Lecture 8: Reputational Bargaining II

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Overview

Last lecture:

- Abreu and Gul (2000): Introduce obstinate types to bargaining.
- Reputational bargaining in discrete time with frequent offers $\approx$ continuous-time war-of-attrition.
- When offers are frequent and players have a rich set of commitment types, each player's payoff $\approx$ their Rubinstein bargaining payoff.

This lecture:

1. What will happen when players have outside options?
2. What will happen when players have private info about payoffs?
3. Alternative formulations of reputational bargaining.
4. Can we use this machinery to deliver sharp predictions in repeated games with two comparably patient players?

Compte and Jehiel (2002): Outside Options

Discrete time bargaining game with one commitment type on each side.

- $t = 0, \Delta, 2\Delta, \ldots$
- In even periods, P1 either takes the outside option (which ends the game), or makes a new offer.

  P2 either accepts P1's offer and ends the game, or rejects the offer.
- In odd periods, P2 either takes the outside option or makes a new offer.

  P1 either accepts P2's offer or rejects.
- If a player takes the outside option, then payoffs are (β_1^*, β_2^*), satisfying

$$1 - \alpha_2^* < \beta_1^* < \frac{1 - e^{-r_2 \Delta}}{1 - e^{-(r_1 + r_2)\Delta}} \approx \frac{r_2}{r_1 + r_2},$$

$$1 - \alpha_1^* < \beta_2^* < \frac{1 - e^{-r_1 \Delta}}{1 - e^{-(r_1 + r_2)\Delta}} \approx \frac{r_1}{r_1 + r_2}.$$

For each player, the outside option is better than conceding, but is worse than their Rubinstein payoff.

Benchmark: Game without Commitment Types

Theorem: Binmore, Shaked and Sutton (1987)

Suppose players' payoffs from the outside option are such that

$$\beta_1^* < \frac{r_2}{r_1 + r_2},$$

and

$$\beta_2^* < \frac{r_1}{r_1 + r_2},$$

then the unique subgame perfect equilibrium attains the same outcome as the Rubinstein bargaining game without any outside option.

Intuition: Since the outside option is inferior to the Rubinstein bargaining payoff, taking the outside option is not a credible threat.

Result: No Reputation Building

Theorem: Compte and Jehiel

In every PBE of the reputational bargaining game with outside options,

- *The rational-type of player 1 demands* $\frac{1-e^{-r_2\Delta}}{1-e^{-(r_1+r_2)\Delta}}$ *at time 0 and the rational type player 2 accepts immediately.*
- *If player 1 demands* α_1^**, then the rational-type of player 2 takes the outside option.*
- *If player 1 demands sth greater than* $\frac{1-e^{-r_2\Delta}}{1-e^{-(r_1+r_2)\Delta}}$ *but not* α_1^**, then player 2 rejects and offers* $\frac{1-e^{-r_1\Delta}}{1-e^{-(r_1+r_2)\Delta}}$.
- *If player 2 demands* α_2^* *in period* Δ*, then the rational-type player 1 takes the outside option.*

- When a player imitates the commitment type, his opponent takes the outside option immediately.
- Otherwise, play proceeds as in the Rubinstein bargaining game.

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Why no reputation building?

Rational players have no incentive to imitate the commitment type. Why?

- Outside option $\succ$ conceding $\Rightarrow$ Rational type never concedes.
- If my opponent never concedes, then there is no benefit for me to imitate the commitment type.
- The reputational equilibrium in Abreu and Gul unravels.

  Board and Pycia (14): outside options unravel the Coase conjecture.

Comments:

- What if there is a rich set of commitment types?
- Is the Rubinstein bargaining payoff a robust prediction?
- How should we think about wars, strikes, and so on?

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Private Information about Payoffs

How to incorporate private information about payoffs?

- Private info about discount rate: Abreu, Pearce, and Stacchetti (15).
- Private info about value/cost: Pei and Vairo (22).

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APS: Private Info about Discount Rate

- Time $t \in [0, +\infty)$. Players' interest rates are r_1 and r_2.
- Player i's commitment demand α_i^*.
- With prob z_i, player i is committed, where $\pi_i(\alpha_i^*)$ is prob of committing to $\alpha_i^* \in C_i$ conditional on being committed.
- With prob $1 - z_i$, player i is rational and chooses $\tilde{t}_i \in [0, +\infty]$.
- Rational player 1 has private info about their discount rate:
  - with prob p, $r_1 = r_1^H$,
  - with prob $1 - p$, $r_1 = r_1^L$, with $0 < r_1^L < r_1^H$.
- P2 makes an offer first and then P1 makes an offer.
- **Result:** Fix $p \in (0, 1)$. Suppose C_1 and C_2 are rich and $(z_1, z_2) \to (0, 0)$, players' equilibrium payoffs are close to:

$$\left(\frac{r_2}{r_2 + r_1^L}, \frac{r_1^L}{r_2 + r_1^L} \right).$$

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Intuition: Why only the patient type matters?

