Lecture 8: Repeated Bargaining

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Overview

The bargaining models we have seen so far:

- The game ends right after players trade.

Today: Repeated bargaining games.

- Players may trade multiple times.

Closely related: The literature on the Ratchet effect.

- Guesnerie, Freixas and Tirole (1985 REStud), Hart and Tirole (1988 REStud), Bester and Strausz (2001 ECMA), Skreta (2006 REStud), Gerardi and Maestri (2020 TE), Doval and Skreta (2021 ECMA).

Today: Schmidt (1993 JET)

- Time is finite $t = T, T-1, ..., 2, 1$.
- A buyer with discount $\beta \in (0,1)$ *vs* a seller with discount $\delta \in (0,1)$.

- The buyer's value is common knowledge $b > 1$.
- The seller has n possible costs $1 = c^1 > c^2 > ... > c^n \geq 0$.
- The buyer's prior belief: Type c^i occurs with prob μ^i.

- In period t, the buyer offers $p_t \in \mathbb{R}$, and the seller accepts or rejects.
- Players' continuation values in period t are:

$$V_t^B = \mathbb{E}\Big[\sum_{j=0}^{t-1} \beta^j (b - p_{t-j}) \mathbf{1}\{\text{trade at } t-j\}\Big],$$

$$V_t^S = \mathbb{E}\Big[\sum_{j=0}^{t-1} \delta^j (p_{t-j} - c) \mathbf{1}\{\text{trade at } t-j\}\Big].$$

Refinements

Schmidt focuses on PBEs that satisfy two refinements:

1. The highest-cost type accepts an offer if and only if $p \geq 1$.
2. The seller's decision in period t depends only on their cost, the buyer's belief about their cost, and the buyer's current offer.

Schmidt calls the second refinement a *weak Markov property*.

Main Result

Let G^T denote the T-period repeated bargaining game.

Theorem

For every $\beta \in [0,1)$, $\delta \in (1/2,1)$, and $\mu^1 > 0$, there exists $Z \in \mathbb{N}$ such that for every $T > Z$ and every equilibrium σ^T of G^T,

1. *The buyer offers $p_t = 1$ in all except for the last Z periods.*
2. *The buyer's payoff converges to $\frac{b-1}{1-\beta}$ and type-c seller's payoff converges to $\frac{1-c}{1-\delta}$ as $T \to +\infty$.*

Compared to Kreps and Wilson, Milgrom and Roberts:

- Both players can be patient.
- The informed player has more than one rational type.
- The informed player's discount factor can be anything more than $1/2$.

Lemma: Upper Bound on the Buyer's Offer

Lemma

Let $\overline{c}_t$ be the highest type in the support of the buyer's belief in period t.

1. *Any type $c \leq \overline{c}_t$ strictly prefers to accept any offer $p_t > \overline{c}_t$.*
2. *The buyer's offer p_t is no more than $\overline{c}_t$.*
3. *The buyer's continuation value in period t is at least*

$$\underline{V}_t^B(\overline{c}_t) \equiv \sum_{j=0}^{t-1} \beta^j (b - \overline{c}_t).$$

This conclusion relies on finite horizon and backward induction.

Proof of Lemma

In the last period, i.e., $t = 1$:

- Seller with cost c strictly prefers to accept any price $> c$.
- The buyer will never offer anything strictly more than $\overline{c}_1$.

Suppose the conclusion holds for every $s \leq t$, then in period $t + 1$:

- Suppose by way of contradiction that an offer $p_{t+1} > \overline{c}_{t+1}$ is rejected with positive prob in equilibrium.
- Let $\overline{c}$ be the highest type that rejects this offer.
- By induction hypothesis, after rejecting p_{t+1}, type $\overline{c}$'s continuation value is no more than 0 starting from period t.
- Hence, type $\overline{c}$ strictly prefers to accept $p_{t+1} > \overline{c}_{t+1}$.
- This leads to a contradiction.

