The discrete-time second-best day-to-day dynamic pricing scheme

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Introduction

Introduction

Traffic congestion is one of the most challenging problems faced by many major cities. It is believed that congestion pricing is an efficient way to reduce traffic congestion.

Practically implemented in cities like Singapore, Stockholm.

Introduction

The traditional road pricing schemes are developed based on the concept of static traffic equilibrium, assuming that the desired traffic equilibrium state can eventually be achieved in a long run.

However, as [Horowitz, 1984] demonstrated that, for a traffic system whose equilibrium solution was known to exist, depending on the dynamic route adjustment process, the system might still fail to converge to equilibrium.

Therefore, it is very natural to doubt whether the traffic system can reach the desirable traffic equilibrium state or not, from any feasible initial traffic state under the congestion pricing based on the static equilibrium.

Introduction

Day-to-day dynamic pricing scheme, which varies on a day-to-day basis, was proposed. Indeed, the advent of new technologies for the travel information collection and dissemination makes it more practically possible to implement dynamic congestion pricing.

Existing researches on dynamic congestion pricing scheme

- Continuous time: [Sandholm, 2002][Friesz et al., 2004] [Yang and Szeto, 2006], [Tan et al., 2015][Yang et al., 2007].
- Discrete time: [Guo et al., 2015a]: Converge to the restraint stable traffic flow state, the convergence proof is based on the assumption of the uniqueness of the traffic equilibrium.

Introduction

Considering multiple traffic equilibrium states, e.g., asymmetric travel cost function

As reported in [Bie and Lo, 2010], if the initial traffic state does not fall into the attraction domain of the given objective traffic equilibrium state, without external force, traffic system cannot reach the given objective traffic equilibrium state through a day-to-day routing adjustment process

Introduction

Research Question: How to propose a dynamic pricing scheme that satisfied the following characteristics:

- is a discrete-time day-to-day dynamic congestion pricing scheme;
- is developed for multiple traffic equilibria case;
- is implemented on a subset of links;
- drives traffic system to converge to the given second-best objective traffic equilibrium state desired by traffic management authorities from any initial traffic state.

Model Description

Day-to-day traffic dynamic model

This study applied the very general day-to-day traffic dynamic model proposed by [Guo et al., 2015b] to develop the dynamic pricing scheme:

$$\mathbf{x}^{(t+1)} = (1-\alpha^{(t)})\mathbf{x}^{(t)} + \alpha^{(t)}\mathbf{y}^{(t)},\ t = 0, 1, 2, \ldots, \tag{1}$$

$$\mathbf{y}^{(t)} \begin{cases} \in \Psi^{(t)},\ \text{if } \Psi^{(t)} \neq \emptyset, \\ = \mathbf{x}^{(t)},\ \text{if } \Psi^{(t)} = \emptyset. \end{cases} \tag{2}$$

$$\Psi^{(t)} = \{\mathbf{y} | \mathbf{y} \in \Omega_{\mathbf{x}}, (\mathbf{y} - \mathbf{x}^{(t)})^T (\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) < 0\} \tag{3}$$

where $0 < \alpha^{(t)} \leq 1$ is the parameter of step size or adjustment ratio of link flow, and $\mathbf{y}^{(t)} = \mathbf{y}(\mathbf{x}^t, \tau^t)$ is the objective adjustment vector of link flow at time $t + 1$.

The Problem of Existing Pricing Scheme for second-best case

- The dynamic pricing scheme of [Guo et al., 2015b] implemented on a fixed subset of links can drive the traffic dynamic in equations (1)-(3) to converge to its unique traffic equilibrium.
- For multiple traffic equilibria case, if the charged subset of links is fixed, the traffic dynamic system in equations (1) and (3) may not converge to the desired second-best objective equilibrium state.

The problem of existing congestion pricing scheme

A small example is given to demonstrate the conclusion on multiple equilibria case, which consists of one OD pair connected with two links. The travel cost c_1 is $f_1 + 3f_2 + 1$, and $c_2 = 2f_1 + f_2 + 2$. The traffic demand is 1 unit. When a second-best pricing is only charged on link 1 and assumed to be 1 unit. It can also be found that there are three user equilibrium states under this second-best toll:

$$\mathbf{f}^{I} = (1, 0), \quad \mathbf{f}^{II} = (0, 1), \quad \mathbf{f}^{III} = (1/3, 2/3) \tag{4}$$

