Lecture 16: Community Enforcement Models with Complete Information

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From Rich Information to Limited Information

Models we have seen so far make a rich information assumption:

- Players can observe the entire history of actions (e.g., Fudenberg and Maskin 1986, Fudenberg and Levine 1989, Smith and Sorensen 2000).
- Players can observe the entire history of some informative signals (e.g., Fudenberg, Levine and Maskin 1994, Fudenberg and Levine 1992).

In practice, people have limited info about others' past behaviors, e.g.,

- People do not recall events in the distant past.
- People are unfamiliar with their partners (e.g., Maghribi traders in Medieval Europe, online dating apps).
- People don't know who they are playing with (e.g., journal refereeing).

Can people (or the society) sustain good outcomes?

Model

- Time $t = 0, 1, 2, \ldots$
- $N \equiv 2n$ players, discount factor $\delta \in (0, 1)$.
- In each period, players are matched uniformly at random to play the prisoner's dilemma:

-	Cooperate	**Defect**
Cooperate	$1, 1$	$-l, 1+g$
Defect	$1+g, -l$	$0, 0$

with $g, l > 0$.

The matching process is independent across periods.

- **Monitoring structure:**

Each player only observes the action profile of his own matches.

He cannot observe the identity of his current/past opponents.

He cannot observe what happened in other matches.

How Can Players Sustain Cooperation?

Kandori (1992) proposes the following *contagion strategy*:

- Each player has two private states: c and d.
- The player plays C if his private state is c and plays D if his private state is d.

- All players' private states are c in period 0.
- For each player, his private state is c *if and only if* he hasn't observed anything other than (C, C) in his previous matches.

Lemma

For every $g, l > 0$ and $n \in \mathbb{N}$, there exists $\underline{\delta} \in (0, 1)$ such that when $\delta > \underline{\delta}$, all players using the contagion strategy is a Nash equilibrium.

How Can Players Sustain Cooperation?

Lemma

For every $g, l > 0$ and $n \in \mathbb{N}$, there exists $\underline{\delta} \in (0, 1)$ such that when $\delta > \underline{\delta}$, all players using the contagion strategy is a Nash equilibrium.

Why is the contagion strategy a Nash equilibrium?

- We only need to verify players' incentives on the equilibrium path.

  All players play C on the equilibrium path.

- For any given player i, if he deviates to D, then
  - He obtains a one-period gain of g.
  - But he infects others in the community and spreads contagion.
  - Eventually, he will encounter someone who is in state d.

  When δ is large enough, his one-period gain is less than his long-term loss from spreading contagion.

How Can Players Sustain Cooperation?

Kandori (1992) proposes the following *contagion strategy*:

- Each player has two private states: c and d.
- The player plays C if his private state is c and plays D if his private state is d.

- All players' private states are c in period 0.
- For each player, his private state is c *if and only if* he hasn't observed anything other than (C, C) in his previous matches.

Question: Is the contagion strategy a sequential equilibrium?

- Not necessarily!
- Suppose you observe D in period 0, do you play D in period 1?

  You think that at most one person is infected.

  Playing C is attractive since it slows down contagion.

Theorem 1 in Kandori (1992)

Kandori (1992) shows that the contagion strategy is a sequential equilibrium when l is large enough relative to g and n.

Theorem 1 in Kandori (1992)

For every $g > 0$ and $n \in \mathbb{N}$, there exist $\underline{l} > 0$ and $\underline{\delta} \in (0, 1)$,

such that when $l > \underline{l}$ and $\delta > \underline{\delta}$,

all players using the contagion strategy is a sequential equilibrium.

Intuition: How to motivate players to play D in state d?

- When l is large relative to n and g,

  the loss from playing C while encountering someone playing D

  is much larger relative to the benefit from slowing down contagion.

Limitations of Kandori's result

Theorem 1 in Kandori (1992)

For every $g > 0$ and $n \in \mathbb{N}$, there exist $\underline{l} > 0$ and $\underline{\delta} \in (0, 1)$,

such that when $l > \underline{l}$ and $\delta > \underline{\delta}$,

all players using the contagion strategy is a sequential equilibrium.