Why won't the buyer offer anything strictly more than $\overline{c}_{t+1}$?

- Offering anything strictly more than $\overline{c}_{t+1}$ will lead to the same posterior belief, and hence, the same continuation value.

Lemma: Seller will reject all prices below their cost

Lemma

If the buyer offers p_t, then every type of the seller with cost $c > p_t$ strictly prefers to reject p_t.

Suppose by way of contradiction that some type $c > p_t$ accepts p_t with positive prob.

- Let $\overline{c}$ be the highest type that accepts p_t.
- Type $\overline{c}$'s continuation value after accepting p_t is no more than 0.
- Anticipating this, type $\overline{c}$ has no incentive to incur any loss.
- This leads to a contradiction.

Lemma: offer $p_t = 1$ will be accepted for sure

Lemma

If the buyer offers $p_t = 1$, then all types of the seller will accept for sure.

Previous lemmas:

- All types with $c < 1$ will accept for sure.

Refinement: Type c^1 accepts 1 for sure.

Lemma: Lower Bound on the Speed of learning

Let

$$M > \frac{\log(1-\beta) + \log(b-1) - \log b}{\log \beta}$$

and

$$\varepsilon \equiv \frac{(1-\beta)(b-1)}{b} - \beta^M > 0.$$

Lemma

If in equilibrium, the buyer makes M offers with $p < 1$, then there exists at least one of them that will be accepted with probability more than ε.

Implication: If the seller rejects M offers in a row, then the prob assigned to type c^1 is multiplied by at least $\frac{1}{1-\varepsilon}$.

Intuition Behind the Learning Lemma

Suppose there are t periods left.

The buyer's continuation value is at least $\sum_{j=0}^{t-1} \beta^j (b-1)$.

The buyer's continuation value is at most $\sum_{j=0}^{t-1} \beta^j b$.

Suppose the buyer makes M offers with $p < 1$ and each is accepted with prob less than ε.

- Each time the offer is rejected, the buyer loses at least his payoff from trading with the highest type.
- Each time the offer is accepted, the buyer's gain in continuation value is bounded.

What is missing? Why don't we set $M = 1$?

Proof of the Learning Lemma

Let π_{τ_t} be the prob that the seller accepts offer p_{τ_t} at time τ_t.

The buyer's continuation value starting from τ_1:

$$\pi_{\tau_1}\left\{b - p_{\tau_1} + \beta V^B_{\tau_1 - 1}(p_{\tau_1}, A)\right\} + (1-\pi_{\tau_1}) \sum_{j=1}^{\tau_1-\tau_2-1} \beta^j (b-1)$$

$$+(1-\pi_{\tau_1})\pi_{\tau_2}\beta^{\tau_1-\tau_2}\left\{b - p_{\tau_2} + \beta V^B_{\tau_2 - 1}(p_{\tau_2}, A)\right\}$$

$$+(1-\pi_{\tau_1})(1-\pi_{\tau_2}) \sum_{j=\tau_1-\tau_2+1}^{\tau_1-\tau_3-1} \beta^j (b-1) + \ldots$$

$$+\prod_{j=1}^{M-1}(1-\pi_{\tau_j})\pi_{\tau_M}\beta^{\tau_1-\tau_M}\left\{b - p_{\tau_M} + \beta V^B_{\tau_M - 1}(p_{\tau_M}, A)\right\}$$

$$+\prod_{j=1}^{M}(1-\pi_{\tau_j})\beta^{\tau_1-\tau_M+1} V^B_{\tau_M - 1}(p_{\tau_M}, R).$$

Bound this continuation value from above

For each blue term, note that

$$b - p_{\tau_j} + \beta V^B_{\tau_j - 1}(p_{\tau_j}, A) < \frac{b}{1-\beta}$$

For the last term, we have:

$$V^B_{\tau_M - 1}(p_{\tau_M}, R) < \frac{b}{1-\beta}.$$

Intuition:

- Even if the buyer successfully screens the seller, his payoff per period is no more than b.
- Even if the buyer may get a high continuation value after M offers are rejected, his continuation value is no more than $\frac{b}{1-\beta}$.