The problem of existing congestion pricing scheme for second-best case

Supposing $\mathbf{f}_I^* = (1, 0)$ be the objective traffic equilibrium state, and the another equilibrium flow $\mathbf{f}_{II}^* = (0, 1)$ is the initial flow pattern. The initial travel cost corresponding to the initial flow without road pricing is

$$c_1(\mathbf{f}_I^*) = 4, c_2(\mathbf{f}_{II}^*) = 3. \tag{5}$$

This means that the initial state $\mathbf{f}_{II}^* = (0, 1)$ also is a UE flow without any road pricing. So, for any road pricing $\tau = (\tau_1 \geq 0, \tau_2 = 0)$, $\mathbf{f}_{II}^* = (0, 1)$ is a UE equilibrium flow, i.e., for any feasible flow $\mathbf{f}$ of this example,

$$(\mathbf{f} - \mathbf{f}_{II}^*)^T \mathbf{C}(\mathbf{f}_{II}^*) \geq 0, \tag{6}$$

and

$$c_1(\mathbf{f}_{II}^*) > c_2(\mathbf{f}_{II}^*). \tag{7}$$

The problem of existing congestion pricing scheme for second-best case

Therefore, for any nonnegative second-best road pricing $\tau = (\tau_1 \geq 0, \tau_2 = 0)$ implemented on link 1, $c_1(\mathbf{f}_{II}^*) + \tau_1 > c_2(\mathbf{f}_{II}^*) + \tau_2$, and

$$(\mathbf{f} - \mathbf{f}_{II}^*)^T(\mathbf{C}(\mathbf{f}_{II}^*) + \tau) \geq 0. \tag{8}$$

Further, there is no feasible flow $\mathbf{y}$ defined in equation (3) which satisfies

$$(\mathbf{y} - \mathbf{f}_{II}^*)^T(\mathbf{C}(\mathbf{f}_{II}^*) + \tau) < 0, \tag{9}$$

In the dynamic process in equations (1)-(3), the flow $\mathbf{f}^{(t)}$ $(t = 0, 1, \dots)$ is always equal to $\mathbf{f}_{II}^*$ instead of converging to the objective traffic equilibrium $(1, 0)$.

The problem of existing congestion pricing scheme for second-best case

The dynamic pricing scheme of [Guo et al., 2015a] even levied on a dynamic subset of links cannot also guarantee that the traffic system can be driven to converge to the objective equilibrium if the traffic system has multiple equilibria. Consider an simple example which has only one OD pair and three links. The travel cost of each link is: $c_1 = f_1 + 3f_2 + 1$, $c_2 = 2f_1 + f_2 + 0.5$ and $c_3 = f_3 + 1$. Assume traffic demand is 2 unit. Under a toll $\tau = 0.5$ implemented on link 2, there are three user equilibrium states: $(1, 0, 1)$, $(0, 1, 1)$ and $(1/2, 1/4, 5/4)$. Assume the objective traffic state to be achieved is $(0, 1, 1)$.

The problem of existing congestion pricing scheme for second-best case

The specific pricing scheme applied is the Walrasian scheme of [Garcia et al., 2012], which also belongs to the dynamic pricing scheme of [Guo et al., 2015b]:

$$\tau^{(t+1)} = [\tau^{(t)} + \rho^{(n)}(\mathbf{x}^{(t)} - \bar{\mathbf{x}})]_{+}, \quad t = 0, 1, 2, \ldots$$

where $\rho^{(t)}$ is the sensitive parameter and determines the speed at which the tolls are updated. According to the assumption in [Garcia et al., 2012], let $\rho^{(t)} = 1/(t+1)^{1/4}$ and $\alpha^{(t)} = 1/(t+1)^{0.51}$ in Eq.1. The dynamic pricing is implemented on the dynamic charged link set $\bar{A}^{(t+1)}$ as defined by this study.

The problem of existing congestion pricing scheme for second-best case

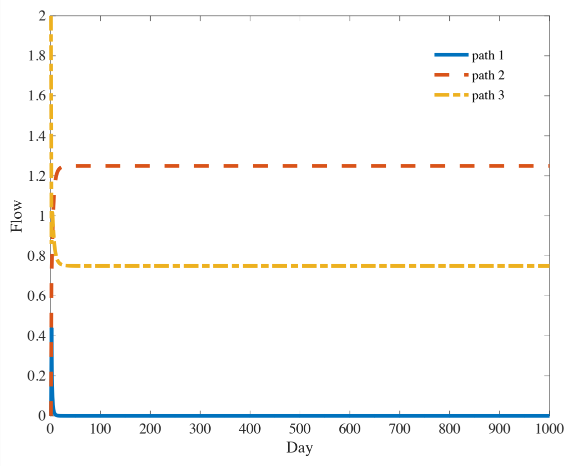

Figure 1: The dynamics of flow with initial state $(0, 0, 2)$ and objective equilibrium $(0, 1, 1)$

The problem of existing congestion pricing scheme for second-best case

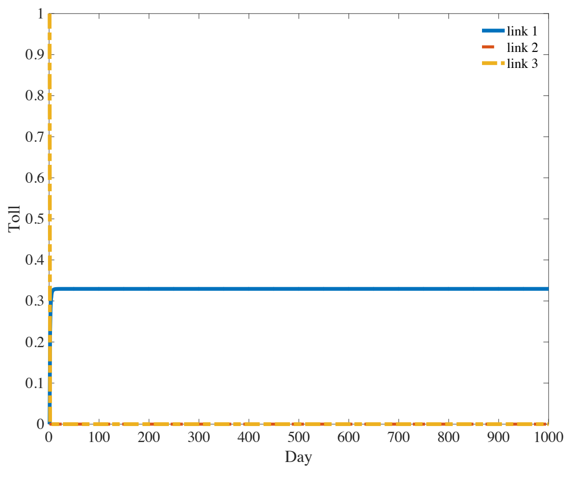

Figure 2: The dynamics of toll with initial state $(0, 0, 2)$ and objective equilibrium $(0, 1, 1)$