It does not imply that players can cooperate in *all* prisoner's dilemma.

- l needs to be implausibly large as $n \to +\infty$.

Cooperation is very fragile and is not robust to trembles.

- One defection causes cooperation to breakdown all together.

Ellison (1994)

Ellison (1994) sharpens Kandori's results in three steps:

1. He assumes that players have a public randomization device and show that cooperation is feasible for all g and l.
2. He shows that the public randomization device is dispensable, i.e., sufficiently patient players can sustain cooperation without public randomization.
3. He shows that the equilibria he constructed are robust to trembles.

Community Enforcement with Public Randomization

- Time $t = 0, 1, 2, \ldots$
- $N \equiv 2n$ players, all share the same discount factor $\delta \in (0, 1)$.
- In each period, players are matched uniformly at random to play the prisoner's dilemma:

-	Cooperate	Defect
Cooperate	$1, 1$	$-l, 1+g$
Defect	$1+g, -l$	$0, 0$

with $g, l > 0$.

The matching process is independent across periods.

- By the end of each period, a random variable $q_t \sim U[0, 1]$.
- Each player only observes the action profile of his own matches as well as the realization of q_t in the period before.

Contagion Strategy with Moderate Punishment

Ellison (1994) proposes the following modified version of Kandori's contagion strategy, parameterized by $\widehat{q} \in [0, 1]$, call it $\sigma_{\widehat{q}}$:

- Each player has two private states: c and d.
- The player plays C if his private state is c and plays D if his private state is d.

- All players' private states are c in period 0.
- For every $t \geq 1$, a player's period t state is c *if and only if* period $t-1$ action profile was (C, C) *or* $q_{t-1} \geq \widehat{q}$.

Intuition: When actions other than (C, C) occur, punish with prob $\widehat{q}$.

- An amnesty with prob $1 - \widehat{q}$ in each period after contagion has began.
- Kandori (1992)'s contagion strategy corresponds to $\widehat{q} = 1$.

Why can moderating punishment help?

Two ICs are required to sustain cooperation in sequential equilibrium.

- Incentive to play C when their private state is c, which is stronger when $\widehat{q}$ is larger.
- Incentive to play D when their private state is d, which is stronger when $\widehat{q}$ is smaller.

  Why? The only benefit from playing C is to slow down contagion, and $\widehat{q} = 0$ kills contagion all together.

Does there exist $\widehat{q} \in [0, 1]$ that can satisfy both constraints?

- Ellison's answer: Yes for all δ large enough.

Proposition 1 in Ellison (1994)

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In the community enforcement game with public randomization.
For every g, l, n, there exists $\underline{\delta} \in (0,1)$ such that when $\delta > \underline{\delta}$,
there exists a sequential equilibrium where (C, C) is always played on-path.

In fact, for every δ large enough, there exists $\widehat{q} \in [0, 1]$ such that all players playing $\sigma_{\widehat{q}}$ is such a sequential equilibrium.

Thought experiment: Fix any $\widehat{q}$ and suppose others play $\sigma_{\widehat{q}}$,

- Suppose you think that k other players are in state d,

  does your incentive to play D increase or decrease in k?

Ellison (1994) shows that it is increasing in k.

- **Intuition:** When there are more infected players already, an additional infected player makes less difference.

Proof: Increasing Incentives to Defect

Let ω be a realization of the matching process (for everyone from 0 to $+\infty$).

Let $f(k, \delta, \widehat{q}, \omega)$ be player 1's continuation value when

- He plays D until observing $q \geq \widehat{q}$.
- Other players use strategy $\sigma_{\widehat{q}}$.
- k of the other players are in private state d.
- The realized matching process is ω.