After P1 demands α_1^* and P2 demands α_2^*.

- Suppose $\alpha_1^* + \alpha_2^* > 1$ and both r_1^H and r_1^L occur with positive prob.
- The impatient type r_1^H concedes first and the patient type r_1^L starts to concede only after r_1^H finishes conceding
- Player 1's concession rate:

$$\lambda_1 = \frac{(1-\alpha_1^*)r_2}{\alpha_1^* + \alpha_2^* - 1}.$$

- Player 2's concession rate when the impatient type is conceding:

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- Player 2 wastes most of her time conceding with the patient type, so her time average concession rate $\approx \lambda_2^L$.

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Kambe (1999)

- Time $t \in [0, +\infty)$. Two players with discount rates r_1 and r_2.
- At time 0, players announce their demands $\alpha_1^*, \alpha_2^* \in [0, 1]$.
- If $\alpha_1^* + \alpha_2^* \leq 1$, then the game ends at 0 where player i receives $\alpha_i^* + \frac{1}{2}(1 - \alpha_1^* - \alpha_2^*)$.
- If $\alpha_1^* + \alpha_2^* > 1$, then play enters a *war-of-attrition phase*.

  Player i becomes committed at time 0 with prob $\varepsilon_i > 0$.

  At every $t \in [0, +\infty)$, each flexible type decides whether to concede.
- Player i chooses α_i^* in order to maximize their expected payoff.

  Important: Player i takes their payoff when they are committed into account when they choose their demand.

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- If $\alpha_1^* + \alpha_2^* \leq 1$, then the game ends at 0 where player i receives $\alpha_i^* + \frac{1}{2}(1 - \alpha_1^* - \alpha_2^*)$.
- If $\alpha_1^* + \alpha_2^* > 1$, then play enters a *war-of-attrition phase*.

  Player i becomes committed at time 0 with prob $\varepsilon_i > 0$.

  At every $t \in [0, +\infty)$, each flexible type decides whether to concede.
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  Important: Player i takes their payoff when they are committed into account when they choose their demand.

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Result

Theorem 1 in Kambe (1999)

When $\varepsilon_1, \varepsilon_2 \to 0$ while keeping $\frac{\varepsilon_1}{\varepsilon_2}$ fixed, every equilibrium converges to:

- *Players' initial demands are their Rubinstein payoffs $(\frac{r_2}{r_1+r_2}, \frac{r_1}{r_1+r_2})$.*
- *Players will reach a deal without any delay.*

Intuition: Player i secures payoff close to $\frac{r_{-i}}{r_i+r_{-i}}$ by demanding $\frac{r_{-i}}{r_i+r_{-i}}$.

- Player $-i$ has an incentive to make a compatible offer in order to avoid the loss once they become committed.

Kambe (1999) also considers the case in which whether player i is committed is known to player i before choosing α_i^*.

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Kambe (1999) *vs* Abreu and Gul (2000)

Advantages of Kambe's formulation.

- The commitment types' demands are endogenous.
- Avoid requirements on rich type spaces.
- Convenient in context with incomplete info about values/costs/quality, or when players can make complicated commitments.
- Examples: Wolitzky (2012).

Disadvantages of Kambe's formulation:

- Why players do not know whether they are committed or not when choosing their initial demands? Stories?

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Motivation: Repeated Games with Contracts

In general, it is hard to make sharp predictions in repeated games with two equally patient players.

Abreu and Pearce (2007): Sharp predictions in repeated games when

- players can sign a binding contract,

  after which future play is pinned down by the terms of the contract.

Example:

-	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

Before agreeing on a contract, player 1 chooses $\alpha_{1,t} \in \Delta\{T, B\}$ and player 2 chooses $\alpha_{2,t} \in \Delta\{L, R\}$. A contract specifies what payoffs players receive in future periods, subject to feasibility constraints.

Model

Stage game: Two-player finite game $\mathcal{G} = (I, A, U)$.

In each integer time $t = 0, 1, 2, \ldots$, player i chooses $\alpha_i \in \Delta(A_i)$ and offers a binding contract (v_1, v_2) to player j.

- If player j accepts, then the continuation values are (v_1, v_2).
- We focus on contracts on the Pareto frontier.

Players' mixed actions are perfectly monitored.

At every $t \in [0, +\infty]$, players can accept the other player's contract.

Player i's payoff if an agreement (v_1, v_2) is reached at τ:

$$r \int_0^\tau e^{-rt} u_i(\alpha_{1,t}, \alpha_{2,t}) dt + e^{-r\tau} v_i,$$

where $\alpha_{i,t}$ is player i's action at time $\lfloor t \rfloor$.

