ECON 480-3
LECTURE 4: ENDOGENEITY

Ivan A. Canay
Northwestern University

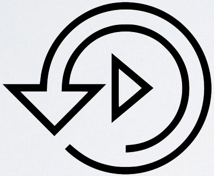
Past & Future

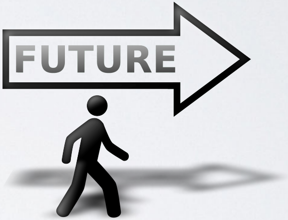
So Far

- Three Interpretations of β
- Solving and estimating sub-vectors of β
- Properties of LS
- Estimating $\mathbb{V}$
- Classical Problems that lead to $E[XU] \neq 0$

TODAY

- Instrumental Variables
- The IV Estimator
- The 2SLS Estimator
- Properties of 2SLS
- Estimating $\mathbb{V}$





Instrumental Variables

- Let (Y, X, U) be a random vector where Y and U take values in $\mathbf{R}$ and X takes values in $\mathbf{R}^{k+1}$. Assume further that the first component of X is constant and equal to one, i.e., $X = (X_0, X_1, \ldots, X_k)'$ with $X_0 = 1$. Let $\beta = (\beta_0, \beta_1, \ldots, \beta_k)' \in \mathbf{R}^{k+1}$ be such that
$$Y = X'\beta + U\ .$$
- We do not assume $E[XU] = 0$. Any X_j such that $E[X_jU] = 0$ is said to be *exogenous*; any X_j such that $E[X_jU] \neq 0$ is said to be *endogenous*. Normalizing β_0 if necessary, we view X_0 as exogenous.
- **Instrument**: to overcome the difficulty associated with $E[XU] \neq 0$, we assume that there is an additional random vector Z taking values in $\mathbf{R}^{\ell+1}$ with $\ell + 1 \geqslant k + 1$ such that $E[ZU] = 0$.
- Any exogenous component of X is contained in Z (the so-called included instruments). In particular, we assume the first component of Z is constant equal to one, i.e., $Z = (Z_0, Z_1, \ldots, Z_\ell)'$ with $Z_0 = 1$.
- We also assume that $E[ZX'] < \infty$, $E[ZZ'] < \infty$ and that there is no perfect collinearity in Z.

Instrumental Variables

- We assume 1 $E[ZU] = 0$, 2 $E[ZX'] < \infty$, 3 $E[ZZ'] < \infty$, and 4 there is no perfect collinearity in Z.
- The requirement that $E[ZU] = 0$ is termed *instrument exogeneity*.
- We further assume 5 the rank of $E[ZX']$ is $k+1$. This is termed *instrument relevance* or *rank condition*.
- A necessary condition for 5 to be true is $\ell \geqslant k$. This is referred to as the *order condition*.
- Using that $U = Y - X'\beta$ and $E[ZU] = 0$, we see that β solves the system of equations

$$E[ZY] = E[ZX']\beta \ .$$

- Since $\ell + 1 \geqslant k + 1$, this may be an over-determined system of equations.

A Useful Lemma

Lemma

Suppose there is no perfect collinearity in Z *and let* Π *be such that BLP*$(X|Z) = \Pi' Z$.
$E[ZX']$ *has rank* $k+1$ *if and only if* Π *has rank* $k+1$. *Moreover, the matrix* $\Pi' E[ZX']$ *is invertible.*

Solving for β

β solves: $E[ZY] = E[ZX']\beta$ or $\Pi' E[ZY] = \Pi' E[ZX']\beta$

Using the previous lemma and $\Pi = E[ZZ']^{-1}E[ZX']$, we can derive three formulae for β

Interpreting the rank condition

Interpretation: Consider the case where $k = \ell$ and only X_k is endogenous. Let $Z_j = X_j$ for all $0 \leqslant j \leqslant k-1$. In this case,

$$\Pi' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \pi_0 & \pi_1 & \cdots & \pi_{\ell-1} & \pi_\ell \end{pmatrix}.$$

The rank condition therefore requires $\pi_\ell \neq 0$: the instrument Z_ℓ must be "correlated with X_k after controlling for $X_0, X_1, \ldots, X_{k-1}$."

Partition of β: endogenous components

- Partition X into X_1 and X_2, where X_2 is exogenous. Partition Z into Z_1 and Z_2 and β into β_1 and β_2 analogously.