Lemma

For every $k' > k$, we have

$$f(k, \delta, \widehat{q}, \omega) - f(k+1, \delta, \widehat{q}, \omega) \geq f(k', \delta, \widehat{q}, \omega) - f(k'+1, \delta, \widehat{q}, \omega).$$

Proof: Increasing Incentives to Defect

Lemma

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$$f(k, \delta, \widehat{q}, \omega) - f(k+1, \delta, \widehat{q}, \omega) \geq f(k', \delta, \widehat{q}, \omega) - f(k'+1, \delta, \widehat{q}, \omega).$$

Let us compare $f(k, \delta, \widehat{q}, \omega)$ to $f(k+1, \delta, \widehat{q}, \omega)$:

- They are the same after period t if $q_s \geq \widehat{q}$ for some $s < t$.
- If $q_s < \widehat{q}$ for all $s < t$, then player 1's period t stage-game payoffs are different only when he is matched with someone
  1. who will not be infected before t by the first k players,
  2. who will be infected before t by the $k+1$th player,

  in which case his payoff is reduced by $1 + g$.
- The red set is independent of k while the blue set shrinks with k.

Proof: Expression for the Payoff Difference

Lemma

$\frac{f(k,\delta,\widehat{q},\omega)-f(k+1,\delta,\widehat{q},\omega)}{1-\delta}$ *depends on* $\widehat{q}$ *and* δ *only through* $\delta\widehat{q}$.

Let us compare $f(k,\delta,\widehat{q},\omega)$ to $f(k+1,\delta,\widehat{q},\omega)$:

- They are the same after period t if $q_s \geq \widehat{q}$ for some $s < t$.
- If $q_s < \widehat{q}$ for all $s < t$, then player 1's period t stage-game payoffs are different only when he is matched with someone:
  1. who will not be infected before t by the first k players,
  2. who will be infected before t by the $k+1$th player.

Hence,

$$f(k,\delta,\widehat{q},\omega)-f(k+1,\delta,\widehat{q},\omega) = (1-\delta)\sum_{t=0}^{+\infty}\delta^t\widehat{q}^t(1+g)\mathbf{1}\{\ldots\}$$

where "..." stands for the event that "you encounter someone that belongs to both the red and the blue set".

Proof of the Ellison Theorem

Proposition

For every δ large enough, there exists $\widehat{q} \in [0, 1]$ such that all players using strategy $\sigma_{\widehat{q}}$ is a sequential equilibrium.

Choose $\widehat{q}$ such that players are indifferent between C and D when $k = 0$:

$$(1-\delta)g = \delta \cdot \widehat{q} \cdot \mathbb{E}_{\omega}[f(0, \delta, \widehat{q}, \omega) - f(1, \delta, \widehat{q}, \omega)]. \tag{1}$$

A player's incentive to play D after he is infected:

- If his current partner is infected, then playing D and playing C leads to the same continuation value, yet playing D leads to a benefit l.
- If his current partner is not infected, then the payoff difference between playing D and playing C is:

$$(1-\delta)g - \delta \cdot \widehat{q} \cdot \mathbb{E}_{k,\omega}[f(k, \delta, \widehat{q}, \omega) - f(k+1, \delta, \widehat{q}, \omega)],$$

  which is positive given (1) and $f(k, ..) - f(k+1, ...)$ is decreasing in k.

Proof of the Ellison Theorem

Choose $\widehat{q}$ such that players are indifferent between C and D when $k = 0$:

$$(1-\delta)g = \delta \cdot \widehat{q} \cdot \mathbb{E}_\omega[f(0, \delta, \widehat{q}, \omega) - f(1, \delta, \widehat{q}, \omega)]. \qquad (2)$$

Question: Does there exist such a $\widehat{q}$?

Yes! Why? Let $\widehat{q} = 1$.

- By continuity, there exists $\widehat{\delta} \in (0, 1)$ such that when $\delta > \widehat{\delta}$, inequality (2) is true when $\widehat{q} = 1$.

Since $\frac{f(k,\delta,\widehat{q},\omega)-f(k+1,\delta,\widehat{q},\omega)}{1-\delta}$ depends on $\widehat{q}$ and δ only through $\delta\widehat{q}$, for every $\delta > \widehat{\delta}$, we can set $\widehat{q} = \widehat{\delta}/\delta$ which satisfies (2).

Remove the Public Randomization Device

Ellison's construction relies on a public randomization device.

- Grants an amnesty after each period with probability $1 - \widehat{q}$.
- Moderate the punishment to provide players incentives to punish.

Can we moderate the punishment without any public randomization?