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Commitment Types

Player $i \in \{1, 2\}$ is either rational (w.p. $1 - z_i$) or committed (w.p. z_i).

A finite set of commitment types Γ_i for player i.

- Every $\gamma_i \in \Gamma_i$ specifies $\alpha_i \in \Delta(A_i)$ and (v_1, v_2), s.t. commitment type γ_i takes action α_i until their contract (v_1, v_2) is accepted.

Conditional on committed, player i's type distribution is $\pi_i \in \Delta(\Gamma_i)$.

Before the game starts, players simultaneously announce which commitment type they want to imitate.

- Every commitment type truthfully announces their type.
- Every rational type decides which commitment type to announce, or announces that they are rational.

Important: Once the game starts at time 0, each player's belief assigns positive prob to at most one commitment type.

How to solve this model?

Directly solving this model is hard.

- If there exists some particular commitment type for each player, then players' payoffs are pinned down regardless of other types.

Detour: Nash Bargaining (Nash 1950)

Convex bargaining set $\Pi \subset \mathbb{R}^2$, and disagreement point $(d_1, d_2) \in \Pi$.

- Let
$$\Pi(d_1, d_2) \equiv \left\{ (d_1', d_2') \in \Pi \middle| d_1' \geq d_1, d_2' \geq d_2 \right\}.$$
- Nash bargaining payoff:
$$u^N(d_1, d_2) \equiv \arg \max_{(u_1, u_2) \in \Pi(d_1, d_2)} \left\{ (u_1 - d_1)(u_2 - d_2) \right\}.$$

One can show that $u^N(d_1, d_2)$ is uniquely defined and is Pareto efficient.

Detour: Nash Bargaining with Threat (Nash 1953)

Normal-form game $\mathcal{G} \equiv (A, U)$, let Π be the convex hull of feasible payoffs.

1. Players simultaneously choose $\alpha_1 \in \Delta(A_1)$ and $\alpha_2 \in \Delta(A_2)$.
2. Players' payoffs are given by $u^N\Big(u_1(\alpha_1, \alpha_2), u_2(\alpha_1, \alpha_2)\Big)$,

   i.e., Nash bargaining payoff with threat point $\Big(u_1(\alpha_1, \alpha_2), u_2(\alpha_1, \alpha_2)\Big)$.

Theorem: Nash Bargaining with Threat

Suppose $\mathcal{G} \equiv (A, U)$ is finite, the game where players payoffs are $u^N\Big(u_1(\alpha_1, \alpha_2), u_2(\alpha_1, \alpha_2)\Big)$ admits at least one Nash equilibrium.

All Nash equilibria lead to the same payoff $u^*(\mathcal{G}) \in \mathbb{R}^2$.

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Assumptions on the Set of Commitment Types

We assume that NBWT posture is adopted by at least one commitment type.

Assumption: NBWT Posture Exists

For every $i \in \{1,2\}$, there exists $\gamma_i^* \in \Gamma_i$ such that γ_i^* offers contract $u^*(\mathcal{G}) \equiv (u_1^*, u_2^*)$ and plays his equilibrium strategy in the NBWT game α_i^*.

We assume that after a player has a perfect reputation for being any commitment type, their opponent has a strict incentive to concede.

Assumption: NBWT Type Penalizes Rejection

For every $i \in \{1,2\}$ and $\gamma_i \equiv (\alpha_i^*, u_1^*, u_2^*) \in \Gamma_i$,

$$u_j^* > \max_{a_j \in A_j} u_j(a_j, \alpha_i^*).$$

Theorem: Repeated Games with Contracts

Theorem: Abreu and Pearce (2007)

Under the two assumptions on Γ_i. For every $\varepsilon, R > 0$, there exists $\bar{z} > 0$, such that if $\max\{z_1, z_2\} < \bar{z}$ and $\max\{\frac{z_1}{z_2}, \frac{z_2}{z_1}\} \leq R$, then players' payoffs in any PBE of the repeated game with contracts is within ε of $u^*(\mathcal{G})$.

Proof: Suppose P1 announces NBWT bargaining posture $\gamma_1^* \equiv (\alpha_1^*, u_1^*, u_2^*)$ and never accepts any contract that offers less than his NBWT payoff u_1^*.

- If P2 offers a contract that gives P1 $\geq u_1^*$, then P1's payoff $\geq u_1^*$.
- Next: If P2 takes action α_2 and offers contract (v_1^*, v_2^*) s.t. $v_1^* < u_1^*$ and $v_2^* > u_2^*$, we show that P1's concession rate is higher than P2's.

- Similar to Abreu and Gul, if a player's concession rate is higher, then his opponent concedes with prob close to 1 at time 0 when $z_1, z_2 \to 0$.

