

A crash course on $\text{HF}^*(L, K)$ and $\text{Fuk}(M)$.

(REFS!)

(M^{2n}, ω) - symplectic mfld : $d\omega = 0$, $\omega^n \neq 0$ everywhere

- ex: $(T^*X, \sum_i y_i dx_i)$
- ex: smooth subvariety in $\mathbb{C}P^n$

$L^n \subset M^{2n}$ - Lagrangian submfld : $\omega|_L = 0$

- ex: $T_p^*X \subset T^*X$, $\Gamma(\text{closed 1-form}) \subset T^*X$
- ex: $\mathbb{R}P^n \subset \mathbb{C}P^n$, $\{[z_0 : \dots : z_n] \mid |z_i| = 1 \forall i\} \subset \mathbb{C}P^n$.

GIANT OVERARCHING GOAL : Understand the L 's $\subset M$,

intersection theory, rigidity phenomena.

$H : M \rightarrow \mathbb{R}$ ("Hamiltonian") $\rightsquigarrow X_H \in \mathcal{X}(M)$, $\omega(X_H, -) = dH$.

(Note, $L_{X_H} \omega \stackrel{\text{CMF}}{=} X_H \lrcorner \underline{\frac{d\omega}{=0}} + d(\underline{\frac{X_H \lrcorner \omega}{=dH}}) = 0$.)

$(H_t)_{t \in [0,1]}$ \rightsquigarrow Hamiltonian isotopy $\Psi : M \supseteq$, time-1 flow of the X_{H_t} .

Conjecture (Arnold-Givental), thm (Floer) :

Assume $[\omega] \cdot \pi_2(M, L) = 0$, and $L \pitchfork \Psi(L)$. Then :

$$\#(L \cap \Psi(L)) \geq \sum_{i=0}^n \text{rk } H_*^i(L; \mathbb{Z}/2).$$

(mention the t -independent case)

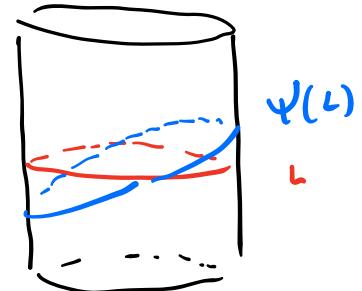
□

• $H_t \equiv H$

Proof uses Floer cohomology.

intersection theory of
Lagrangians; made into
Ham isotopy invariant by
counts of pseudoholomorphic
curves.

Ex. $M = \mathbb{R} \times S^1$, $\omega = \text{area form}$.

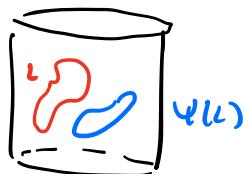


Ψ Hamiltonian \Rightarrow signed area of 2-chain
told by L . $\Psi(L) = 0$

$$\Rightarrow \#(L \cap \Psi(L)) \geq 2.$$

△

Non-ex.



$$\text{Now. } \#(L \cap \Psi(L)) = 0$$

Why? Because L bounds a disk.

△

Floer cohomology $\text{HF}^*(L, K)$ is a graded vector space associated to $L, K \subset M$.

Properties :

- (1) $\text{HF}^*(L, L) \cong H_{\text{sing}}^*(L)$ when $[\omega] \cdot \pi_2(M, L) = 0$.
- (2) $\text{HF}^*(L, \Psi(L)) \cong \text{HF}^*(L, L)$ for Ψ a Ham diffeo.
- (3) when $L \pitchfork K$, $\text{CF}^*(L, K) = \mathbb{k}\langle p \rangle_{p \in L \cap K}$.

Proof of Arnold-Givental conj :

$$\begin{aligned} \text{HF}^*(L, \Psi(L)) &\stackrel{(2)}{\cong} \text{HF}^*(L, L) \\ &\stackrel{(1)}{\cong} H_{\text{sing}}^*(L) \end{aligned} \quad (*)$$

$$\begin{aligned} \rightarrow \#(L \cap \Psi(L)) &\stackrel{(3)}{=} \text{rk } \text{CF}^*(L, \Psi(L)) \\ &\geq \text{rk } \text{HF}^*(L, \Psi(L)) \\ &\stackrel{(*)}{=} \sum_{i=0}^n \text{rk } H_i(L; \mathbb{Z}/2). \end{aligned}$$

□

Def of $\text{HF}^*(L, K)$. Say $L \cap K$, then $\text{HF}^*(L, K) := H(CF^*(L, K) := \Lambda \langle p \rangle_{p \in L \cap K}, d)$.