The problem of existing congestion pricing scheme for second-best case

From Figure 1 and 2, one can find that the traffic dynamic system under the dynamic pricing scheme of [Garcia et al., 2012] with dynamic charged link set converges to the steady flow pattern $(0, 1.25, 0.75)$ rather than the objective equilibrium flow pattern $(0, 1, 1)$. The steady toll is 0.329, under which the steady flow $(0, 1.25, 0.75)$ is a UE flow pattern.

A dynamic pricing scheme for multiple equilibrium case

The dynamic charged links $\bar{A}^{(t+1)}$ at time $t+1$ $(t = 0, 1, \ldots)$ is defined as follows:

$$\bar{A}^{(t+1)} \subseteq \bar{A}_{+}^{(t+1)}, \tag{10}$$

where

$$\bar{A}_{+}^{(t+1)} = \{a \in A | x_a^{(t)} - x_a^{obj} > 0\} \tag{11}$$

A dynamic pricing scheme for multiple equilibrium case

The charged toll $\tau^{(t+1)}$ at time $t+1$ $(t = 0, 1, dots)$ can be expressed as follows:

$$\tau_a^{(t+1)} \begin{cases} \geq 0, \text{if } a \in \bar{A}^{(t+1)} \\ = 0, \text{otherwise}, \end{cases} \tag{12}$$

$$\bar{A}^{(t+1)} \begin{cases} = \bar{A}^{(t+1)}, \text{if } \|\mathbf{x}^{obj} - \mathbf{x}^{(t)}\| \geq \delta \\ = \bar{A}, \text{otherwise}, \end{cases} \tag{13}$$

and

$$\tau^{(t+1)} \begin{cases} \in Q^{(t+1)}, \text{if } \|\mathbf{x}^{obj} - \mathbf{x}^{(t)}\| \geq \delta, \\ = \tau^*, \text{otherwise}, \end{cases} \tag{14}$$

where

$$Q^{(t+1)} = \{\tau | \tau \geqslant 0, (\mathbf{c}(\mathbf{x}^{(t)}) + \tau)^T (\mathbf{x}^{obj} - \mathbf{x}^{(t)}) < 0\}. \tag{15}$$

A dynamic pricing scheme for multiple equilibrium case

Assumption 1

$\alpha^{(t)}$ ($t = 0, 1, \dots$) is a sequence satisfying

$$\sum_{t=0}^{+\infty} \alpha^{(t)} = +\infty, \sum_{t=0}^{+\infty} (\alpha^{(t)})^2 < +\infty. \tag{16}$$

Assumption 2

For any $\mathbf{x} \in \Omega_{\mathbf{x}} \backslash \mathbf{H}(\mathbf{x}^{obj}, \delta)$, the pricing scheme in equations (12)-(15) can drive the function $\mathbf{y} = \mathbf{y}(\mathbf{x}, \tau)$ in equation (2) to satisfy,

$$(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T (\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) < 0. \tag{17}$$

A dynamic pricing scheme for multiple equilibrium case

Theorem 1

With Assumption 1 and Assumption 2, the dynamic second-best dynamic congestion pricing scheme in equations (12)-(15) *can drive the traffic dynamic system in equations* (1)-(3) *to converge to the objective traffic equilibrium* $\mathbf{x}^{obj}$.

A dynamic pricing scheme for multiple equilibrium case

Proof.

First, we prove that the day-to-day traffic dynamic system eventually enter into the neighborhood $\mathbf{H}(\mathbf{x}^{obj}, \delta)$ of the objective traffic equilibrium state $\mathbf{x}^{obj}$. The proof is by contradiction. For any $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{H}(\mathbf{x}^{obj}, \delta)$, then the distance $D(\mathbf{x}^{(t+1)}, \mathbf{x}^{obj})$ between $\mathbf{x}^{(t+1)}$ and the objective traffic equilibrium $\mathbf{x}^{obj}$ can be described by Euclidean norm as follows:

$$D(\mathbf{x}^{(t+1)}, \mathbf{x}^{obj}) = \|\mathbf{x}^{(t+1)} - \mathbf{x}^{obj}\|^2. \tag{18}$$

Substituting equation (1) into equation (18), it can be obtained that

$$\begin{aligned} D(\mathbf{x}^{(t+1)}, \mathbf{x}^{obj}) = {} & \|\mathbf{x}^{(t)} - \mathbf{x}^{obj}\|^2 + 2\alpha^{(t)}(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) \\ & + (\alpha^{(t)})^2\|(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}\|^2, \end{aligned}$$

i.e.,

$$\begin{aligned} D(\mathbf{x}^{(t+1)}, \mathbf{x}^{obj}) - D(\mathbf{x}^{(t)}, \mathbf{x}^{obj}) = {} & 2\alpha^{(t)}(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) \\ & + (\alpha^{(t)})^2||(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}||^2. \end{aligned} \tag{19}$$

