SUPPORTING INFORMATION

Appendix for "Varieties of Clientelism: Machine Politics During Elections"

Proofs of Propositions 1 - 3

We refer to opposing voters as OV; to supporting nonvoters as SNV; and to opposing nonvoters as ONV. Also, for notational simplicity, let $h = g(c)f(x)dc\,dx$, $r = x - x^M$, and $s = -x - x^M$.

The proofs to Propositions 1 and 3 make use of the following lemma:

Lemma 1: For any allocation of budget B, a machine could buy more citizens if it had additional resources of any positive amount.

Proof. Let A be an allocation of budget B. Define $M(A)$ to be the set of citizens who vote for a machine given this allocation: $M(A) \equiv \{(x_i, c_i) : b_i \geq \overline{b}_i\}$, where b_i is the payment received by citizen i under allocation A and $\overline{b}_i$ is the payment required to buy this citizen. Limited resources means that for any allocation A, a machine cannot afford to buy all citizens: $\int\int \overline{b}_i h > B$. It follows that there exists a set $Q \notin M(A)$ of positive measure such that $\overline{b}_i > b_i$ for all $(x_i, c_i) \in Q$. Let $(\dot{x}_i, \dot{c}_i)$ be any point on the interior of Q and select η sufficiently small such that $\Delta(\eta) \equiv [\dot{x}_i, \dot{x}_i + \eta] \times [\dot{c}_i, \dot{c}_i + \eta] \subset Q$. Let $\theta > 0$ represent some nonzero amount of resources. Then by the continuity of $f(x)$ and $g(c)$, there exists a $\eta_0 < \eta$ such that for any θ, a machine can afford to buy all citizens in $\Delta(\eta_0)$: $\int_{\Delta(\eta_0)} \overline{b}_i h \leq \theta$. $\square$

Proposition 1: In an optimal allocation of resources, a machine sets $b^*_{VB} = 2b^*_{TB} = 2b^*_{DP} = 2b^*_{AB}$.

Proof. We will show (i) $b^*_{TB} = b^*_{DP}$ and (ii) $b^*_{VB} = 2b^*_{TB}$. (The proof to $b^*_{TB} = b^*_{AB}$ follows identical logic).

(i) Let b^*_{TB} and b^*_{DP} be the upper bounds on a machine's payments to SNV and ONV, respectively. For contradiction, assume A is an optimal allocation in which $b^*_{TB} \neq b^*_{DP}$. Without loss of generality, say $b^*_{TB} > b^*_{DP}$. We will show there exists an allocation A' that is affordable and produces a strictly greater number of net votes. Thus, A cannot be optimal.

Let S be a set with positive measure of SNV such that all citizens in set S have a required payment $\overline{b}_i = b^*_{TB}$. Let $(\hat{x}, \hat{c})$ be any point on the interior of S and take δ small enough such that $\Delta(\delta) \equiv [\hat{x}, \hat{x} + \delta] \times [\hat{c}, \hat{c} + \delta] \subset S$. Recall from Lemma 1 that Q is a set of citizens who remain unbought under allocation A. Let $R \subset Q$ be a set with positive measure of ONV such that all citizens in set R have a required payment $b^*_{TB} > \overline{b}_i > b^*_{DP}$. Let $(\tilde{x}, \tilde{c})$ be any point on the interior of R. Take μ small enough such that $\Delta(\mu) \equiv [\tilde{x}, \tilde{x} + \mu] \times [\tilde{c}, \tilde{c} + \mu] \subset R$. By the continuity of $f(x)$ and $g(c)$, there exists a $\delta_0 < \delta$ and a $\mu_0 < \mu$ such that $\int_{\Delta(\delta_0)} h = \int_{\Delta(\mu_0)} h$ (call this Equation A1). Observe that $\Delta(\delta_0)$ and $\Delta(\mu_0)$ have the same number of citizens, so buying either set produces the same net votes. Let $\theta \equiv \int_{\Delta(\delta_0)} \overline{b}_i h - \int_{\Delta(\mu_0)} \overline{b}_i h$ and note $\theta > 0$ because citizens on $\Delta(\delta_0)$ are more expensive than those on $\Delta(\mu_0)$. Finally, let $\Delta(\eta_0)$ be a set of citizens who are mutually exclusive of set $\Delta(\mu_0)$ and who do not receive rewards under allocation A. Formally, $\Delta(\eta_0) \subset Q$ and $\Delta(\mu_0) \cap \Delta(\eta_0) = \emptyset$.