With w/ Novikov coeffs: $\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lambda_i \xrightarrow{i \rightarrow \infty} \infty \right\}$

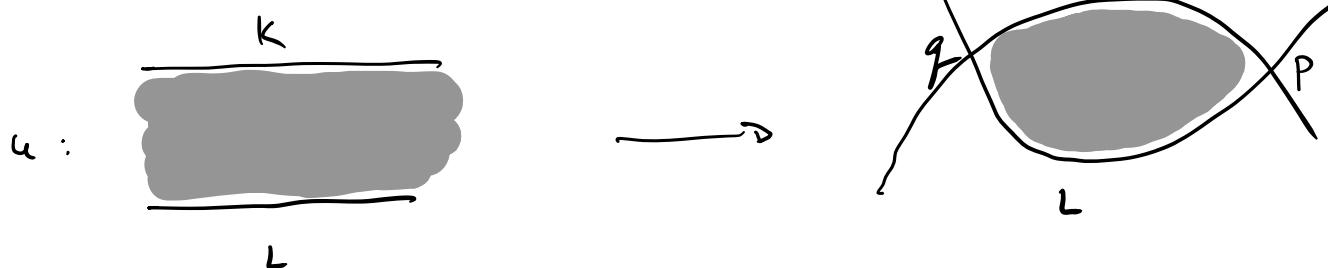
\uparrow
 $\pi/2\mathbb{Z}, \mathbb{Q}, \dots$

To define d , choose $J : TM \supseteq$, $J^2 = -1$, $\omega(-, J-)$ a metric.

A J -holomorphic strip is an element of the following moduli space:

$$\mathcal{M}(p_+, p_-, [u], J) := \left\{ u : \mathbb{R} \times [0, 1] \rightarrow M \mid \begin{array}{l} \partial_s u + J(u) \partial_t u = 0, \\ u(s, 0) \in L, \quad u(s, 1) \in K, \\ \lim_{s \rightarrow \pm\infty} u(s, t) = p_{\pm}, \quad \{u^*[\omega]\}^\infty \end{array} \right\}$$

/ translation



When M is cut out transversally, it's stratified by Maslov index

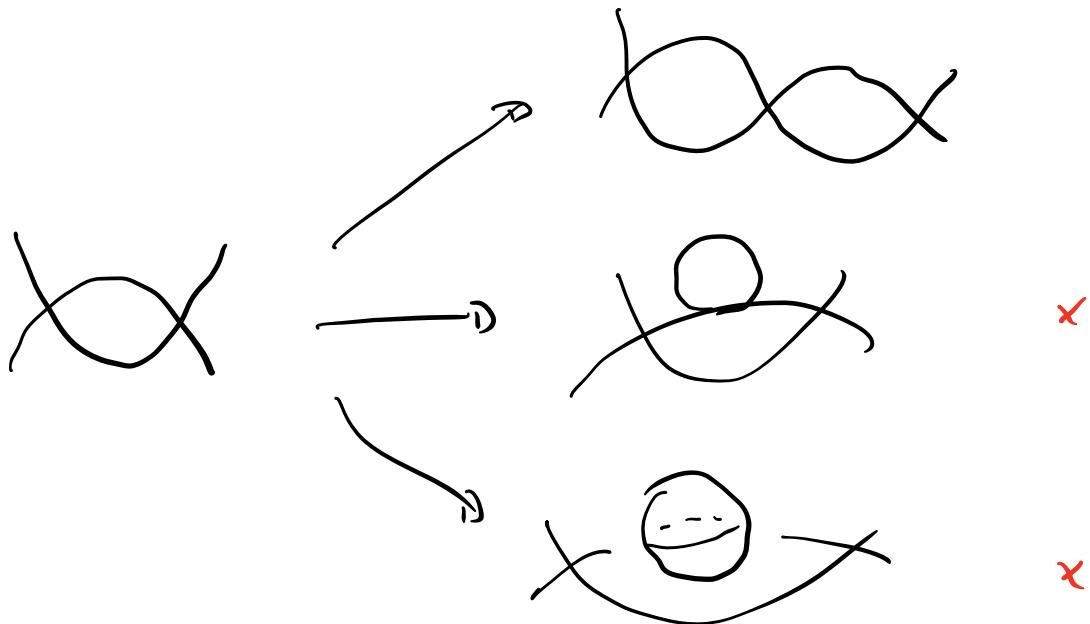
$$\rightsquigarrow d_p := \sum_{\substack{q \in L \cap K \\ \text{ind}[u] = 1}} \# \mathcal{M}(p, q, [u], J) T^{w([u])} q.$$

Compactness. $M(p, q, [u], J)$ admits a well-behaved compactification: "Gromov compactification".