From equation (19), it can be obtained that

$$\begin{aligned} D(\mathbf{x}^{(t+1)}, \mathbf{x}^{obj}) - D(\mathbf{x}^{(0)}, \mathbf{x}^{obj}) = {} & 2\sum_{n=0}^{t} \alpha^{(n)}(\mathbf{x}^{(n)} - \mathbf{x}^{obj})^T(\mathbf{y}^{(n)} - \mathbf{x}^{(n)}) \\ & + (\alpha^{(n)})^2||(\mathbf{y}^{(n)} - \mathbf{x}^{(n)}||^2. \end{aligned} \tag{20}$$

A dynamic pricing scheme for multiple equilibrium case

Since $\Omega_{\mathbf{x}}/\mathbf{H}(\mathbf{x}^{obj}, \delta)$ is a compact set, and $D(\mathbf{x}, \mathbf{x}^{obj})$ is continuous with respect to $\mathbf{x} \in \Omega_{\mathbf{x}}$, the distance function $D(\mathbf{x}, \mathbf{x}^{obj})$ is bounded. Therefore, the right-hand side of equation (20) and $\|\mathbf{y} - \mathbf{x}\|^2$ are both bounded in $\Omega_{\mathbf{x}}/\mathbf{H}(\mathbf{x}^{obj}, \delta)$. Further, from Assumption 1 and Assumption 2, it can be derived that

$$\sum_{t=0}^{+\infty} (\alpha^{(n)})^2 \|(\mathbf{y}^{(n)} - \mathbf{x}^{(n)}\|^2 < +\infty \tag{21}$$

and,

$$\sum_{n=0}^{+\infty} \alpha^{(n)} (\mathbf{x}^{(n)} - \mathbf{x}^{obj})^T (\mathbf{y}^{(n)} - \mathbf{x}^{(n)}) > -\infty. \tag{22}$$

Let

$$Z(\mathbf{x}^{(t)}, \mathbf{y}^{(t)}) = (\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T (\mathbf{y}^{(t)} - \mathbf{x}^{(t)}). \tag{23}$$

Then, from Assumption 2, we have $Z(\mathbf{x}^{(t)}, \mathbf{y}^{(t)}) < 0$ for any $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{H}(\mathbf{x}^{obj}, \delta)$. Thus, by Theorem 5.2.12 in [Trench, 2013], there is a constant $h < 0$ such that $Z(\mathbf{x}^{(t)}, \mathbf{y}^{(t)}) \leqslant h < 0$ for $\forall \mathbf{x}^{(t)} \in \mathbf{H}(\mathbf{x}^{obj}, \delta)$. As $0 < \alpha^{(t)} \leqslant 1$,

$$Z(\mathbf{x}^{(t)}, \mathbf{y}^{(t)}) = (\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T (\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) \leqslant \alpha^{(t)} Z(\mathbf{x}^{(t)}, \mathbf{y}^{(t)}). \tag{24}$$

If the day-to-day traffic dynamic system under the second-best dynamic pricing scheme cannot enter into the neighborhood $\mathbf{H}(\mathbf{x}^{obj}, \delta)$, then we have

$$\sum_{t=0}^{+\infty} \alpha^{(n)} (\mathbf{x}^{(n)} - \mathbf{x}^{obj})^T (\mathbf{y}^{(n)} - \mathbf{x}^{(n)}) \leqslant \sum_{t=0}^{+\infty} \alpha^{(n)} h < -\infty \tag{25}$$

A dynamic pricing scheme for multiple equilibrium case

This contradicts with equation (22). Therefore, the dynamic pricing scheme can drive the traffic dynamic system enter into the neighborhood $\mathbf{H}(\mathbf{x}^{obj}, \delta) \subset \mathbf{B}(\mathbf{x}^{obj})$.

By the definition of attraction domain of a stable equilibrium in [Bie and Lo, 2010], the attraction domain $\mathbf{B}(\mathbf{x}^{obj})$ of $\mathbf{x}^{obj}$ can be described as follows:

$$\mathbf{B}(\mathbf{x}^{obj}) = \{\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}} | \lim_{t \to +\infty} \mathbf{x}^{(t)} = \mathbf{x}^{obj}\}.$$

Because the neighborhood $\mathbf{H}(\mathbf{x}^{obj}, \delta) \subset \mathbf{B}(\mathbf{x}^{obj})$, one can derive that, for any $\mathbf{x}^{(t)} \in \mathbf{H}(\mathbf{x}^{obj}, \delta)$, $\lim_{t \to \infty} \mathbf{x}^{(t)} = \mathbf{x}^{obj}$. Therefore, the dynamic second-best dynamic congestion pricing scheme in equations (12)-(15) can drive the traffic dynamic system in equations (1)-(3) to converge to the objective traffic equilibrium $\mathbf{x}^{obj}$. The proof is completed.