Consider an allocation A' in which a machine buys all citizens in $\Delta(\mu_0)$, reduces payments to citizens on $\Delta(\delta_0)$ to zero, and redistributes the savings to citizens in $\Delta(\eta_0)$. Recall from Lemma 1 that citizens on $\Delta(\eta_0)$ can be be bought with resources θ. Formally, define $\Omega \equiv [\underline{X}, \overline{X}] \times [0, \overline{C}] - (\Delta(\delta_0) \cup \Delta(\mu_0) \cup \Delta(\eta_0))$. Let $A' = A$ for all (x_i, c_i) on Ω, $A' = 0$ for all (x_i, c_i) on $\Delta(\delta_0)$, and $A' = \overline{b}_i$ for all (x_i, c_i) on $\Delta(\mu_0)$ and for all (x_i, c_i) on $\Delta(\eta_0)$. The cost of A' is $\leq$ the cost of allocation A, and A' buys $\int_{\Delta(\eta_0)} h$ more citizens. Thus A cannot be an optimal allocation.

(ii) To show $b^*_{VB} = 2b^*_{TB}$ (or, equivalently, $b^*_{VB} = 2b^*_{DP}$ or $b^*_{VB} = 2b^*_{AB}$), we repeat the proof that $b^*_{TB} = b^*_{DP}$, replacing Equation (A1) with $\int_{\Delta(\delta_0)} h = 2\int_{\Delta(\mu_0)} h$, where $\Delta(\delta_0)$ is a subset of OV for whom $\overline{b}_i = b^*_{VB} > 2b^*_{TB}$, and where $\Delta(\mu_0)$ is a subset of SNV for whom $\frac{1}{2}b^*_{VB} > \overline{b}_i > b^*_{TB}$. □

Proposition 2: If a machine engages in electoral clientelism, then optimally it allocates resources across all three strategies of vote buying, turnout buying, and double persuasion.

Proof. Let $b^*_{VB} = b^{**}$ and $b^*_{TB} = b^*_{DP} = b^*_{AB} = b^*$. In an optimal allocation, the number of vote-buying recipients is $VB = N\int_{-\frac{b^{**}}{2}}^{0} \int_{\underline{C}}^{x^O} h$ (Equation A2), the number turnout-buying recipients is $TB = N\int_0^{\overline{X}} \int_r^{r+b^*} h$ (Equation A3), the number of double-persuasion recipients is $DP = N\int_{-\frac{b^*}{2}}^{0} \int_s^{r+b^*} h$ (Equation A4), and the number of abstention buying recipients is $AB = N\int_{\underline{X}}^{-\frac{b^{**}}{2}} \int_{s-b^*}^{s} h + N\int_{-\frac{b^{**}}{2}}^{0} \int_{x^O}^{s} h$ (Equation A5). By Proposition 1, $b^{**} = 2b^*$, so $b^* > 0 \iff b^{**} > 0$. It then follows from equations A2, A3, A4, and A5 that $VB > 0 \iff TB > 0 \iff DP > 0 \iff AB > 0$. □

Proposition 3: If $\overline{b}_i^{VB} \leq b^{**}$ and $c_i \leq x^O$, a machine pays $\overline{b}_i^{VB}$ to a OV. If $\overline{b}_i^{AB} \leq b^*$ and $c_i > x^O$, a machine pays $\overline{b}_i^{AB}$ to a OV. If $\overline{b}_i^{TB} \leq b^*$, a machine pays $\overline{b}_i^{TB}$ to a SNV. If $\overline{b}_i^{DP} \leq b^*$, a machine pays $\overline{b}_i^{DP}$ to a ONV. All other citizens receive no payment.

Proof. We prove the TB case; identical logic holds for other strategies. We show (i) if $\overline{b}_i^{TB} \leq b^{**}$, a machine pays $\overline{b}_i^{TB}$ to a SNV; (ii) if $\overline{b}_i^{TB} > b^*$, a machine offers $b_i = 0$ to a SNV.

