# ECON 481-3
# LECTURE 11: UNIFORMITY OF THE $t$-TEST

Ivan A. Canay
Northwestern University

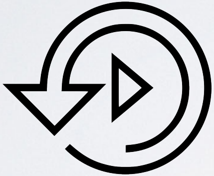
# Past & Future

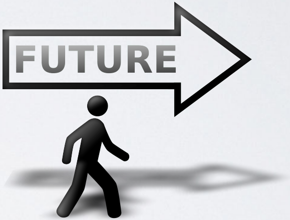
## LAST CLASS

- Uniformity and definition of "Size"
- The Bahadur-Savage (1956) Problem
- Extension by Romano (2004)
- Applications

## TODAY

- Dist. with compact support
- Dist. with $2 + \delta$ moments
- Uniformity of the $t$-test
- Power of the $t$-test





# Introduction

- Last class: covered the important result by **Bahadur and Savage**, and some generalizations.
- The class of distributions that satisfy the conditions of their theorem is **large**.
- **Example**: the class of distributions with finite second moment satisfies the requirements of the theorem, as does the class of distributions with infinitely many moments.
- We concluded that the failure of the $t$-test is not special to the $t$-test; in this setting, there simply exist no "reasonable" tests.
- **Problem**: the mean $\mu(P)$ is quite sensitive for the tails of $P$, and one sample yields little information about the tails.
- **TODAY**: Let's study conditions to **save** our good friend the $t$-test.

# Distributions with Compact Support

- **Observation**: restricting attention to distributions with **compat support** does not save the $t$-test.

- Let $X_1, \ldots, X_n$ be i.i.d. $P$.

- ***t*-test**: one sided-version $\phi_n = I\{\sqrt{n}X_n > \hat{\sigma}_n z_{1-\alpha}\}$ for testing $\mu(P) = 0$ versus $\mu(P) > 0$.

- Let $\mathbf{P}$ be the set of distributions supported on $[-1, 1]$, and $\mathbf{P}_0 \subseteq \mathbf{P}$ those with mean 0. Using the same exact arguments as in the previous class, we obtain

$$\sup_{P \in \mathbf{P}_0} E_P[\phi_n] = 1, \quad \forall n \geqslant 2 .$$

- **Proof**: Fix $n > 1$ and any $c < 1$. Then, choose $p_n > 0$ so that $(1 - p_n)^n = c$. Let $P = P_{n,c}$ be the distribution that places mass $1 - p_n$ at $p_n$ and mass $p_n$ at $-(1 - p_n)$, so that $\mu(P) = 0$.
(exactly the same example as before)

# Skewness

- The problem here is that we have no control over the **skewness** in the class $P_{n,c}$.

$$\frac{E_{P_{n,c}}[X^3]}{\sigma^3(P_{n,c})} = \frac{p_n^2 + (1-p_n)^2}{\sqrt{p_n(1-p_n)}} \to \infty \,.$$

- In fact, this problem appears the moment we exceed the second moment: "$2+\delta$"

- Let $\delta > 0$ and note that,

$$\frac{E_{P_{n,c}}[|X|^{2+\delta}]}{\sigma^{2+\delta}(P_{n,c})} = \frac{p_n^{1+\delta} + (1-p_n)^{1+\delta}}{(\sqrt{p_n(1-p_n)})^{\delta}} \to \infty \,.$$

  This condition that will be meaningful in the next section.

- **On the Example**: Before moving on, let's make the previous example more convincing to ensure that the underlying distribution is continuous and the observations are distinct.

# Extending the example

Let $X^*_{n,i} = X_{n,i} + U_{n,i}$, where $X_{n,i} \sim P_{n,c}$ as before and, independently, $U_{n,i} \sim U[-\tau_n, \tau_n]$ with

$$\tau_n < \frac{\sqrt{n} p_n}{\sqrt{n} + z_{1-\alpha}} \; .$$

# Extending the example

With prob. at least $c$: (1) $\bar{X}^*_{n,n} > \tau_n z_{1-\alpha}/\sqrt{n}$ and (2) $\hat{\sigma}^{*2}_n \leqslant \tau^2_n$

# QUESTIONS?

# Distributions with $2+\delta$ Moments

- **Good news**: the $t$-test is uniformly consistent over certain large subfamilies of dist. with two finite moments.

