Lecture 17: Community Enforcement Models with Incomplete Information

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Spring Quarter, 2023

Community Enforcement Models with Complete Info

A large group of players randomly matched to play game G.

- Each player only observes the actions in his own match.
- They cannot observe their partners' identities and cannot observe what's going on in other matches.

A special class of repeated games with private monitoring.

- Each player's private signal is the actions in his match.
- The folk theorem in Sugaya (2021) does not apply since the private signals cannot statistically identify the (entire) action profile.

Kandori (1992), Ellison (1994), Deb and Gonzalez-Diaz (2019), Deb (2020), and Deb, Sugaya and Wolitzky (2020):

- Folk theorems in community enforcement with complete info.

Question: What happens when there is incomplete info?

Community Enforcement with Incomplete Information

To fix ideas, consider a large population of players playing the prisoner's dilemma:

- A fraction of the population are bad types who always play D,

  e.g., each player is normal w.p. $1-\varepsilon$ and is bad w.p. ε.
- In each period, players are randomly matched and can only observe the actions in their own match.

Two key findings:

- Sugaya and Wolitzky (2020): Anti-folk theorem.
- Sugaya and Wolitzky (2021): Folk theorem when players can communicate via cheap talk messages.

A General Anonymous Repeated Game with Bad Types

- Discrete time $t = 0, 1, 2, ...$
- N players with discount factor δ.
- Each player's action set A, with $a_t \in A^N$ the action profile at t.

- Player i's type $\theta_i \in \{R, B\}$, with type B taking a^* in every period.
- Type distribution $p \in \Delta\left(\{R, B\}^N\right)$.

- Player i's private signal $y_{i,t} \sim F(\cdot|(a_\tau, y_\tau)_{\tau=0}^{t-1}, a_t)$.
- Public randomization device $\xi_t \sim U[0, 1]$.
- Player i's private history in period t consists of θ_i and $(a_{i,\tau}, y_{i,\tau}, \xi_\tau)_{\tau=0}^{t-1}$.

- Players' stage-game payoffs $(u_1, ..., u_N) : A^N \to [0, 1]^N$.

Symmetry Assumptions

Assumption: Symmetric Type Distribution

$p(\theta_1, ..., \theta_n)$ depends only on the number of bad types in $(\theta_1, ..., \theta_n)$.

Assumption: Symmetric Payoff Function

Fix $i, j \in \{1, 2, ..., N\}$. We have $u_i(a_i, a_{-i}) = u_j(a'_j, a'_{-j})$ if

- $a_i = a'_j$,
- *the number of other players playing each action is the same under a_{-i} and under a'_{-j}.*

Prisoner's Dilemma with Uniform Random Matching

Leading example: $N = 2n$ players are uniformly matched into pairs in each period to play the prisoner's dilemma.

- Payoffs are symmetric since matching is uniform and anonymous.

  Each opponent's action matters for your payoff with prob $\frac{1}{N-1}$.
- The private signal $y_{i,t}$ is the action profile in agent i's match, i.e., agent i perfect observes each opponent's action with prob $\frac{1}{N-1}$.
- The type distribution is symmetric when each player is bad w.p. ε.

Analysis

- With symmetry and public randomization, focusing on symmetric equilibrium is w/o loss of generality.

- Let $\mathcal{B}_n$ be the event that there are n bad players, with $p_n \equiv \Pr(\mathcal{B}_n)$.

- Let $q_n \equiv \Pr\left(n \text{ out of } N-1 \text{ other players are bad} \middle| \text{player i is rational}\right)$.
- Let $q_n^- \equiv q_{n-1}$. Let $q_N \equiv 0$ and $q_0^- \equiv 0$.

  Both $q \equiv (q_0, \ldots, q_N)$ and $q^- \equiv (q_0^-, \ldots, q_N^-)$ are prob distributions.

- The total variation distance between q and q^- is:

$$\Delta \equiv \max_{\mathcal{N} \subset \{0,1,\ldots,N\}} \left| \sum_{n \in \mathcal{N}} (q_n - q_n^-) \right|.$$

Analysis

Interpretations of the two distributions q and q^-:

- Let $q_n \equiv \Pr\left(n \text{ out of } N-1 \text{ other players are bad} \middle| \text{player i is rational}\right)$.
- Let $q_n^- \equiv q_{n-1}$.