(i) Let b^* be the upper bound on payments a machine makes to SNV. Define $M(A)$ to be the set of SNV who vote for the machine given the payment allocation A. For contradiction, assume A is an optimal allocation in which the machine does not buy all SNV who are cheaper than b^*. Formally, there exists a set Z with positive measure of SNV receiving $b_i < \overline{b}_i < b^*$. We will show there exists a A' that is affordable and produces a strictly greater number of net votes. Thus, A cannot be optimal.

Let $(\hat{x}, \hat{c})$ be any point on the interior of $M(A)$ and take δ small enough such that $\Delta(\delta) \equiv [\hat{x}, \hat{x} + \delta] \times [\hat{c}, \hat{c} + \delta] \subset M(A)$. Let $(\tilde{x}_i, \tilde{c}_i)$ be any point in Z and select μ sufficiently small such that $\Delta(\mu) \equiv [\tilde{x}_i, \tilde{x}_i + \mu] \times [\tilde{c}_i, \tilde{c}_i + \mu] \subset Z$. By the continuity of $f(x)$ and $g(c)$ there exists a $\delta_0 < \delta$ and $\mu_0 < \mu$ such that $\int_{\Delta(\delta_0)} h = \int_{\Delta(\mu_0)} h$. Observe that $\Delta(\delta_0)$ and $\Delta(\mu_0)$ have the same number of SNV, so buying either set produces the same net votes. Let $\theta \equiv \int_{\Delta(\delta_0)} b_i h - \int_{\Delta(\mu_0)} \bar{b}_i h$ and note that $\theta > 0$ because citizens in $\Delta(\mu_0)$ are cheaper than those in $\Delta(\delta_0)$. Consider an allocation A' in which a machine buys all citizens in $\Delta(\mu_0)$, reduces payments to citizens in $\Delta(\delta_0)$ to zero, and redistributes the savings to citizens in $\Delta(\eta_0)$. Recall from Lemma 1 that $\Delta(\eta_0)$ is a set of citizens who remain unbought under allocation A, and who could be bought with resources θ. Formally, define $\Omega \equiv [\underline{X}, \overline{X}] \times [0, \overline{C}] - (\Delta(\delta_0) \cup \Delta(\mu_0) \cup \Delta(\eta_0))$. Let $A' = A$ for all (x_i, c_i) on Ω, $A' = 0$ for all (x_i, c_i) on $\Delta(\delta_0)$, and $A' = \bar{b}_i$ for all (x_i, c_i) on $\Delta(\mu_0)$ and for all (x_i, c_i) on $\Delta(\eta_0)$. The cost of A' is less than or equal to the cost of allocation A and A' buys $\int_{\Delta(\eta_0)} h$ more citizens. Thus A cannot be an optimal allocation.

(ii) Recall that b^* is the upper bound on payments a machine makes to SNV. Offering b^* to a citizen for whom $\bar{b}_i^{TB} > b^*$ is insufficient to induce turnout (i.e., it is an underpayment). Formally, underpayment can be defined as a set of positive measure P of SNV receiving rewards b_i such that $\bar{b}_i > b_i > 0$. For contradiction, assume A is an optimal allocation in which a machine underpays some SNV. We show there exists an affordable allocation A'' that produces strictly more net votes than A. Thus, A cannot be optimal.

Define $\theta \equiv \int_P b_i h$ as the resources the machine devotes to citizens in set P. In allocation A, $\theta > 0$. Observe that since the machine underpays these citizens, it receives 0 net votes in return. Recall from Lemma 1 that a machine can purchase all citizens on set $\Delta(\eta_0)$ for resources θ, where $\Delta(\eta_0)$ are citizens who remain unbought under allocation A. Consider an allocation A'' in which a machine reduces payments to citizens on set P to 0 and uses the savings to purchase citizens on set $\Delta(\eta_0)$. Formally, define $\Omega \equiv [\underline{X}, \overline{X}] \times [0, \overline{C}] - (P \cup \Delta(\eta_0))$. Let $A'' = A$ for all (x_i, c_i) on Ω, $A'' = 0$ for all (x_i, c_i) on P, and $A'' = \bar{b}_i$ for all (x_i, c_i) on $\Delta(\eta_0)$. Then the costs of A'' are $\leq$ the costs of A, and A'' buys $\int_{\Delta(\eta_0)} h$ more citizens. Thus A cannot be an optimal allocation. $\square$

