>} J_{ij} \hat{S}_i \hat{S}_j \tag{1}$$

The disordered Ising chain can model two kinds of disordered systems.

- In an instance where the couplings J_{ij} are the only disordered terms in the Hamiltonian, and the couplings are fortuitously all positive, the disorder still prefers the ferromagnetic phase at sufficiently low temperatures, and the paramagnetic phase, at sufficiently high temperatures. The effect of the disorder is simply to change the local tendancy towards ferromagnetism, or the "local critical temperature". Such disorders are labelled Random-T_c disorders, or random mass disorders.
- In an instance where the disorder results from a random magnetic field B_i acting on each spin, the spin-flip symmetry is broken locally, and we label these as random-field disorders.

More intricate realizations of disorder are possible. Random anisotropy disorder results from the disorder breaking the rotational invariance in a Heisenberg chain. A disordered spin-glass system results from the interaction picking up random signs causing the system to end up in a frustrated state[FH91].

Harris[Har74] studied the condition for a critical point to be 'clean' against random-mass disorders, and came up with the Harris criterion $d\nu > 2$, where d is the dimensionality of the lattice, and ν is the correlation length critical exponent. If the Harris criterion is satisfied, a random-mass disordered system looks less and less random at larger length scales, and the randomness vanishes asymptotically at criticality, leading to a clean phase transition.

In a random-field disordered Ising ferromagnet, the random field breaks the spin flip symmetry, and individual sites may prefer to align with the local field or their neighbouring spins depending on the relative strength of B_i and J. Imry and Ma[IM75] developed a heuristic argument to state a criteria for the stability of the ferromagnetic phase against domain formations caused by local field fluctuations. Demanding the energy cost to build up a domain wall to be more than compensated with the decrease in energy by the entire domain flipping to align with the rest of the lattice, we arrive at an estimate

$$\sqrt{W} L^{d/2} < J L^{d-1} \tag{2}$$

where L is the length of the domain, W is the variance of the random field, and d is the dimension of the lattice. Hence, there is a critical dimension, here $d = 2$, above which domain formation is always unfavorable and the ferromagnetic phase is stable against random-field disorders. This was proved rigorously by Aizenman and Wehr[AW89].

Disordered transverse Ising chain

The paradigmatic example for SDRG is that of a disordered one-dimensional transverse Ising chain, with the Hamiltonian

$$\mathcal{H} = -\sum_i J_i \sigma_i^z \sigma_{i+1}^z - \sum_i B_i \sigma_i^x \tag{3}$$

We assume the J_i's and the B_i's to be broadly distributed positive valued quantities. Each of the coupling constants and field strengths define the energy scale associated with anti-aligning adjacent spins or flipping a spin. The renormalization trick is to successively decimate the highest energy levels and replace them with effective interactions to eventually obtain an effective low-energy Hamiltonian. There are two possibilities here:

1. The largest energy scale is a field, say B_i. This essentially locks the spin σ_i along the positive x-direction. We can then replace σ_i by an effective interaction $\tilde{J}_i$ between σ_{i-1} and σ_{i+1}. Consider the three-site Hamiltonian

$$\mathcal{H}_i = -B_i \sigma_i^x - J_{i-1} \sigma_{i-1}^z \sigma_i^z - J_i \sigma_i^z \sigma_{i+1}^z \tag{4}$$

   To lowest order in perturbation theory, we can replace this by the effective term $\mathcal{H}_i' = -\tilde{J}_i \sigma_{i-1}^z \sigma_{i+1}^z$, with

$$\tilde{J}_i = J_{i-1} J_i / B_i \tag{5}$$

   which is smaller than both J_{i-1} and J_i.

2. The largest energy scale is a coupling, say J_i. This means that in the four-dimensional subspace spanned by σ_i and σ_{i+1}, there are two high-energy states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, and two low-energy states $|\uparrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$. For large enough J_i, excitations to the high-energy states can be neglected, and we can define an effective two state operator $\tilde{\sigma}_i$, with an effective field strength $\tilde{B}_i$. To lowest order in perturbation theory, we obtain

$$\tilde{B}_i = \frac{B_{i-1} B_i}{J_i} \tag{6}$$

   which is again smaller than either of the previous terms.

It is crucial to the usefulness of this prescription that the new terms added upon decimation be smaller than the terms we removed, so that every iteration actually takes us to a lower energy scale. A broad distribution in fields and couplings is essential as well, so that we are guaranteed that the terms adjacent to the largest term are statistically small enough to be treated perturbatively. This prescription is then run iteratively to bring down the highest energy scale $\Omega = \mathrm{Max}\{J_i, B_i\}$.