- Yes, when δ is close enough to 1.
- Ellison introduces a cool trick to do this.

Ellison's Trick: How to Lower the Discount Factor

Theorem: Lower the Discount Factor

Let $G(\delta)$ be any repeated complete info game.

Suppose there exists a non-empty interval (δ_0, δ_1) such that for every $\delta \in (\delta_0, \delta_1)$, $G(\delta)$ has an equilibrium $s^*(\delta)$ with outcome $\alpha \in \Delta(A)$.

Then there exists $\underline{\delta} < 1$ such that for every $\delta^* \in (\underline{\delta}, 1)$, there also exists a strategy profile $s^{**}(\delta^*)$ which is an equilibrium in $G(\delta^*)$ and implements α.

There exists $\underline{\delta} \in (0,1)$ such that for every $\delta > \underline{\delta}$, there exists $N(\delta) \in \mathbb{N}$ such that $\delta^{N(\delta)} \in (\delta_0, \delta_1)$.

- Treat the entire repeated game as $N(\delta)$ separate repeated games.
- Repeated game 1 is played in period $0, N(\delta), 2N(\delta), \ldots$
- Repeated game 2 is played in period $1, N(\delta)+1, 2N(\delta)+1, \ldots$

Ellison's Theorem without Public Randomization

Proposition 4 in Ellison (1994)

In the community enforcement game without public randomization.

For every g, l, n, there exists $\underline{\delta} \in (0,1)$ such that when $\delta > \underline{\delta}$,

there exists a sequential equilibrium where (C, C) is always played on-path.

Strategy $\sigma_{\widehat{q}}$ with $\widehat{q} = 1$ does not require public randomization.

Recall that for every $k' > k$, we have

$$f(k, \delta, 1, \omega) - f(k+1, \delta, 1, \omega) \geq f(k', \delta, 1, \omega) - f(k'+1, \delta, 1, \omega).$$

Therefore, we know that for every $k' > k$

$$\mathbb{E}_\omega[f(k, \delta, 1, \omega) - f(k+1, \delta, 1, \omega)] > \mathbb{E}_\omega[f(k', \delta, 1, \omega) - f(k'+1, \delta, 1, \omega)].$$

Ellison's Theorem without Public Randomization

Recall that fix $\widehat{q} = 1$, there exists $\widehat{\delta} \in (0,1)$ such that

$$(1-\widehat{\delta})g = \widehat{\delta} \cdot \mathbb{E}_{\omega}[f(0,\widehat{\delta},1,\omega) - f(1,\widehat{\delta},1,\omega)],$$

i.e., indifferent between C and D when discount factor is $\widehat{\delta}$, no other player is infected, and all players use strategy σ_1.

Since for every $k' > 0$,

$$\mathbb{E}_{\omega}[f(0,\delta,1,\omega) - f(1,\delta,1,\omega)] > \mathbb{E}_{\omega}[f(k',\delta,1,\omega) - f(k'+1,\delta,1,\omega)],$$

there exists an open set of discount factors $(\widehat{\delta}, \widehat{\delta}+\varepsilon)$ such that every player

- prefers C to D when no other player is infected,
- prefers D to C when at least one other player is infected.

Applying the Ellison's trick, we know that (C,C) can be attained in sequential equilibrium for δ large enough even w/o public randomization.

Robustness to Trembles

Recall that a major critique of Kandori's construction is that the equilibrium is not robust to small trembles.

Ellison examines two types of trembles.

1. Independent trembles:
   - Each agent is forced to play D with prob $\varepsilon > 0$ in each period, and trembles are independent across players and across periods.
2. Correlated trembles:
   - Each agent is a commitment type with prob ε, and commitment types always play D. Players' types are independent of each other.

Robustness to Independent Trembles

Independent trembles: Each agent is forced to play D with prob $\varepsilon > 0$ in each period, and trembles are independent across players and across periods.