- **Uniform Integrability**: consider the family of distributions $\mathbf{P}$ on the real line satisfying,

$$\lim_{\lambda\to\infty} \sup_{P\in\mathbf{P}} E_P\left[\frac{|X-\mu(P)|^2}{\sigma^2(P)} I\left\{\frac{|X-\mu(P)|}{\sigma(P)} > \lambda\right\}\right] = 0\,. \qquad (\clubsuit)$$

- **$2+\delta$ moments**: let $\mathbf{P}^{2+\delta}$ be the set of distributions satisfying

$$\mathbf{P}^{2+\delta} = \left\{P : E_P\left[\frac{|X-\mu(P)|^{2+\delta}}{\sigma^{2+\delta}(P)}\right] \leqslant M\right\}\,,$$

for some $\delta > 0$ and $M < \infty$. **Result**: $\mathbf{P}^{2+\delta} \subseteq \mathbf{P}$.

# Uniformity of the t-test

Let $\mathbf{P}_0$ be the set of distributions in $\mathbf{P}$ with $\mu(P) = 0$.

For testing $\mu(P) = 0$ versus $\mu(P) > 0$, the $t$-test $\phi_n = I\{T_n > z_{1-\alpha}\}$, where

$$T_n = \frac{\sqrt{n}\bar{X}_n}{\hat{\sigma}_n} ,$$

is **uniformly asymptotically level** $\alpha$ over $\mathbf{P}_0$ as shown in the next theorem.

## Theorem

*Suppose $X_{n,1}, \ldots, X_{n,n}$ are i.i.d. with distribution $P_n \in \mathbf{P}$, where $\mathbf{P}$ satisfies (♣). Then, under $P_n$*

$$\frac{\sqrt{n}(\bar{X}_{n,n} - \mu(P_n))}{\hat{\sigma}_{n,n}} \xrightarrow{d} N(0,1) .$$

*In addition, for testing $\mu(P) = 0$ versus $\mu(P) > 0$, the $t$-test is uniformly asymptotically level $\alpha$ over $\mathbf{P}_0$; that is,*

$$\lim_{n\to\infty} \sup_{P \in \mathbf{P}_0} E_P[\phi_n] = \alpha .$$

# Proof

$$\frac{n^{1/2}(\bar{X}_{n,n} - \mu(P_n))}{\hat{\sigma}_{n,n}} = \frac{n^{1/2}(\bar{X}_{n,n} - \mu(P_n))}{\sigma(P_n)} \times \frac{\sigma(P_n)}{\hat{\sigma}_{n,n}} \quad \text{and let} \quad Y_{n,i} = \frac{X_{n,i} - \mu(P_n)}{\sigma(P_n)} .$$

# Proof

$$\frac{n^{1/2}(\bar{X}_{n,n} - \mu(P_n))}{\hat{\sigma}_{n,n}} = \frac{n^{1/2}(\bar{X}_{n,n} - \mu(P_n))}{\sigma(P_n)} \times \frac{\sigma(P_n)}{\hat{\sigma}_{n,n}} \quad \text{and let} \quad Y_{n,i} = \frac{X_{n,i} - \mu(P_n)}{\sigma(P_n)} .$$

# Last Step

- Complete the proof by showing that

$$\lim_{n\to\infty} \sup_{P\in\mathbf{P}_0} E_P[\phi_n] = \alpha \, .$$

- Note that, if the result failed, one could extract a subsequence $\{P_n\}$ with $P_n \in \mathbf{P}_0$ such that,

$$E_{P_n}[\phi_n] \to \alpha' \neq \alpha \, .$$

  This would contradict $T_n$ being asymptotically standard normal under $P_n$.

- **Lesson**: to prove the theorem all we need is a CLT for triangular arrays and a law of large numbers for triangular arrays. The latter is handled by the following two lemmas.

# QUESTIONS?

# Two Lemmas - LLNs

## Lemma (1)

*Let* $Y_{n,1}, \ldots, Y_{n,n}$ *be i.i.d. with cdf* $G_n$ *and finite mean* $\mu(G_n)$ *satisfying*

$$\lim_{\beta \to \infty} \limsup_{n \to \infty} E_{G_n}\left[|Y_{n,i} - \mu(G_n)| I\{|Y_{n,i} - \mu(G_n)| \geqslant \beta\}\right] = 0 \,. \qquad (\spadesuit)$$

*Let* $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_{n,i}$*. Then, under* $G_n$*,* $\bar{Y}_n - \mu(G_n) \to 0$ *in probability.*

## Lemma (2)

*Let* $\mathbf{P}$ *be a family of distributions satisfying (♣). Suppose* $X_{n,1}, \ldots, X_{n,n}$ *are i.i.d.* $P_n \in \mathbf{P}$ *and* $\mu(P_n) = 0$*. Then, under* $P_n$*,*

$$\frac{1}{n}\sum_{i=1}^n \frac{X_{n,i}^2}{\sigma^2(P_n)} \to 1 \text{ in probability} \,.$$

The proof of Lemma 2 follows from applying Lemma 1 to

$$Y_{n,i} = \frac{X_{n,i}^2}{\sigma^2(P_n)} - 1 \,.$$

We therefore only prove Lemma 1.