Suppose the rational type's equilibrium strategy is *not* a^* in every period.

- If I am rational and play my equilibrium strategy, then q is my belief about the total number of people playing a^* in every period.
- If I am rational but I deviate to a^* in every period, then q^- is my belief about the total number of people playing a^* in every period.

- Therefore, Δ measures the *detectability* of a rational type's deviation to the bad type's strategy.

Lower Bound on Rational Type's Payoff

Let $U_i(\theta)$ be player 1's equilibrium payoff conditional on type profile θ.

Let

$$u_n^R \equiv \mathbb{E}[U_i(\theta)|\theta_i = R, \mathcal{B}_n] \text{ and } u_n^B \equiv \mathbb{E}[U_i(\theta)|\theta_i = B, \mathcal{B}_n].$$

Lemma

In every equilibrium of the repeated game, we have

$$\sum_{n=0}^{N-1} q_n u_n^R \geq \sum_{n=0}^{N-1} q_n u_n^B - \Delta.$$

What is the rational type's expected payoff when he plays his equilibrium strategy?

- $\sum_{n=0}^{N-1} q_n u_n^R$.

Lower Bound on Rational Type's Payoff

Let $U_i(\theta)$ be player 1's equilibrium payoff conditional on type profile θ.

Let

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Lemma

In any equilibrium,

$$\sum_{n=0}^{N-1} q_n u_n^R \geq \sum_{n=0}^{N-1} q_n u_n^B - \Delta.$$

What is the rational type's expected payoff when he deviates and plays a^* in every period?

- $\sum_{n=0}^{N-1} q_n u_{n+1}^B = \sum_{n=0}^{N} q_n^- u_n^B$.

  (comes directly from $q_n^- = q_{n-1}$)

Proof: Lower Bound on Payoff

Let

$$u_n^R \equiv \mathbb{E}[U_i(\theta)|\theta_i = R, \mathcal{B}_n] \text{ and } u_n^B \equiv \mathbb{E}[U_i(\theta)|\theta_i = B, \mathcal{B}_n].$$

Lemma

In any equilibrium,

$$\sum_{n=0}^{N-1} q_n u_n^R \geq \sum_{n=0}^{N-1} q_n u_n^B - \Delta.$$

Rational type's payoff from deviating to a^* in every period is given by $\sum_{n=0}^{N-1} q_n u_{n+1}^B = \sum_{n=0}^{N} q_n^- u_n^B$. Therefore,

$$\sum_{n=0}^{N-1} q_n u_{n+1}^B = \sum_{n=0}^{N-1} q_n u_n^B - \sum_{n=0}^{N} (q_n - q_n^-) u_n^B \geq \sum_{n=0}^{N-1} q_n u_n^B - \Delta$$

The blue term is no more than his equilibrium payoff $\sum_{n=0}^{N-1} q_n u_n^R$.

Pairwise Dominant Action

This lemma is useful in games where a^* is a pairwise dominant action:

Assumption: Pairwise Dominance

Action $a^* \in A$ is a pairwise dominant action if there exists $c > 0$ such that for every $a \neq a^*$ and $a_{-ij} \in A^{N-2}$, we have

$$u_i(a_i = a^*, a_j = a, a_{-ij}) - u_j(a_j = a, a_i = a^*, a_{-ij}) > c.$$

This neither implies nor is implied by a^* being a dominant action.

- Find two counterexamples to convince yourself.

In the prisoner's dilemma game with uniform random matching:

- D is a pairwise dominant action since

$$\frac{x+1}{N-1}(1+g) \geq \frac{x}{N-1} - l \cdot \frac{N-1-x}{N-1} + \underbrace{\min\{g, l\}}_{\equiv c},$$

where x is the number of people playing C other than i and j.

Upper Bound on Rational Type's Payoff

Fix an equilibrium. When the rational type plays his equilibrium strategy,

- let γ_n be the occupation measure with which he plays actions other than a^* conditional on there are n bad types in the population.