Comparative Statics

For analysis of comparative statics, we assume f and g are distributed uniformly. The machine's constrained optimization problem, where λ is the Lagrangian multiplier, is: $\underset{b_{\text{TB}},\, b_{\text{DP}},\, b_{\text{VB}},\, b_{\text{AB}}}{max} V^M - V^O - \lambda(E - B)$.

The machine maximizes the difference between its votes (V^M) and opposition votes (V^O), given that total expenditures (E) must be less than or equal to its budget B. Note that $V^O = \int_{\underline{X}}^{-\frac{b^{VB}}{2}} \int_{\underline{C}}^{s - b^{AB}} h$ and $V^M = VB + TB + DP + S$, where: Vote Buying (VB) $= \int_{-\frac{b^{VB}}{2}}^{0} \int_{\underline{C}}^{x^O} h$, Turnout Buying (TB) $= \int_0^{\overline{X}} \int_r^{r + b^{TB}} h$, Double Persuasion (DP) $= \int_{-\frac{b^{DP}}{2}}^{0} \int_s^{r + b^{DP}} h$, and Supporters (S) $= \int_0^{\overline{X}} \int_0^r h$. Total expenditures for the machine party are $E = E_{VB} + E_{TB} + E_{DP} + E_{AB}$, where: VB Expenditures (E_{VB}) $= \int_{-\frac{b^{VB}}{2}}^{0} \int_{\underline{C}}^{x^O} \bar{b}_i^{VB} h$, TB Expenditures ($E_{TB}$) $= \int_0^{\overline{X}} \int_r^{r + b^{TB}} \bar{b}_i^{TB} h$, DP Expenditures ($E_{DP}$) $= \int_{-\frac{b^{DP}}{2}}^{0} \int_s^{r + b^{DP}} \bar{b}_i^{DP} h$, and AB Expenditures ($E_{AB}$) $= \int_{\underline{X}}^{-\frac{b^{VB}}{2}} \int_{s - b^{AB}}^{s} \bar{b}_i^{AB} h + \int_{-\frac{b^{VB}}{2}}^{0} \int_{x^O}^{s} \bar{b}_i^{AB} h$. Solving the problem yields four first order conditions. Solving all first order conditions for λ yields the results from Proposition 1: $b^*_{\text{VB}} = 2b^*_{\text{TB}} =$

$2b^*_{\text{DP}} = 2b^*_{\text{AB}}$. For the following analyses, let $\Gamma = \frac{1}{(\overline{X}-\underline{X})(\overline{C}-\underline{C})}$. Recall that $\underline{C} < 0$, $\underline{X} < 0$, and $\underline{X} = -\overline{X}$.

Compulsory Voting: Substitute $b^* = \frac{1}{2}b^{**}$ from the FOCs into the budget constraint. Implicit differentiation yields: $\frac{\partial b^{**}}{\partial a} = \frac{-4b^{**}}{8(a+\overline{X}-x^M-\underline{C})-b^{**}} < 0$. Substitute $b^{**} = 2b^*$ into the budget constraint. Implicit differentiation yields: $\frac{\partial b^*}{\partial a} = \frac{-2b^*}{4(a+\overline{X}-x^M-\underline{C})-b^*} < 0$. Comparative statics follow: (1) $\frac{\partial VB}{\partial a} = \frac{\Gamma}{4}\left[2b^{**} + (2(a-x^M-\underline{C}) + b^{**})\frac{\partial b^{**}}{\partial a}\right] - b^*\frac{\partial b^{**}}{\partial a} - b^{**}\frac{\partial b^*}{\partial a} =$
$\frac{\Gamma}{4}\left[2b^{**} - 2b^{**}\left(\frac{4(a-x^M-\underline{C})+2b^{**}}{8(a+\overline{X}-x^M-\underline{C})-b^{**}}\right)\right] - b^*\frac{\partial b^{**}}{\partial a} - b^{**}\frac{\partial b^*}{\partial a} > 0$. (2) $\frac{\partial TB}{\partial a} = \Gamma\overline{X}\frac{\partial b^*}{\partial a} < 0$. (3) $\frac{\partial DP}{\partial a} = \Gamma\frac{b^*}{2}\frac{\partial b^*}{\partial a} < 0$. (4) $\frac{\partial AB}{\partial a} = -\frac{\Gamma}{4}\left[b^*\frac{\partial b^{**}}{\partial a} + (4\underline{X} + b^{**})\frac{\partial b^*}{\partial a}\right] = -\frac{\Gamma}{4}\left[b^{**}\frac{\partial b^{**}}{\partial a} + 2\underline{X}\frac{\partial b^{**}}{\partial a}\right] < 0$ (recall that $\underline{X} < 0$ and that under an optimal allocation of resources, $b^* = \frac{1}{2}b^{**}$ and $\frac{\partial b^*}{\partial a} = \frac{1}{2}\frac{\partial b^{**}}{\partial a}$).