One specific case

The formulation of the function $\mathbf{y}^{(t)} = \mathbf{y}(\mathbf{x}^{(t)}, \tau^{(t+1)})$ $(t = 0, 1, \dots)$ of [He et al., 2010], the link-based network tatonnement process and link-based projected dynamic model of [Guo et al., 2015b] can be formulated as follows:

$$\mathbf{y}^{(t)} = \arg\min_{\mathbf{y}\in\Omega_{\mathbf{x}}} \lambda \mathbf{c}(\mathbf{x}^{(t)})^T \mathbf{y} + (1-\lambda) D(\mathbf{x}^{(t)} - \mathbf{y}) \tag{26}$$

One specific case

Theorem 2

The second-best dynamic pricing scheme $\tau^{(t+1)}$ $(t = 0, 1, \dots)$ *in equations* (12)-(15) *levied on the variable subset* $\bar{A}^{(t+1)}$ *of links can drive the day-to-day traffic dynamic model with function* $\mathbf{y}^{(t)} = \mathbf{y}(\mathbf{x}^{(t)}, \tau^{(t+1)})$ *in equation* (26) *to satisfy Assumption 2.*

One specific case

Proof.

First, for any $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{H}(\mathbf{x}^{obj}, \delta)$, let us prove that, under the dynamic road pricing scheme $\tau^{(t+1)}$ in equations (12)-(15), $\mathbf{x}^{(t)}$, $\mathbf{x}^{obj}$ and the $\mathbf{y}^{(t)}$ in equation (26) have the relation in equation (17), i.e.,

$$(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) < 0. \tag{27}$$

From equation (26), it can be found that, when $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{H}(\mathbf{x}^{obj}, \delta)$, for any feasible flow $\mathbf{y} \in \Omega_{\mathbf{x}}$,

$$\|\beta(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{y}^{(t)} - \mathbf{x}^{(t)})\|^2 \leq \|\beta(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{y} - \mathbf{x}^{(t)})\|^2.$$

Further,

$$\|\beta(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{y}^{(t)} - \mathbf{x}^{(t)})\|^2 \leq \|\beta(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}) - (\mathbf{x}^{(t)} - \mathbf{x}^{(t)})\|^2,$$

and

$$(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) < 0. \tag{28}$$

Therefore, $\mathbf{x}^{obj}, \mathbf{y}^{(t)} \in \Psi^{(t)}$ and $\mathbf{x}^{(t)} \notin \Psi^{(t)}$. If $\mathbf{x}^{obj} = \mathbf{y}^{(t)}$, then

$$(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{x}^{(t)} - \mathbf{y}^{(t)}) = (\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{x}^{(t)} - \mathbf{x}^{obj}) > 0. \tag{29}$$

$\square$

One specific case

If $\mathbf{x}^{obj} \neq \mathbf{y}^{(t)}$, let $Q^{(t)} = \{\mathbf{z}|\mathbf{z} = \mathbf{x}^{(t)} - \mathbf{y}, \mathbf{y} \in \Omega_{\mathbf{x}}\}$. From equation (26), one can find that $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$ is the projection of $\beta(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})$ on set $Q^{(t)}$, and the original point $\mathbf{O}$ also belongs set $Q^{(t)}$. Because points $\mathbf{O}$, $\mathbf{x}^{(t)} - \mathbf{x}^{obj}$ and $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$ are different, they can make a plane or a line. If these three points make a plane, let $\hat{Q}^{(t)} = \{\mathbf{z}|\mathbf{z} = \alpha_1\mathbf{O} + \alpha_2(\mathbf{x}^{(t)} - \mathbf{x}^{obj}) + \alpha_3(\mathbf{x}^{(t)} - \mathbf{y}^{(t)}), \sum_i \alpha_i = 1, \alpha_1, \alpha_2, \alpha_3 \geqslant 0\}$. It can be found that set $\hat{Q}^{(t)}$ is a subset of the plane, and $\hat{Q}^{(t)} \subset Q^{(t)}$. $\hat{Q}^{(t)}$ also is a closed convex set. One can also find that $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$ is the projection of $\beta(\mathbf{c}(\mathbf{x}^{t}) + \tau^{(t+1)})$ onto $\hat{Q}^{(t)}$. Let point $\hat{\mathbf{B}}$ be the projection of $\beta(\mathbf{c}(\mathbf{x}^{t}) + \tau^{(t+1)})$ onto the plane made by points $\mathbf{O}$, $\mathbf{x}^{(t)} - \mathbf{x}^{obj}$ and $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$. Although $\hat{\mathbf{B}}$ may be not equal to $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$, the minimum distance between point of $\hat{Q}^{(t)}$ and $\hat{\mathbf{B}}$ is $\|\hat{\mathbf{B}} - (\mathbf{x}^{(t)} - \mathbf{y}^{(t)})\|^2$. Thus, when $\mathbf{O}$, $\mathbf{x}^{(t)} - \mathbf{x}^{obj}$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}^{(t)}$ make a plane, we can use figure 3 to show the position relation among $\beta(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})$, $\mathbf{x}^{(t)} - \mathbf{x}^*$ and $\mathbf{x}^{(t)} - \hat{\mathbf{y}}$.