- Note that $Z_2 = X_2$ are *included* instruments and Z_1 are *excluded* instruments. Then,
$$Y = X_1'\beta_1 + X_2'\beta_2 + U \ .$$

- We can conveniently re-write this by projecting (BLP) on $Z_2 = X_2$. Consider the case $k = \ell$
$$\text{BLP}(Y|Z_2) = \text{BLP}(X_1|Z_2)'\beta_1 + X_2'\beta_2 \ .$$

- Define $Y^* = Y - \text{BLP}(Y|Z_2)$ and $X_1^* = X_1 - \text{BLP}(X_1|Z_2)$ so that
$$E[Z_1 Y^*] = E[Z_1 X_1^{*\prime}]\beta_1 + E[Z_1 U]$$

- It follows that
$$\beta_1 = E[Z_1 X_1^{*\prime}]^{-1} E[Z_1 Y^*] \ .$$

QUESTIONS?

Estimating β: The IV Estimator

- **Just identified case**: $k = \ell$. Denote by P the marginal distribution of (Y, X, Z).

- Let $(Y_1, X_1, Z_1), \ldots, (Y_n, X_n, Z_n)$ be an i.i.d. sequence of random variables with distribution P.

- By analogy with $\beta = E[ZX']^{-1}E[ZY]$, the natural estimator of β is simply

$$\hat{\beta}_n = \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i X_i' \right)^{-1} \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i Y_i \right) .$$

- This estimator is called the *instrumental variables (IV)* estimator of β. Note that $\hat{\beta}_n$ satisfies

$$\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i (Y_i - X_i' \hat{\beta}_n) = 0 .$$

  In particular, $\hat{U}_i = Y_i - X_i' \hat{\beta}_n$ satisfies

$$\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i \hat{U}_i = 0 .$$

The IV Estimator

Insight on the IV estimator: assume $X_0 = 1$ and $X_1 \in \mathbf{R}$. An interesting interpretation of the IV estimator of β_1 is obtained by multiplying and dividing by $\frac{1}{n}\sum_{i=1}^{n}(Z_{1,i} - \bar{Z}_{1,n})^2$, i.e.,

$$\hat{\beta}_{1,n} = \frac{\frac{1}{n}\sum_{i=1}^{n}(Z_{1,i} - \bar{Z}_{1,n})Y_i \;/\; \frac{1}{n}\sum_{i=1}^{n}(Z_{1,i} - \bar{Z}_{1,n})^2}{\frac{1}{n}\sum_{i=1}^{n}(Z_{1,i} - \bar{Z}_{1,n})X_{1,i} \;/\; \frac{1}{n}\sum_{i=1}^{n}(Z_{1,i} - \bar{Z}_{1,n})^2}$$

The IV Estimator: matrix notation

This estimator may be expressed more compactly using matrix notation. Define

$$\begin{aligned}\mathbb{Z} &= (Z_1, \ldots, Z_n)' \\ \mathbb{X} &= (X_1, \ldots, X_n)' \\ \mathbb{Y} &= (Y_1, \ldots, Y_n)' \,.\end{aligned}$$

In this notation, we have

$$\hat{\beta}_n = (\mathbb{Z}'\mathbb{X})^{-1}(\mathbb{Z}'\mathbb{Y}) \,.$$

The Two-Stage Least Squares (TSLS) Estimator

- **Over-identified case**: $\ell > k$
- The expressions we derived for β in this case, like

$$\beta = E[\Pi' E[ZX']]^{-1} \Pi' E[ZY] \; ,$$

  all involved the matrix Π, where

$$\text{BLP}(X|Z) = \Pi' Z \; .$$

- An estimate of Π can be obtained by OLS.
- Since $\Pi = E[ZZ']^{-1} E[ZX']$, a natural estimator of Π is

$$\hat{\Pi}_n = \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i Z_i' \right)^{-1} \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i X_i' \right) \; .$$

The Two-Stage Least Squares (TSLS) Estimator

Let $X_i = \hat{\Pi}_n' Z_i + \hat{V}_i$ where $\hat{\Pi}_n = \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i Z_i' \right)^{-1} \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i X_i' \right)$.

With this estimator of Π, a natural estimator of β is simply

The TSLS Estimator

- Note that $\hat{\beta}_n$ satisfies

$$\frac{1}{n}\sum_{1\leqslant i\leqslant n}\hat{\Pi}_n' Z_i(Y_i - X_i'\hat{\beta}_n) = 0\ .$$

- In particular, $\hat{U}_i = Y_i - X_i'\hat{\beta}_n$ satisfies

$$\frac{1}{n}\sum_{1\leqslant i\leqslant n}\hat{\Pi}_n' Z_i\hat{U}_i = 0\ .$$

- This implies that $\hat{U}_i$ is orthogonal to all of the instruments equal to an exogenous regressors, but may not be orthogonal to the other regressors.

- It is termed the TSLS estimator because it may be obtained in the following way:
  1. regress (each component of) X_i on Z_i to obtain $\hat{X}_i = \hat{\Pi}_n' Z_i$;
  2. regress Y_i on $\hat{X}_i$ to obtain $\hat{\beta}_n$. However, in order to obtain proper standard errors, it is recommended to compute the estimator in one step (see the following section).