Renormalization Flow equations

Let us now mathematically formulate the SDRG flow equation derived from this process. Since the fields and couplings are random variables, we will talk in terms of probability distributions. Let $P(J,\Omega)$ and $R(B,\Omega)$ be the probability distributions of J_i and B_i respectively as a function of the current energy scale Ω. We also assume a knowledge of the bare values of these distributions, $P_I(J)$ and $R_I(B)$, from the un-renormalized Hamiltonian.

To derive the flow equation, consider decreasing the scale Ω by an amount $d\Omega$. As a result,

$$\begin{aligned} -dP(J,\Omega) = d\Omega R(\Omega,\Omega)\left[-2P(J,\Omega) + \int dJ_1 \int dJ_2 P(J_1,\Omega)P(J_2,\Omega)\delta\left(J - \frac{J_1 J_2}{\Omega}\right)\right] \\ + d\Omega\left[R(\Omega,\Omega) + P(\Omega,\Omega)\right]P(J,\Omega) \end{aligned} \quad (7)$$

The expression above demands some elaboration.

- The terms in the first line are due to decimation of the strong fields. The terms in the second line compensate for the decimation to restore the normalization of the probability distributions.
- The occurance of a decimation depends on the existence of a field variable between energies Ω and $\Omega - d\Omega$. The probability of there being such terms is given by the product $d\Omega R(\Omega,\Omega)$.
- Each decimation removes two couplings from the Hamiltonian $-2P(J,\Omega)$, and introduces a new coupling, according to the relation 5, which is denoted by the integral.
- Each instance of decimation thus reduces the number of field variables and the number of couplings by one, and hence the additive terms in the second line.

The flow equation for the field can also be written analogously, and we obtain the differential equations

$$\begin{aligned} -\frac{\partial P}{\partial \Omega} &= \left[P_\Omega - R_\Omega\right]P + R_\Omega \int dJ_1 \int dJ_2\, P(J_1,\Omega)P(J_2,\Omega)\delta\left(J - \frac{J_1 J_2}{\Omega}\right), \\ -\frac{\partial R}{\partial \Omega} &= \left[R_\Omega - P_\Omega\right]R + P_\Omega \int dB_1 \int dB_2\, R(B_1,\Omega)R(B_2,\Omega)\delta\left(B - \frac{B_1 B_2}{\Omega}\right) \end{aligned} \quad (8)$$

where P and P_Ω stand for $P(J,\Omega)$ and $P(\Omega,\Omega)$ respectively, and so on. The solutions of these integro-differential equations give us the SDRG flows.

A complete solution to the flow equations was provided by Fisher[Fis94][Fis95], which starts with shifting to logarithmic variables

$$\Gamma = \ln\left(\frac{\Omega_I}{\Omega}\right), \quad \zeta = \ln\left(\frac{\Omega}{J}\right), \text{and} \quad \beta = \ln\left(\frac{\Omega}{B}\right) \quad (9)$$

where Ω_I is the initial value of the cutoff. The new probability distributions are related to the distributions of the old variables as $P(J,\Omega) = \bar{P}(\zeta,\Gamma)/J$, and $R(B,\Omega) = \bar{R}(\beta,\Gamma)/B$. In terms of the new distributions,

$$\begin{aligned} \frac{\partial P}{\partial \Gamma} &= \frac{\partial P}{\partial \zeta} + \left[P_0 - R_0\right]P + R_0 \int_0^\zeta d\zeta_1 P(\zeta_1,\Gamma)P(\zeta - \zeta_1,\Gamma), \\ \frac{\partial R}{\partial \Gamma} &= \frac{\partial R}{\partial \beta} + \left[R_0 - P_0\right]R + P_0 \int_0^\beta d\beta_1 R(\beta_1,\Gamma)R(\beta - \beta_1,\Gamma) \end{aligned} \quad (10)$$

where P and P_0 stand for $P(\zeta,\Gamma)$ and $P(0,\Gamma)$ respectively, and so on, and the bars have been omitted for brevity.

Although a general solution of these equations presents considerable challenges, a simple ansatz can be plugged in to obtain a fixed point. Consider the exponentials

$$P(\zeta,\Gamma) = p_0(\Gamma)\exp(-p_0(\Gamma)\zeta), \text{and} \quad R(\beta,\Gamma) = r_0(\Gamma)\exp(-r_0(\Gamma)\beta) \quad (11)$$

These turn out to be valid solutions provided the inverse widths p_0 and r_0 satisfy the coupled differential equations

$$\frac{dp_0}{d\Gamma} = -r_0 p_0, \qquad (12)$$
$$\frac{dr_0}{d\Gamma} = -r_0 p_0$$

The resulting solution can be shown to be a global attractor of the SDRG flow.

Further Reading

1. References for this article [Voj13][RA13] [Sac11]
2. Griffiths-McCoy Singularity
3. Infinite disorder critical point
4. Percolation networks

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