From equations (27) and (28), it can be found that the angle $\gamma \in [0, \pi/2)$ between vectors $\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}$ and $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$, and the angle $\zeta \in [0, \pi/2)$ between $\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)}$ and $\mathbf{x}^{(t)} - \mathbf{x}^{obj}$. Thus the angle between vectors $\mathbf{O}\hat{\mathbf{B}}$ and $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$ is a cute angle, and the angle between vectors $\mathbf{O}\hat{\mathbf{B}}$ and $\mathbf{x}^{(t)} - \mathbf{x}^{obj}$ also is a cute angle. Further, the angle between vectors $\mathbf{x}^{(t)} - \mathbf{x}^{obj}$ and $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$ also is a cute angle. So, when $\mathbf{x}^{(t)}$, $\mathbf{x}^*$ and $\hat{\mathbf{y}}^{(t)} \in \Psi^{(t)}$ make a plane,

$$(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{x}^{(t)} - \mathbf{y}^{(t)}) > 0. \tag{30}$$

One specific case

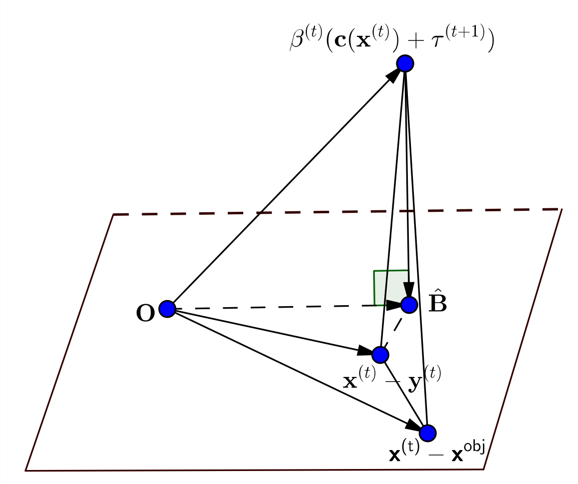

Figure 3: Relation among $\beta^{(t)}(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})$, $\mathbf{x}^{(t)} - \mathbf{x}^{obj}$ and $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$

One specific case

If points $\mathbf{O}$, $\mathbf{x}^{(t)} - \mathbf{x}^{obj}$ and $\mathbf{x}^{(t)} - \mathbf{y}^{(t)}$ are collinear, then $\mathbf{x}^{(t)}$, $\mathbf{x}^{obj}$ and $\mathbf{y}^{(t)}$ are collinear. As $\mathbf{x}^{(t)} \notin \Psi^{(t)}$, and $\Psi^{(t)}$ is a convex set,

$$(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{x}^{(t)} - \mathbf{y}^{(t)}) > 0. \tag{31}$$

Combining equations (29), (30) and (31), it can be obtained that

$$(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{x}^{(t)} - \mathbf{y}^{(t)}) > 0,$$

i.e.,

$$(\mathbf{x}^{(t)} - \mathbf{x}^{obj})^T(\mathbf{y}^{(t)} - \mathbf{x}^{(t)}) < 0. \tag{32}$$

By equation (32) and the proof of Theorem 1, the conclusion of Theorem 2 is immediately obtained. The proof is completed.

Some Discussions

In our second-best dynamic road pricing scheme, the subset $\bar{A}^{(t+1)}$ in equations (10)-(11) ,which is used to levy road pricing, varies with day-to-day traffic dynamic when $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{H}(\mathbf{x}^{obj}, \delta)$. The variability of $\bar{A}^{(t+1)}$ is used to guarantee that the objective equilibrium $\mathbf{x}^{obj}$ is a feasible adjustment flow of $\mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{H}(\mathbf{x}^{obj}, \delta)$, i.e.,

$$(\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})^T(\mathbf{x}^{obj} - \mathbf{x}^{(t)}) < 0. \tag{33}$$

It should be noted that set $\bar{A}^{(t+1)}$ may be not identical with the set $\bar{A}_{+}^{(t+1)}$ in equation (11).

Some Discussions

We can consider to minimize the number of links charged on day $t+1$ by solving the following problem:

$$\min_{(\tau,\mathbf{r})} \sum_{a \in \bar{A}_{+}^{(t+1)}} r_a \tag{34a}$$

$$s.t. \quad (\mathbf{c}(\mathbf{x}^{(t)}) + \tau)^T (\mathbf{x}^{obj} - \mathbf{x}^{(t)}) < 0, \tag{34b}$$

$$\tau_a \begin{cases} 0 \leq \tau_a \leq \tilde{\tau}_a, a \in \bar{A}_{+}^{(t+1)}, \\ = 0, \text{ otherwise}, \end{cases} \tag{34c}$$

$$r_a \begin{cases} \in \{0, 1\}, a \in \bar{A}_{+}^{(t+1)} \\ = 0, \text{ otherwise}, \end{cases} \tag{34d}$$

where $\tilde{\tau}_a$ is a given boundary of road pricing charged on link $a \in \bar{A}_{+}^{(t+1)}$.