"thm": Given a sequence of J -hol curves $u_i: \Sigma \rightarrow M$ w/ $\int u_i^* \omega \leq E$,
by f'dy conditions, can pass to a subsequence s.t.:

- away from p_1, \dots, p_k , have C_{loc}^∞ -convergence
- can rescale at the p_i 's to produce a "bubble tree" of disks and spheres

In the case of $M(p, q, [u], J)$:



Proof of $d^2 = 0$.

We just saw that the only degeneration in M is  \rightarrow .

"gluing" \Rightarrow if $\text{ind}[u] = 2$, $\partial M(p, q, [u], J) = \bigsqcup_{r \in L \cap K} M(p, r, [u'], J) \times M(r, q, [u''], J)$

$$[u] = [u'] + [u'']$$

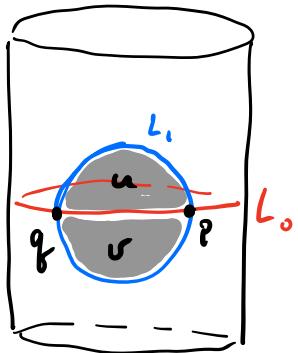
$$\text{ind } u' = \text{ind } u'' = 1$$

$$\Rightarrow \sum \# M(p, r, [u'], J) \# M(r, q, [u''], J) T^{w([u']) + w([u''])} = 0$$

$$\Rightarrow d^2 = 0.$$

□

example where $d^2=0$ fails:



$$\left. \begin{array}{l} dp = \pm T^{w(u)} q \\ dq = \pm T^{w(u)} p \end{array} \right\} \implies d^2 \neq 0.$$

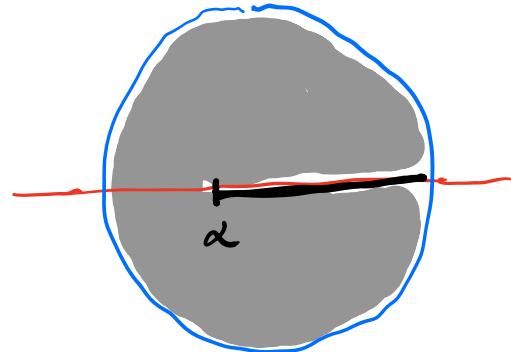
why:

$$u_\alpha : \{ |z| \leq 1, \operatorname{im} z \geq 0 \} \setminus \{ \pm 1 \} \rightarrow \mathbb{C}$$

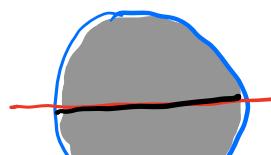
$$u_\alpha(z) \in L_0, \quad u_\alpha(e^{i\theta}) \in L_1, \quad \alpha \in (-1, 1).$$

$$u_\alpha(z) := \frac{z^2 + \alpha}{1 + \alpha z^2}$$

index -2 strips from $p \times p$

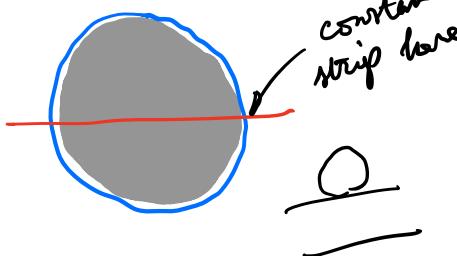


$$\alpha \xrightarrow{\alpha} \alpha$$



contributes to d^2

$$\alpha \xrightarrow{\alpha}$$



constant strip here

obstruction to $d^2=0$.

Two approaches to prove $\text{HF}^*(L, \Psi(L)) \simeq H_{\text{sing}}^*(L)$.

(1) Floer, Hofer-Selamoni:

Product on HF^* .