Ballot Secrecy: In the constrained optimization problem above, replace E_{VB} with βE_{VB} and E_{DP} with βE_{DP}. The FOCs become $\beta b^*_{\text{VB}} = 2\beta b^*_{\text{DP}} = 2b^*_{\text{TB}} = 2b^*_{\text{AB}}$. Substitute $b^*_{\text{DP}} = \frac{1}{2}b^*_{\text{VB}}$ and $b^*_{\text{TB}} = b^*_{\text{AB}} = \frac{\beta}{2}b^*_{\text{VB}}$ from the FOCs into the budget constraint. Implicit differentiation yields: $\frac{\partial b^*_{\text{VM}}}{\partial \beta} = \frac{b^*_{\text{VB}}((5-12\beta)2b^*_{\text{VB}}-12(\beta x^M+\underline{C}-2\beta\overline{X}))}{3\beta(4\beta\overline{X}(6\beta-5)b^*_{\text{VB}}+8(x^M+\underline{C}))} < 0$. Substitute $b^*_{\text{VB}} = 2b^*_{\text{DP}}$ and $b^*_{\text{TB}} = b^*_{\text{AB}} = \beta 2b^*_{\text{DP}}$ and implicit differentiation yields: $\frac{\partial b^*_{\text{DP}}}{\partial \beta} = \frac{b^*_{\text{VB}}((5-12\beta)b^*_{\text{VB}}-6(\beta x^M+\underline{C}-2\beta\overline{X}))}{3\beta(2\beta\overline{X}(6\beta-5)b^*_{\text{VB}}+4(x^M+\underline{C}))} < 0$. Let $b_{\text{TB}} = b_{\text{AB}}$ and substitute $b^*_{\text{VB}} = \frac{2}{\beta}b^*_{\text{TB}}$ and $b^*_{\text{DP}} = \frac{1}{\beta}b^*_{\text{TB}}$ and implicit differentiation yields: $\frac{\partial b^*_{\text{TB}}}{\partial \beta} = \frac{\partial b^*_{\text{AB}}}{\partial \beta} = \frac{2b^*_{\text{TB}}((3\beta-5)b^*_{\text{TB}}+3\beta(x^M+\underline{C}))}{3\beta((6\beta-5)b^*_{\text{TB}}+2\beta(2(x^M+\underline{C})-2\beta\overline{X}))} > 0$. Comparative statics follow: (1) $\frac{\partial VB}{\partial \beta} = \frac{\Gamma}{4}\left[(b^*_{\text{VB}} - 2(x^M+\underline{C}))\frac{\partial b^*_{\text{VB}}}{\partial \beta} - b^*_{\text{AB}}\frac{\partial b^*_{\text{VB}}}{\partial \beta} - b^*_{\text{VB}}\frac{b^*_{\text{AB}}}{\partial \beta}\right] =$
$\frac{\Gamma}{4}\left[(b^*_{\text{VB}} - 2(x^M+\underline{C}))\frac{\partial b^*_{\text{VB}}}{\partial \beta} - \frac{\beta}{2}b^*_{\text{VB}}\frac{\partial b^*_{\text{VB}}}{\partial \beta} - b^*_{\text{VB}}(\frac{1}{2}(b^*_{\text{VB}} + \beta\frac{\partial b^*_{\text{VB}}}{\partial \beta})\right] < 0$ (using the fact that in an optimal allocation of resources, $b^*_{\text{AB}} = \frac{\beta}{2}b^*_{\text{VB}}$ and $\frac{\partial b^*_{\text{AB}}}{\partial \beta} = \frac{1}{2}(b^*_{\text{VB}} + \beta\frac{\partial b^*_{\text{VB}}}{\partial \beta})$). (2) $\frac{\partial TB}{\partial \beta} = \Gamma\overline{X}\frac{\partial b^*_{\text{TB}}}{\partial \beta} > 0$. (3) $\frac{\partial DP}{\partial \beta} = \frac{\Gamma b^*_{\text{DP}}}{2}\frac{\partial b^*_{\text{DP}}}{\partial \beta} < 0$. (4)
$\frac{\partial AB}{\partial \beta} = -\frac{\Gamma}{4}\left[b^*_{\text{AB}}\frac{\partial b^*_{\text{VB}}}{\partial \beta} + (4\underline{X} + b^*_{\text{VB}})\frac{\partial b^*_{\text{AB}}}{\partial \beta}\right] = -\frac{\Gamma}{4}\left[\frac{\beta}{2}b^*_{\text{VB}}\frac{\partial b^*_{\text{VB}}}{\partial \beta} + \frac{1}{2}(4\underline{X} + b^*_{\text{VB}})(b^*_{\text{VB}} + \beta\frac{\partial b^*_{\text{VB}}}{\partial \beta})\right] > 0$ (again substituting $b^*_{\text{AB}} = \frac{\beta}{2}b^*_{\text{VB}}$ and $\frac{\partial b^*_{\text{AB}}}{\partial \beta} = \frac{1}{2}(b^*_{\text{VB}} + \beta\frac{\partial b^*_{\text{VB}}}{\partial \beta})$).