Some discussions

For a given subset $\bar{A}^{(t+1)}$ links charged, the pricing $\tau^{(t+1)}$ satisfies condition (33) is not unique. So, after the decision on $\bar{A}^{(t+1)}$, we can consider to minimize the total road pricing charged on day $t+1$. This problem can be formulated as the following minimum problem:

$$\min_{\tau \geq 0} \sum_{a \in A} \tau_a \tag{35a}$$

$$s.t. \quad \tau_a \begin{cases} = 0, \text{ if } a \in A/\bar{A}^{(t+1)}, \\ 0 \leq \tau_a \leq \tilde{\tau}_a, \text{otherwise}; \end{cases} \tag{35b}$$

$$(\mathbf{c}(\mathbf{x}^t) + \tau)^T (\mathbf{x}^{obj} - \mathbf{x}^{(t)}) < 0. \tag{35c}$$

Numerical test

Numerical test

The test example only has a OD pair and three links. The travel cost of these three links are respectively:

$$c_1 = f_1 + f_2 + 2, \quad c_2 = 2f_1 + f_2 + 1, \quad c_3 = f_3 + 6. \tag{36}$$

The traffic demand is 2. Under the second-best pricing $(0, 2, 0)$ levied on link 2, there are three traffic equilibrium states in this example:

$$\mathbf{f}^{I} = (2, 0, 0), \quad \mathbf{f}^{II} = (0, 2, 0), \quad \mathbf{f}^{III} = (1, 1, 0). \tag{37}$$

The link-based traffic dynamic model of [He et al., 2010] is applied in this numerical tests.

Numerical test

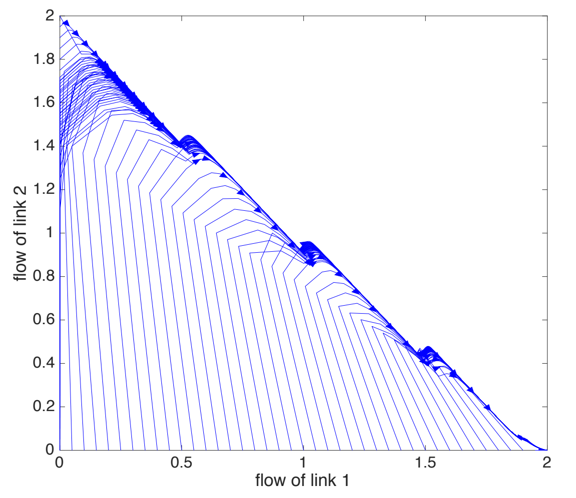

Figure 4: Flow trajectories with objective equilibrium flow $(2, 0, 0)$

Numerical test

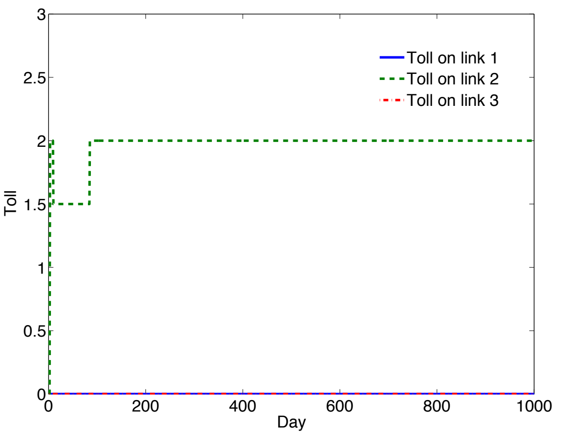

Figure 5: The dynamics of toll with initial flow $(0, 0, 2)$ and objective equilibrium $(2, 0, 0)$

Numerical test

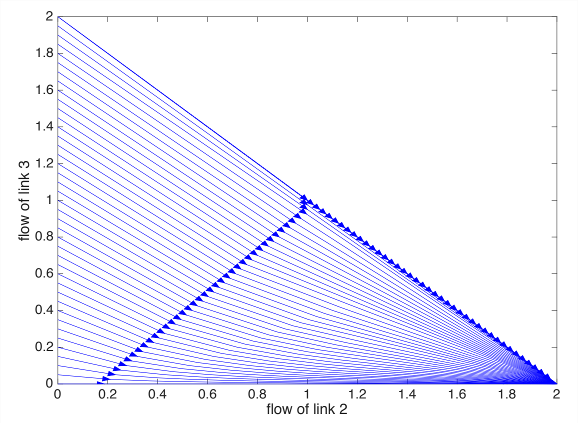

Figure 6: Flow trajectories with objective equilibrium flow $(0, 2, 0)$

Numerical test

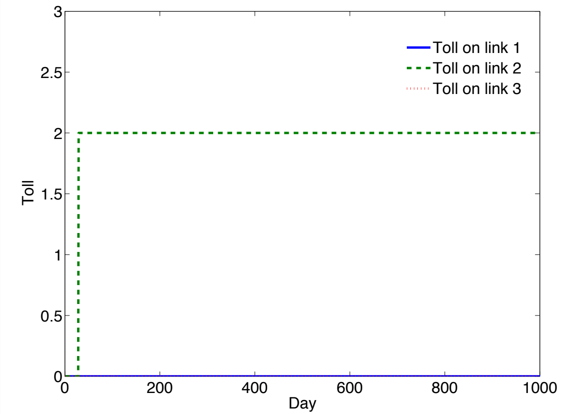

Figure 7: The dynamics of toll with initial state $(0, 0, 2)$ and objective equilibrium $(0, 2, 0)$