Concession Rates when $v_1^* < u_1^*$ and $v_2^* > u_2^*$

Recall that

- P1 offers NBWT payoffs (u_1^*, u_2^*) and takes the NBWT action α_1^*.
- (v_1^*, v_2^*) is P2's offer with $v_1^* < u_1^*$ and $v_2^* > u_2^*$, and P2 commits to α_2.

Let λ_i be player i's concession rate.

P2 is indifferent between accepting P1's contract and waiting:

$$\lambda_1(v_2^* - u_2^*) = r\Big(u_2^* - u_2(\alpha_1^*, \alpha_2)\Big).$$

P1 is indifferent between accepting P2's contract and waiting:

$$\lambda_2(u_1^* - v_1^*) = r\Big(v_1^* - u_1(\alpha_1^*, \alpha_2)\Big).$$

P1 has an advantage *iff* $\lambda_1 > \lambda_2$, which is equivalent to

$$\frac{u_2^* - u_2(\alpha_1^*, \alpha_2)}{v_1^* - u_1(\alpha_1^*, \alpha_2)} > \frac{v_2^* - u_2^*}{u_1^* - v_1^*}.$$

Concession Rates when $v_1^* < u_1^*$ and $v_2^* > u_2^*$

We want to show that:

$$\frac{u_2^* - u_2(\alpha_1^*, \alpha_2)}{v_1^* - u_1(\alpha_1^*, \alpha_2)} > \frac{v_2^* - u_2^*}{u_1^* - v_1^*}.$$

Since (α_1^*, α_2^*) is an equilibrium of the NBWT game, P2's payoff in the NBWT game is weakly lower than u_2^* when the threat point is (α_1^*, α_2).

$$(w_1^*, w_2^*) \equiv \arg \max_{(w_1,w_2) \geq (u_1(\alpha_1^*,\alpha_2),u_2(\alpha_1^*,\alpha_2))} \left\{(w_1 - u_1(\alpha_1^*, \alpha_2))(w_2 - u_2(\alpha_1^*, \alpha_2))\right\}.$$

Therefore, $w_2^* \leq u_2^*$ and $w_1^* \geq u_1^*$.

- Either $w_2^* < u_2^*$ and $w_1^* > u_1^*$, in which case

$$l \equiv \frac{u_2^* - w_2^*}{w_1^* - u_1^*} \quad \underbrace{\geq \frac{v_2^* - u_2^*}{u_1^* - v_1^*}}_{\text{since the bargaining set is convex}}$$

- or $w_2^* = u_2^*$ and $w_1^* = u_1^*$

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Case 1: $w_2^* < u_2^*$, $w_1^* > u_1^*$, and $l \equiv \frac{u_2^* - w_2^*}{w_1^* - u_1^*} \geq \frac{v_2^* - u_2^*}{u_1^* - v_1^*}$.

- Since (w_1^*, w_2^*) maximizes $(w_1 - u_1(\alpha_1^*, \alpha_2))(w_2 - u_2(\alpha_1^*, \alpha_2))$, and $(w_1^* - \Delta, w_2^* + l\Delta)$ belongs to the bargaining set for small Δ,

$$l(w_1^* - u_1(\alpha_1^*, \alpha_2)) - (w_2^* - u_2(\alpha_1^*, \alpha_2)) \leq 0 \quad \Rightarrow \quad l \leq \frac{w_2^* - u_2(\alpha_1^*, \alpha_2)}{w_1^* - u_1(\alpha_1^*, \alpha_2)}.$$

- Since $u_2^* > w_2^*$, $u_1^* < w_1^*$, and $v_1^* < u_1^*$, we have

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Proof: Suppose P1 announces NBWT bargaining posture γ_1^* and never accepts anything that offers less than his NBWT payoff u_1^*.

- If P2 offers a contract that gives P1 $\geq u_1^*$, then P1's payoff $\geq u_1^*$.
- If P2 takes action α_2 and offers contract (v_1^*, v_2^*) s.t. $v_1^* < u_1^*$ and $v_2^* > u_2^*$, then P1's concession rate is higher than P2's.

P1 can guarantee payoff $\approx u_1^*$ by imitating their NBWT type.

Similarly, P2 can guarantee payoff $\approx u_2^*$ by imitating their NBWT type.

- Since (u_1^*, u_2^*) is Pareto optimal, players' payoffs must be close to (u_1^*, u_2^*) in every equilibrium.

Non-Stationary Bargaining Postures

Abreu and Pearce (2007) also consider non-stationary bargaining postures.

- Their payoff prediction remains robust.
- The announcement stage (or the transparent commitment type assumption) is very important.
- Wolitzky (2011) shows a folk theorem in repeated games with contracts without the announcement stage.

What will happen when players have different discount rates?