Salience of Political Preferences: Substituting FOCs into the budget constraint and implicitly differentiating yields: (1) $\frac{\partial b^{**}}{\partial \kappa} = \frac{b^{**}(b^{**}+12\underline{C})}{3\kappa(8(x^M+\kappa(\underline{C}-\overline{X}))+b^{**})} > 0$ and (2) $\frac{\partial b^*}{\partial \kappa} = \frac{b^*(b^*+6\underline{C})}{3\kappa(4(x^M+\kappa(\underline{C}-\overline{X}))+b^*)} > 0$. Comparative statics follow: (1) $\frac{\partial VB}{\partial \kappa} = -\frac{\Gamma}{8\kappa^2}\left[2b^{**}(b^{**}+2\underline{C}) + 2\kappa(2(\kappa x^M + \underline{C}) + b^* - \kappa b^{**})\frac{\partial b^{**}}{\partial \kappa}\right] < 0$ (using the fact that in an optimal allocation of resources, $\frac{\partial b^*}{\partial \kappa} = \frac{1}{2}\frac{\partial b^{**}}{\partial \kappa}$). (2) $\frac{\partial TB}{\partial \kappa} = \Gamma\left[\overline{X}(\frac{\partial b^*}{\partial \kappa})\right] > 0$. (3) $\frac{\partial DP}{\partial \kappa} = \frac{\Gamma}{4\kappa^2}\left[2\kappa\frac{\partial b^*}{\partial \kappa} - b^*\right] = \frac{\Gamma}{4\kappa^2}\left[2\kappa b^*\frac{(b^*+6\underline{C})}{3\kappa(4(x^M+\kappa(\underline{C}-\overline{X}))+b^*)} - b^*\right] > 0$. (4) $\frac{\partial AB}{\partial \kappa} =$
$\frac{\Gamma}{4\kappa^2}\left[b^*(b^{**} - \kappa\frac{\partial b^{**}}{\partial \kappa}) - \kappa(4\underline{X} + b^{**})\frac{\partial b^*}{\partial \kappa}\right] = \frac{\Gamma}{4\kappa^2}\left[b^*(b^{**} - b^{**}\frac{\kappa(b^{**}+12\underline{C})}{3\kappa(8(x^M+\kappa(\underline{C}-\overline{X}))+b^{**})}) - \kappa(4\underline{X} + b^{**})\frac{\partial b^*}{\partial \kappa}\right] > 0$.