Without link constraints case

Without link constraints case

The dynamic pricing scheme in Eq.(10)-Eq.(15) can be applied to without link constraints case, i.e., if the charged link set $\bar{A} = A$, then the pricing scheme in in Eq.(10)-Eq.(15) can drive the dynamic traffic system in Eq.(1)-Eq.(3) to any given equilibrium state.

Without link constraints case

Moreover, for the without link constraints case, a specific dynamic pricing scheme can be develop without considering Assumption 2. The formulation of this pricing scheme:

$$\tau^{(t+1)} \begin{cases} = \hat{\tau}^{(t)}/(2\beta^{(t)}), \text{ if } \mathbf{x}^{(t)} \in \Omega_{\mathbf{x}}/\mathbf{N}(\mathbf{x}^{obj}, \delta), \\ = \tau^{obj}, \text{if } \mathbf{x}^{(t)} \in \mathbf{N}(\mathbf{x}^{obj}, \delta), \end{cases} \tag{38}$$

where $\mathbf{x}^{obj}$ is the given objective traffic equilibrium. τ^{obj} is the road pricing scheme which can support $\mathbf{x}^{obj}$ as an UE.

Without link constraints case

τ^{obj} is the Lagrange multiplier vector of the link flow constraint Eq.(39b) in the following optimization problem Eq.(39) :

$$\min_{\mathbf{y}} \|\beta^{(t)}\mathbf{c}(\mathbf{x}^{(t)}) - (\mathbf{x}^{(t)} - \mathbf{y})\|^2, \tag{39a}$$

$$s.t. \quad y_a \leqslant x_a^*, a \in A, \tag{39b}$$

$$\sum_r f_w^r = d_w, w \in W, \tag{39c}$$

$$y_a = \sum_w \sum_r f_w^r \delta_{ar}^w, \tag{39d}$$

$$f_w^r \geqslant 0, r \in R_w, w \in W, \tag{39e}$$

where parameter $\beta^{(t)} > 0$.

Conclusion

Summary

This study develops a second-best dynamic road pricing scheme implemented on a dynamic subset of links, so that the traffic evolution based on day-to-day adjustment process can be directed to converge to a desired second-best objective traffic equilibrium from any initial traffic state when multiple traffic equilibria exist.

Moreover, for without link constraints case, the dynamic pricing is also applicable.

Thanks!

References I

Bie, J. and Lo, H. K. (2010).
Stability and attraction domains of traffic equilibria in a day-to-day dynamical system formulation.
Transportation Research Part B: Methodological, 44(1):90–107.

Friesz, T. L., Bernstein, D., and Kydes, N. (2004).
Dynamic congestion pricing in disequilibrium.
Networks and Spatial Economics, 4(2):181–202.

Garcia, A., Hong, M., and Barrera, J. (2012).
"cap and trade" for congestion control.
Dynamic Games and Applications, 2(3):280–293.

Guo, R.-Y., Yang, H., Huang, H.-J., and Tan, Z. (2015a).
Day-to-day flow dynamics and congestion control.
Transportation Science.

References II

Guo, R.-Y., Yang, H., Huang, H.-J., and Tan, Z. (2015b).
Link-based day-to-day network traffic dynamics and equilibria.
Transportation Research Part B: Methodological, 71:248–260.

He, X., Guo, X., and Liu, H. X. (2010).
A link-based day-to-day traffic assignment model.
Transportation Research Part B: Methodological, 44(4):597–608.

Horowitz, J. L. (1984).
The stability of stochastic equilibrium in a two-link transportation network.
Transportation Research Part B: Methodological, 18(1):13–28.

Sandholm, W. H. (2002).
Evolutionary implementation and congestion pricing.
The Review of Economic Studies, 69(3):667–689.

References III

Tan, Z., Yang, H., and Guo, R.-Y. (2015).
Dynamic congestion pricing with day-to-day flow evolution and user heterogeneity.
Transportation Research Part C: Emerging Technologies, 61:87–105.

Trench, W. F. (2013).
Introduction to real analysis.

Yang, F. and Szeto, W. Y. (2006).
Day-to-day dynamic congestion pricing policies towards system optimal.
In *First International Symposium on Dynamic Traffic Assignment*, pages 266–275. Leeds, United Kingdom.

References IV

Yang, F., Yin, Y., and Lu, J. (2007).
Steepest descent day-to-day dynamic toll.
Transportation Research Record: Journal of the Transportation Research Board, 2039(1):83–90.