For the construction *with* public randomization:

- For every g, l, n, there exist $\overline{\varepsilon} > 0$ and $\underline{\delta} \in (0, 1)$, such that when $\delta > \underline{\delta}$ and $\varepsilon < \overline{\varepsilon}$, there exists a sequential equilibrium in which (C, C) is played with probability more than $1 - \eta$

For the construction *without* public randomization:

- Cooperation breaks down as $t \to +\infty$, but players' payoffs are close to 1 in the double limit where $\lim_{\varepsilon \to 0} \lim_{\delta \to 1} \ldots$

Ellison's constructions are somewhat robust to small independent trembles.

- What about large trembles or general noisy monitoring?

What about Correlated Trembles?

Correlated trembles: Each agent is a commitment type with prob ε, and commitment types always play D.

Why are correlated trembles different from independent trembles?

- Independent trembles: After an amnesty, you know that everyone will play C with prob ≈ 1.
- Correlated trembles: Your belief about the number of commitment types depends on your private history.

  e.g., you may have no incentive to play C after an amnesty if you believe that many players are commitment types.

We will talk about correlated trembles in the next lecture.

What is special about the prisoner's dilemma?

Early works on community enforcement focus on the prisoner's dilemma.

- The stage-game NE is a dominant strategy equilibrium.
- When a player plays D, he receives a benefit in the stage game at the cost of his continuation value.

What about other games, e.g., the product choice game?

Suppose there are n buyers and n sellers, randomly matched in each period.

Seller Buyer	**T**rust	**N**o trust
High effort	$1, 1$	$-1, 0$
Low effort	$2, -1$	$0, 0$

Contagion strategies cannot sustain (H, T) in sequential equilibrium:

- Suppose a buyer observes L in period 0, will she play N?
- No, since playing N may not even given her a stage game benefit.

Deb and Gonzalez-Diaz (2019)

Community enforcement in games that look like the product choice game.

- There exists a strict Nash equilibrium a^*.
- There exists an action profile s.t. player 1 has a strict incentive to deviate and player 2 has a strict incentive not to deviate.

Theorem in Deb and Gonzalez-Diaz (2019)

There exists $\underline{n}$ such that for every v that Pareto dominates $u(a^*)$, every $\varepsilon > 0$, and every $n > \underline{n}$, there exists $\underline{\delta} \in (0,1)$ such that when $\delta > \underline{\delta}$, there exists a sequential equilibrium that achieves payoff within ε of v.

Any payoff that strictly Pareto dominates $u(a^*)$ can be attained when:

- the community is large enough and players are sufficiently patient.

Idea of Construction: Product Choice Game

Seller Buyer	**Trust**	**No trust**
High effort	$1, 1$	$-1, 0$
Low effort	$2, -1$	$0, 0$

The equilibrium play consists of three phases. On the equilibrium path,

1. In the first T_1 periods, play (H, T).
2. In the next T_2 periods, play (L, T).
3. After that, play (H, T) in every subsequent period.

T_1 and T_2 are large relative to n, but small relative to $\frac{1}{1-\delta}$.

Any deviation in Phase II or III, switch to (L, N) forever.

How to punish deviations in Phase I?

- If a buyer observes the seller's deviation, then she switches to N forever in beginning of Phase II.
- If a seller observes a deviation, then he continues to play as if on path.

Off-Path Beliefs

The equilibrium play consists of three phases:

1. In the first T_1 periods, play (H, T).
2. In the next T_2 periods, play (L, T).
3. After that, play (H, T) in every subsequent period.

The seller's deviation in Phase I is unpunished until Phase II.
The buyer's deviation in Phase I is ignored.

Off-path beliefs that sustain this sequential equilibrium:

- If I observe a deviation, I believe that the deviation starts in period 0.
- As soon as a seller plays L, he will play L forever.

Rough Ideas Behind Deb and Gonzalez-Diaz (2019)

The equilibrium play consists of three phases:

1. In the first T_1 periods, play (H, T).
2. In the next T_2 periods, play (L, T).
3. After that, play (H, T) in every subsequent period.

Any seller's deviation in Phase I is unpunished until Phase II.

A seller finds it optimal to play L after playing L once.

- Playing H increases his payoff in Phase II and III.
- Phase II: Affects his payoff iff he is matched with the person I infected in Phase I. This prob is small when the population is large.
- Phase III: Affects his payoff iff the additional buyer I infected spread contagion beyond the existing buyer that I infected.