Upper Bound for this term when $\pi_{\tau_j} < \varepsilon$

$$\pi_{\tau_1}\left\{b - p_{\tau_1} + \beta V^B_{\tau_1 - 1}(p_{\tau_1}, A)\right\} + (1 - \pi_{\tau_1}) \sum_{j=1}^{\tau_1 - \tau_2 - 1} \beta^j (b - 1)$$

$$+ (1 - \pi_{\tau_1})\pi_{\tau_2}\beta^{\tau_1 - \tau_2}\left\{b - p_{\tau_2} + \beta V^B_{\tau_2 - 1}(p_{\tau_2}, A)\right\}$$

$$+ (1 - \pi_{\tau_1})(1 - \pi_{\tau_2}) \sum_{j=\tau_1 - \tau_2 + 1}^{\tau_1 - \tau_3 - 1} \beta^j (b - 1) + \ldots$$

$$+ \prod_{j=1}^{M-1} (1 - \pi_{\tau_j})\pi_{\tau_M}\beta^{\tau_1 - \tau_M}\left\{b - p_{\tau_M} + \beta V^B_{\tau_M - 1}(p_{\tau_M}, A)\right\}$$

$$+ \prod_{j=1}^{M} (1 - \pi_{\tau_j})\beta^{\tau_1 - \tau_M + 1} V^B_{\tau_M - 1}(p_{\tau_M}, R).$$

Upper Bound for this term when $\pi_{\tau_j} < \varepsilon$

$$\leq \varepsilon \frac{b}{1-\beta} + \sum_{j=1}^{\tau_1-\tau_2-1} \beta^j (b-1) + \varepsilon \beta^{\tau_1-\tau_2} \frac{b}{1-\beta} + \sum_{j=\tau_1-\tau_2+1}^{\tau_1-\tau_3-1} \beta^j (b-1) + \ldots$$

$$\ldots + \varepsilon \beta^{\tau_1-\tau_M} \frac{b}{1-\beta} + \beta^M \frac{b}{1-\beta}. \tag{1}$$

$$= \varepsilon \frac{b}{1-\beta} \sum_{j=1}^{M} \beta^{\tau_1-\tau_j} + \beta^M \frac{b}{1-\beta}$$

$$+ \sum_{j=1}^{\tau_1-\tau_2-1} \beta^j (b-1) + \ldots + \sum_{j=\tau_1-\tau_{M-1}+1}^{\tau_1-\tau_M-1} \beta^j (b-1)$$

This must be no less than the buyer's payoff from always offering 1.

Upper Bound for this term when $\pi_{\tau_j} < \varepsilon$

Compare the buyer's payoff from offering $p < 1$ for M times, each accepted with prob less than ε,

$$= \varepsilon \frac{b}{1-\beta} \sum_{j=1}^{M} \beta^{\tau_1 - \tau_j} + \beta^M \frac{b}{1-\beta}$$

$$+ \sum_{j=1}^{\tau_1-\tau_2-1} \beta^j (b-1) + ... + \sum_{j=\tau_1-\tau_{M-1}+1}^{\tau_1-\tau_M-1} \beta^j (b-1)$$

with his payoff from offering 1 in every period. The former is greater only if

$$\varepsilon \frac{b}{1-\beta} \{1 + \beta^{\tau_1-\tau-2} + ... + \beta^{\tau_1-\tau_M}\} + \beta^M \frac{b}{1-\beta}$$

$$> (b-1)\{1 + \beta^{\tau_1-\tau-2} + ... + \beta^{\tau_1-\tau_M}\}.$$

This cannot be true when $\beta^M \frac{b}{1-\beta} < \frac{b-1}{2}$ and $\frac{\varepsilon b}{1-\beta} < \frac{b-1}{2}$.