Recall that

$$u_n^R \equiv \mathbb{E}[U_i(\theta)|\theta_i = R, \mathcal{B}_n] \text{ and } u_n^B \equiv \mathbb{E}[U_i(\theta)|\theta_i = B, \mathcal{B}_n].$$

Lemma

If a^* is a pairwise dominant action, then $u_n^B \geq u_n^R + \gamma_n c$ for every n.

This follows from the definition of pairwise dominant actions.

Lower Bound on the Occupation Measure of a^*

Combining the two lemmas:

Lemma

In any equilibrium, $\sum_{n=0}^{N-1} q_n u_n^R \geq \sum_{n=0}^{N-1} q_n u_n^B - \Delta$.

Lemma

If a^* *is a pairwise dominant action, then* $u_n^B \geq u_n^R + \gamma_n c$ *for every n.*

we obtain the following inequality:

$$\Delta \geq \sum_{n=0}^{N-1} q_n(u_n^B - u_n^R) \geq c \cdot \sum_{n=0}^{N-1} q_n \gamma_n .$$

The expected occupation measure of actions other than a^*, $\sum_{n=1}^{N-1} q_n \gamma_n$, is no more than $\frac{\Delta}{c}$, i.e., the expected occupation measure of a^* is at least $1 - \frac{\Delta}{c}$.

Anti-Folk Theorem

Recall that the expected occupation measure of actions other than a^*, $\sum_{n=1}^{N-1} q_n \gamma_n$, is no more than $\frac{\Delta}{c}$.

If $\Delta \to 0$, then:

- In every equilibrium, the rational type plays a^* in almost all periods.
- Social welfare is close to the case in which everyone is bad.

This leads to an anti-folk theorem, i.e., all payoffs are close to $U(a^*)$.

When is it the case that $\Delta \to 0$ as $N \to +\infty$?

Leading example: Each player is bad with prob ε, and players' types are independently drawn from the same distribution.

Fix $\varepsilon > 0$.

- $q_n = \binom{N-1}{n}(1-\varepsilon)^{N-n}\varepsilon^n$.
- $q_n^- = q_{n-1} = \binom{N-1}{n-1}(1-\varepsilon)^{N-n+1}\varepsilon^{n-1}$.

Since q_n is single-peaked in n, the total variation distance is

$$\Delta = q_0 + (q_1 - q_0) + ... + (q_k - q_{k-1}) = q_k$$

where $q_k \equiv \max_{n\in\{0,1,..,N\}} q_n$.

As $N \to +\infty$, $\max_{n\in\{0,..,N\}} \binom{N-1}{n}(1-\varepsilon)^{N-n}\varepsilon^n \to 0$.

Therefore, $\Delta \to 0$ as $N \to +\infty$.

Conclusion: Anti-Folk Theorem under Incomplete Info

Sugaya and Wolitzky (2020)'s result implies that:

- In a repeated prisoner's dilemma with uniform random matching and each player is a bad type who always defects with prob ε,

  all equilibrium payoffs converge to the minmax payoff as $N \to +\infty$.

Hence, it is impossible to sustain cooperation in large populations.

Sugaya and Wolitzky (2021) focus on this specific setting.

- Theorem 1 in Sugaya and Wolitzky (2021): Extend the anti-folk theorem to when players can observe their partners' identities.
- As $(1-\delta)N \to +\infty$, every NE payoff is close to 0.

The Role of Communication

Sugaya and Wolitzky (2021) also do the following:

- Repeated prisoner's dilemma with uniform random matching.
- Players can observe their partner's identities.
- Each player is bad with prob $\varepsilon > 0$.
- Players can exchange cheap-talk messages with their partners.

As long as $(1 - \delta) \log N \to 0$, there exist equilibria where players' payoffs are arbitrarily close to their payoffs under (C, C).

- Their proof uses a clever information theory argument.
- With complete info, communication can be replaced via actions and contagion (Horner and Olzewski 2009, Deb, et al 2020).
- With incomplete info, communicating via actions and contagion is too slow to sustain cooperation $\Rightarrow$ cheap talk is needed.