Political Polarization: Note that by the assumption of symmetric party platforms, $x^M - x^O = 2x^M$. Substitute $b^* = \frac{1}{2}b^{**}$ from the FOCs into the budget constraint. Implicit differentiation yields: $\frac{\partial b^{**}}{\partial x^M} = \frac{4b^{**}}{8(\overline{X}-x^M-\underline{C})-b^{**}} > 0$. Substitute $b^{**} = 2b^*$ into the budget constraint. Implicit differentiation yields: (2) $\frac{\partial b^*}{\partial x^M} = \frac{2b^*}{4(\overline{X}-x^M-\underline{C})-b^*} > 0$. Comparative statics then follow: (1).
$\frac{\partial VB}{\partial x^M} = \frac{\Gamma}{4}\left[-(2b^{**} + (2(x^M+\underline{C}) + b^{**}))\frac{\partial b^{**}}{\partial x^M}\right] - b^*\frac{\partial b^{**}}{\partial x^M} + b^{**}\frac{\partial b^*}{\partial x^M} = \frac{\Gamma}{4}\left[-(2b^{**} + (2(x^M+\underline{C}) + b^{**}))\frac{\partial b^{**}}{\partial x^M}\right] < 0$ (where the last two terms of the first equation cancel after substituting $b^* = \frac{1}{2}b^{**}$ and $\frac{\partial b^*}{\partial a} = \frac{1}{2}\frac{\partial b^{**}}{\partial a}$). (2)

$\frac{\partial TB}{\partial x^M} = \Gamma\left[\overline{X}(\frac{\partial b^*}{\partial x^M})\right] > 0$. (3) $\frac{\partial DP}{\partial x^M} = \frac{\Gamma}{2}\left[b^*\frac{\partial b^*}{\partial x^M}\right] > 0$. (4) $\frac{\partial AB}{\partial x^M} = -\frac{\Gamma}{4}\left[b^*\frac{\partial b^{**}}{\partial x^M} + (4\underline{X} + b^{**})\frac{\partial b^*}{\partial x^M}\right] = -\frac{\Gamma}{4}\left[b^{**}\frac{\partial b^{**}}{\partial x^M} + 2\underline{X}\frac{\partial b^{**}}{\partial x^M}\right] > 0$ (recall that $\underline{X} < 0$ and that under an optimal allocation of resources, $b^* = \frac{1}{2}b^{**}$ and $\frac{\partial b^*}{\partial x^M} = \frac{1}{2}\frac{\partial b^{**}}{\partial x^M}$).

Machine Support: Substituting FOCs into the budget constraint and implicitly differentiating yields: $\frac{\partial b^{**}}{\partial \overline{x}} = \frac{\partial b^*}{\partial \overline{x}} = 0$. Comparative statics follow: (1) $\frac{\partial VB}{\partial \overline{x}} = -\frac{\Gamma}{4}\left[(2(x^M + \underline{C}) - b^{**} + b^*)\frac{\partial b^{**}}{\partial \overline{x}} + b^{**}\frac{\partial b^*}{\partial \overline{x}}\right] = 0$. (2) $\frac{\partial TB}{\partial \overline{x}} = \Gamma\left[b^* + (\overline{X} + \overline{x})\frac{\partial b^*}{\partial \overline{x}}\right] = \Gamma b^* > 0$. (3) $\frac{\partial DP}{\partial \overline{x}} = \frac{\Gamma}{2}\left[b^*(\frac{\partial b^*}{\partial \overline{x}})\right] = 0$. (4) $\frac{\partial AB}{\partial \overline{x}} = -\frac{\Gamma}{4}\left[b^*(4 + \frac{\partial b^{**}}{\partial \overline{x}}) + (4(\underline{X} + \underline{x}) + b^{**})\frac{\partial b^*}{\partial \overline{x}}\right] = -\Gamma b^* < 0$.