The TSLS Estimator: matrix notation

This estimator may be expressed more compactly using matrix notation. Define

$$\begin{aligned}
\mathbb{Z} &= (Z_1, \ldots, Z_n)' \\
\mathbb{X} &= (X_1, \ldots, X_n)' \\
\mathbb{Y} &= (Y_1, \ldots, Y_n)' \\
\hat{\mathbb{X}} &= (\hat{X}_1, \ldots, \hat{X}_n)' \\
&= \mathbb{P}_Z \mathbb{X} \,,
\end{aligned}$$

where

$$\mathbb{P}_Z = \mathbb{Z}(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'$$

is the projection matrix onto the column space of $\mathbb{Z}$. In this notation, we have

$$\begin{aligned}
\hat{\beta}_n &= (\hat{\mathbb{X}}'\mathbb{X})^{-1}(\hat{\mathbb{X}}'\mathbb{Y}) \\
&= (\hat{\mathbb{X}}'\hat{\mathbb{X}})^{-1}(\hat{\mathbb{X}}'\mathbb{Y}) \\
&= (\mathbb{X}'\mathbb{P}_Z\mathbb{X})^{-1}(\mathbb{X}'\mathbb{P}_Z\mathbb{Y}) \,.
\end{aligned}$$

QUESTIONS?

Properties of Two-Stage Least Squares

- Let (Y, X, U) be a random vector where Y and U take values in $\mathbf{R}$ and X takes values in $\mathbf{R}^{k+1}$. Assume further that the first component of X is constant and equal to one, i.e., $X = (X_0, X_1, \ldots, X_k)'$ with $X_0 = 1$. Let $\beta = (\beta_0, \beta_1, \ldots, \beta_k)' \in \mathbf{R}^{k+1}$ be such that
$$Y = X'\beta + U\ .$$
- We assume 1 $E[ZU] = 0$, 2 $E[ZX'] < \infty$, 3 $E[ZZ'] < \infty$, and 4 there is no perfect collinearity in Z, and 5 the rank of $E[ZX']$ is $k+1$
- Let $(Y_1, X_1, Z_1), \ldots, (Y_n, X_n, Z_n)$ be an i.i.d. sequence of random variables with distribution P.
- Under these assumptions the TSLS estimator is **consistent** for β, and under the additional requirement that Var$[ZU] < \infty$, it is **asymptotically normal** with limiting variance
$$\mathbb{V} = E[\Pi' ZZ' \Pi]^{-1} \Pi' \text{Var}[ZU] \Pi E[\Pi' ZZ' \Pi]^{-1}\ .$$

Consistency of TSLS

$$\hat{\beta}_n = \left(\hat{\Pi}_n' \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i X_i' \right) \right)^{-1} \hat{\Pi}_n' \left(\frac{1}{n} \sum_{1 \leqslant i \leqslant n} Z_i Y_i \right) \xrightarrow{P} \beta \text{ as } n \to \infty \, .$$

Asymptotic Normality of TSLS

Assume that Var$[ZU] = E[ZZ'U^2] < \infty$. Then, as $n \to \infty$,

$$\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{d} N(0, \mathbb{V}) \ .$$

Estimation of $\mathbb{V}$

A natural estimator of $\mathbb{V}$ is given by

$$\hat{\mathbb{V}}_n = \left(\hat{\Pi}_n' \left(\frac{1}{n}\sum_{1\leqslant i\leqslant n} Z_i Z_i'\right)\hat{\Pi}_n\right)^{-1} \times \hat{\Pi}_n' \left(\frac{1}{n}\sum_{1\leqslant i\leqslant n} Z_i Z_i' \hat{U}_i^2\right)\hat{\Pi}_n \times \left(\hat{\Pi}_n' \left(\frac{1}{n}\sum_{1\leqslant i\leqslant n} Z_i Z_i'\right)\hat{\Pi}_n\right)^{-1},$$

where $\hat{U}_i = Y_i - X_i'\hat{\beta}_n$.

- **Primary difficulty** in establishing the consistency of this estimator lies in showing that

$$\frac{1}{n}\sum_{1\leqslant i\leqslant n} Z_i Z_i' \hat{U}_i^2 \xrightarrow{P} \mathrm{Var}[ZU]$$

  as $n \to \infty$. The complication lies in the fact that we do not observe U_i and therefore have to use $\hat{U}_i$.

- However, the desired result can be shown by arguing exactly as in the second part of this class.

- **Note**: $\hat{U}_i = Y_i - X_i'\hat{\beta}_n \neq Y_i - \hat{X}_i'\hat{\beta}_n$, so the standard errors from two repeated applications of OLS will be incorrect.

QUESTIONS?