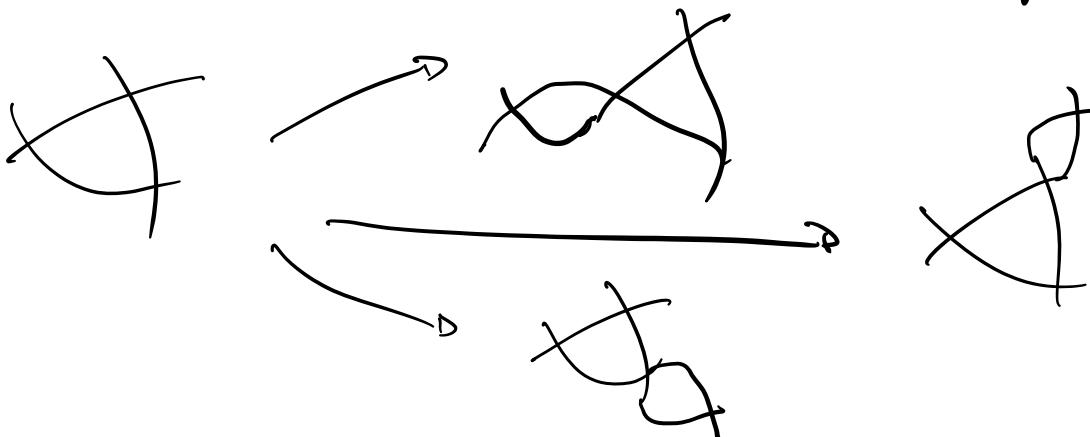
Just as we defined d by counting strips ~~g~~~~X_p~~, define

$$\text{CF}^*(L_1, L_2) \otimes \text{CF}^*(L_0, L_1) \longrightarrow \text{CF}^*(L_0, L_2)$$

by counting triangles

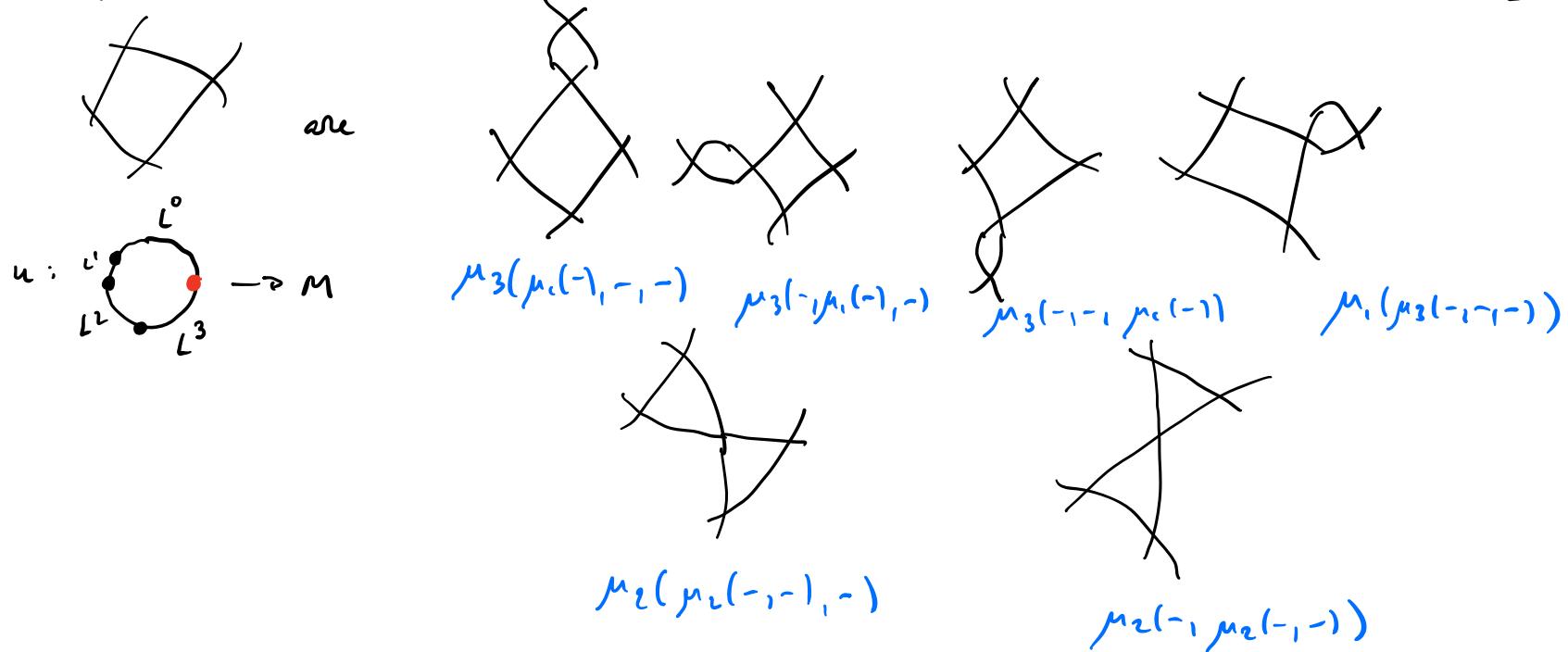
$$u : \begin{array}{c} \bullet \\ \circlearrowleft \\ \bullet \\ \circlearrowright \\ \bullet \end{array} \xrightarrow{\quad J \quad} \begin{array}{c} \bullet \\ \nearrow p_2 \\ f \times g \\ \searrow p_1 \end{array} .$$