Reputation Result

Let

$$K \equiv M \cdot \frac{\log \mu^1}{\log(1-\varepsilon)}.$$

Lemma

In any equilibrium, there can be at most K periods in which the buyer offers $p < 1$ and gets rejected.

After M rejections, the prob of type c^1 is multiplied by at least $\frac{1}{1-\varepsilon}$.

- The prob of type c^1 cannot exceed 1.

Location of Bad Periods

What we know: If the seller imitates the highest-cost type.

- After the seller rejects K offers strictly lower than 1, the buyer will offer 1 in all subsequent periods.
- Type c seller's payoff in the beginning of the game is at least

$$\sum_{j=K+1}^{T} \delta^j (1-c).$$

What we don't know yet is the location of these K periods.

The payoff lower bound is not tight when δ is bounded below 1.

- When $\delta \approx 1/2$, the discounted average payoff is bounded below $1-c$ even when $T \to +\infty$.

A Useful Observation

The buyer may not want to offer $p < 1$ in the beginning.

- Why? When seller's has a high continuation value,

  all types of the seller have strict incentives to reject $p < 1$ in order to build a reputation for having a high cost.
- Hence, offering $p < 1$ cannot make any type of the seller to accept.
- Knowing that no one will accept, the buyer will offer 1 due to the restriction to Markov strategies.

No Screening in the Beginning

For every $\delta > 1/2$, let L be the smallest integer s.t.

$$\sum_{j=1}^{L} \delta^j > 1.$$

Lemma

If $T > KL$, then the buyer offers 1 in the first $T - KL$ periods.

Proof

Suppose the seller has rejected $K-1$ offers with $p<1$ and there are τ_1 periods left with $\tau_1 > L$.

Suppose the buyer offers $p<1$ again,

- Type c's payoff from rejecting is $\sum_{j=1}^{\tau_1} \delta^j (1-c)$.
- Type c's payoff from accepting is at most $p - c + \sum_{j=1}^{\tau_1} \delta^j (\bar{c} - c)$, where $\bar{c}$ is the highest type that accepts with positive prob.

Since $p<1$, type $\bar{c}$ has an incentive to accept p only if:

$$p - \bar{c} \geq \sum_{j=1}^{\tau_1} \delta^j (1-\bar{c}),$$

which implies that $1-\bar{c} \geq \sum_{j=1}^{\tau_1} \delta^j (1-\bar{c})$. This is not true given our definition of L.

Therefore, after being rejected $K-1$ times and there are more than L periods left, the buyer strictly prefers to offer 1.

Proof: By Induction

Suppose the seller has rejected $K-2$ offers with $p<1$ and there are τ_2 periods left, where $\tau_2 > 2L$.

Suppose the buyer offers $p<1$ again,

- Type $\overline{c}$'s payoff from rejecting the offer is at least $\sum_{j=1}^{\tau_2-L-1} \delta^j(1-\overline{c})$.
- The seller's payoff from accepting the offer is at most $1-\overline{c}$.

Type-$\overline{c}$ seller has no incentive to accept offers less than 1.

After being rejected $K-2$ times and there are more than $2L$ periods left, the buyer strictly prefers to offer 1.

Proof: By Induction

By induction, we know that if there are more than KL periods left, the buyer has no incentive to offer anything less than 1.

Main Result

Theorem

For every $\beta \in [0,1)$, $\delta \in (1/2,1)$, and $\mu^1 > 0$, there exists $Z \in \mathbb{N}$ such that for every $T > Z$ and every equilibrium σ^T of G^T,

1. *The buyer offers $p_t = 1$ in all except for the last Z periods.*
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Concluding Remarks

Schmidt assumes that the buyer's offer belongs to a discrete grid.

- With a continuum of offers, the characterization results are more elegant but he does not have a proof for existence.

With two types, we do not need the refinement.