  This effect is small when Phase II is long since one buyer is enough to spread contagion to almost all sellers.

Rough Ideas Behind Deb and Gonzalez-Diaz (2019)

The equilibrium play consists of three phases:

1. First T_1 periods, play (H, T).
2. Next T_2 periods, play (L, T).
3. Then play (H, T) in every subsequent period.

The seller's deviation in Phase I is unpunished until Phase II.

No buyer has an incentive to punish in Phase I:

- Since she thinks that there is only one infected seller.
- Punishing has no stage-game benefit and not affecting future payoffs.

Rough Ideas Behind Deb and Gonzalez-Diaz (2019)

The equilibrium play consists of three phases:

1. First T_1 periods, play (H, T).
2. Next T_2 periods, play (L, T).
3. Then play (H, T) in every subsequent period.

The seller's deviation in Phase I is unpunished until Phase II.

Each infected buyer has a strict incentive to punish in Phase II:

- Recall that she thinks that deviation started in period 0.
- In expectation, lots of buyers have been infected by the end of Phase I.

  These buyers will spread contagion in Phase 2.
- Since all sellers choose L in Phase 2, playing N gives the buyer a higher stage-game payoff while not affecting much his continuation value.

Rough Ideas Behind Deb and Gonzalez-Diaz (2019)

The equilibrium play consists of three phases:

1. First T_1 periods, play (H, T).
2. Next T_2 periods, play (L, T).
3. Then play (H, T) in every subsequent period.

The seller's deviation in Phase I is unpunished until Phase II.

Another role of Phase II: What if a buyer observes L throughout Phase I?

- She thinks that only one seller is infected.
- She knows that she is the only buyer who learns this deviation.
- Will she play N in Phase II and spread contagion?

Rough Ideas Behind Deb and Gonzalez-Diaz (2019)

The equilibrium play consists of three phases:

1. First T_1 periods, play (H, T).
2. Next T_2 periods, play (L, T).
3. Then play (H, T) in every subsequent period.

The seller's deviation in Phase I is unpunished until Phase II.

Another role of Phase II: What if a buyer observes L throughout Phase 1?

- Will she play N in Phase II and spread contagion?
- Yes, because the infected seller does not know this, he will spread contagion in Phase III.
- In expectation, lots of buyers will be infected soon after Phase III, and each buyer's action in Phase II is not that pivotal.

Rough Ideas Behind Deb and Gonzalez-Diaz (2019)

The equilibrium play consists of three phases:

1. First T_1 periods, play (H, T).
2. Next T_2 periods, play (L, T).
3. Then play (H, T) in every subsequent period.

The seller's deviation in Phase I is unpunished until Phase II.

Phase III: Will a player switch to (L, N) after observing a deviation?

- When both T_1 and T_1/T_2 are large, the player believes that with high prob, contagion is widely spread and most players are playing (L, N). This encourages them to play (L, N).

Rough Ideas Behind Deb and Gonzalez-Diaz (2019)

The equilibrium play consists of three phases:

1. First T_1 periods, play (H,T).
2. Next T_2 periods, play (L,T).
3. Then play (H,T) in every subsequent period.

The seller's deviation in Phase I is unpunished until Phase II.

Summary of the Main Idea: Delayed punishment of the seller helps since buyers are asked to punish only when

- they believe that there are enough infected sellers,
- they believe that they are not that pivotal since many others are spreading contagion.

Literature: Community Enforcement with Complete Info

What's coming after Deb and Gonzalez-Diaz (2019)?

- Deb (2020): Folk theorem in community enforcement with general payoffs but players can send cheap-talk messages.
- Deb, Sugaya and Wolitzky (2020): Folk theorem in community enforcement with general payoffs and *without* communication.

A separate literature: Community enforcement with a continuum of players.

- Cooperation cannot be sustained w/o any info about partner's records.
- Intuition: Contagion takes a long time to come back.
- Takahashi (2010) shows this more generally and establishes a folk theorem when players can observe partners' past records.
- Clark, Fudenberg and Wolitzky (2021): Sustaining cooperation with 1st order record vs 2nd order record.