When bubbling excluded, Gromov \Rightarrow ends of 1-diml moduli space given by



\Rightarrow descends to product $\text{HF}^*(L_1, L_2) \otimes \text{HF}^*(L_0, L_1) \longrightarrow \text{HF}^*(L_0, L_2)$,

The product on HF^* is associative: ends of 1-dim'l moduli spaces of "squares"



But this does suggest considering what structure we have on the chain level.

($d \geq 0$)

$\forall d \geq 1$, define $\mu^d : \text{CF}^*(L_{d-1}, L_d) \otimes \dots \otimes \text{CF}^*(L_0, L_1) \rightarrow \text{CF}^*(L_0, L_d)$

by counting J-hol. $(d+1)$ -gons $\rightsquigarrow \sum_{a,d} \mu^{e-d+1}(x_1, \dots, x_a, \mu^d(x_{a+1}, \dots, x_{a+d}), x_{a+d+1}, \dots, x_e) = 0$.

- μ^1 a differential
- μ^2 a chain map
- μ^2 associative up to homotopy given by μ^3

One version of the Fukaya category.

(M, ω) a sympl mfld ω $2c_1(TM) = 0$



$\text{Fuk}(M)$, a (\mathbb{N} -linear) \mathbb{Z} -graded A_∞ -category:

- objects are $L \subset M$ Lag, Oriented, spin, $[\omega] \cdot \pi_2(M, L) = 0$, $\mu_L = \sigma \in H^*(L, \mathbb{Z})$, w/ spin structure, graded lift.
- after fixing perturbation data & tuples, define A_∞ -operations μ^δ $\forall d \geq 1$.

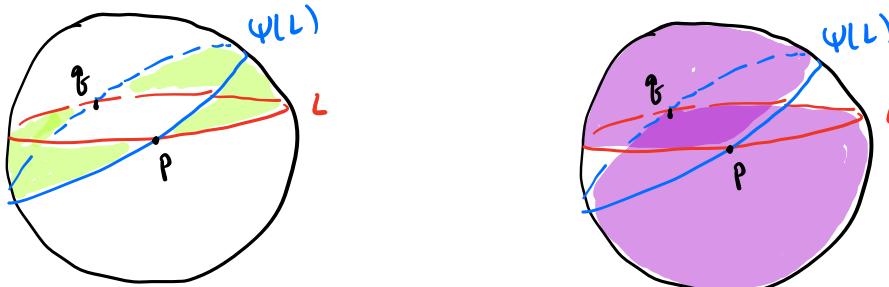
- blue: can drop; get $\mathbb{Z}/2$ -graded cat
- red: " " enough if we're using Novikov coefficients over $\mathbb{Z}/2$.
- green: can use non-Novikov coeffs.

Example : $\text{Fuk}(S^2)$.

$$\text{HF}^*(L, \psi(L))$$

First, let's compute $\text{HF}^*(L, L)$, for $L \subset S^2$ an embedded closed curve.

Say L divides S^2 into two halves of equal area. Work w/ $\mathbb{K} = \mathbb{Z}/2$.



$$dp = T_q^a f + T_q^a g \longleftrightarrow dq = T_p^b p + T_p^b q = 0$$

$$\Rightarrow \text{HF}^*(S'_{eq}, S'_{eq}) = \Lambda^2.$$

Similarly, if S' divides S^2 into two unequal halves, $\text{HF}^*(S'_1, S'_2) = \infty$.

$\rightsquigarrow \text{HFah}(S^2)$ has one object S'_{eq} . $\hookrightarrow \text{HF}(S'_{eq}, S'_{eq}) = \Lambda \langle \times \rangle / \{x^2 = T^{\frac{1}{2}\omega(S^2)}\}$

In practice, typically complete $\text{Fuk}(M)$ wrt. mapping cones, formal summands.