# Proof

STEP 1 : for $Z_{n,i} = Y_{n,i} I\{|Y_{n,i}| \leqslant n\}$ show that $P\{|\bar{Y}_n - m_n| > \epsilon\} \leqslant P\{|\bar{Z}_n - m_n| > \epsilon\} + nP\{|Y_{n,i}| > n\}$ .

# Proof

STEP 2 : show that $P\{|\bar{Y}_n - m_n| > \epsilon\} \leqslant \epsilon^{-2}\kappa_n(n) + \tau_n(n)$

where

$\tau_n(t) = tP\{|Y_{n,i}| > t\}$ and $\kappa_n(t) = \frac{1}{t}E[Y_{n,i}^2 I\{|Y_{n,i}| \leqslant t\}]$ .

STEP 3 : $\tau_n(t) = tP\{|Y_{n,i}| > t\}$ and $\kappa_n(t) = \frac{1}{t}E[Y_{n,i}^2 I\{|Y_{n,i}| \leqslant t\}]$ .

**Claim:** using integration by parts it is possible to show that

$$\kappa_n(t) = -\tau_n(t) + \frac{2}{t}\int_0^t \tau_n(x)dx \; .$$

Therefore, in order to show that $P\{|\bar{Y}_n - m_n| > \epsilon\} \to 0$, it suffices to argue that

$$\frac{2}{n}\int_0^n \tau_n(x)dx \to 0 \quad \text{and} \quad \tau_n(n) \to 0 \; .$$

**Easy:** $\tau_n(n) \to 0$ by (♠).

# Proof

STEP 4 : $\frac{2}{n}\int_0^n \tau_n(x)dx \to 0$ where $\tau_n(t) = tP\{|Y_{n,i}| > t\} \leqslant E[|Y_{n,i}|I\{|Y_{n,i}| > t\}]$ .

# Finish the Argument

STEP 5 : from $P\{|\bar{Y}_n - m_n| > \epsilon\} \leqslant \epsilon^{-2}\kappa_n(n) + \tau_n(n) \to 0$ to $\bar{Y}_n - \mu(G_n) \xrightarrow{G_n} 0$

# QUESTIONS?

# Power of the $t$-test

- So far we know that the $t$-test behaves uniformly well across a fairly large class of distributions.
- We now study some **power properties** of the $t$-test.
- In particular, we will show that the $t$-test is uniformly consistent in level, and derive a limiting power calculation. The result is summarized in the following Theorem.

**Theorem**

*Let $\mathbf{P}$ be a family of distributions satisfying (♣) and let $\mathbf{P}_0$ be the set of distributions in $\mathbf{P}$ with $\mu(P) = 0$ (assumed non-empty). Then, for testing $\mu(P) = 0$ versus $\mu(P) > 0$, the limiting power of the $t$-test against $P_n \in \mathbf{P}$ with $n^{1/2}\mu(P_n)/\sigma(P_n) \to \delta$ is given by*

$$\lim_{n\to\infty} E_{P_n}[\phi_n] = 1 - \Phi(z_{1-\alpha} - \delta) \ .$$

*Furthermore,*

$$\lim_{n\to\infty} \inf_{\{P\in\mathbf{P}: n^{1/2}\mu(P)/\sigma(P)\geqslant\delta\}} E_P[\phi_n] = 1 - \Phi(z_{1-\alpha} - \delta) \ .$$

# Proof (last one!)

- Let $X_{n,1}, \dots, X_{n,n}$ be i.i.d. with distribution $P_n$ and consider the $t$-statistic $T_n = \bar{X}_{n,n}/\hat{\sigma}_{n,n}$. Write

$$T_n = \frac{n^{1/2}(\bar{X}_{n,n} - \mu(P_n))}{\hat{\sigma}_{n,n}} + \frac{n^{1/2}\mu(P_n)/\sigma(P_n)}{\hat{\sigma}_{n,n}/\sigma(P_n)} \ .$$

- By Theorem 1 the first term converges weakly to $N(0,1)$ under $P_n$, and by the proof of the same Theorem, the denominator of the second term converges to 1 in probability under $P_n$.

- It follows that $T_n \xrightarrow{d} N(\delta, 1)$ under $P_n$ and so (4) follows.

- To prove the second part, argue by contradiction and assume there exists a subsequence $\{P_n\}$ with $n^{1/2}\mu(P_n)/\sigma(P_n) \geqslant \delta$ such that

$$E_{P_n}[\phi_n] \to \gamma < 1 - \Phi(z_{1-\alpha} - \delta) \ .$$

- This, however, would violate the first part if $n^{1/2}\mu(P_n)/\sigma(P_n)$ has a limit. If it does not have a limit, pass to any convergent subsequence and apply the same argument.

# THE END