Lecture 7: Reputational Bargaining I

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Lessons from Reputation Models

Reputation models lead to sharp predictions on players' payoffs when the uninformed players are impatient.

- Fudenberg and Levine (89,92), Gossner (11).
- Informed player obtains his optimal commitment payoff.

It is hard to deliver sharp predictions when both players are patient.

- Cripps and Thomas (97), Chan (00).
- Folk theorems in reputation models.

What if both players are forward-looking and can build reputations?

- In general, this problem is not tractable.
- Today: "*dividing a dollar*" bargaining game.

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Recall: Issues with Rubinstein Bargaining Model

1. Division of surplus is sensitive to the *bargaining protocol*.
   - What if P1 makes 2 offers in a row and then P2 makes 1 offer?
   - What if P1 makes 2 offers in a row and then P2 makes 3 offers?

2. Intractable once introducing incomplete info.
   - Sobel and Takahashi (83), Fudenberg, Levine and Tirole (85), Gul, Sonnestein and Wilson (86), Adamati and Perry (87), Chatterjee and Samuelson (87), Ausubel and Deneckere (89,92).
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Myerson (1991), Kambe (1999), Abreu and Gul (2000)

Robust predictions on players' payoffs in bargaining games:

1. does not depend on details of the type distribution
2. does not depend on details of the *bargaining protocol*.

Incomplete info: Uncertainty about other player's bargaining posture.

- Unlike existing works that focus on uncertainty about the payoff relevant fundamentals (e.g., value, cost, discount factor)

Motivation: Rubinstein Bargaining with Incomplete Info

Two players decide how to divide a dollar.

- Time: $t = 0, \Delta, 2\Delta, \ldots$. Player i's discount factor $\delta_i \equiv e^{-r_i \Delta}$.
  Interpret Δ as period length and r_i as player i's interest rate.

In period $2k\Delta$, P1 makes an offer $\alpha_1 \in [0, 1]$.

- If P2 accepts, then the game ends.
  Payoffs: $\alpha_1 \delta_1^{2k}$ for player 1, and $(1 - \alpha_1)\delta_2^{2k}$ for player 2.
- If P2 rejects, then the game moves on to the next period.

In period $(2k+1)\Delta$, P2 makes an offer $\alpha_2 \in [0, 1]$.

- If P1 accepts, then the game ends.
  Payoffs: $(1 - \alpha_2)\delta_1^{2k+1}$ for player 1, and $\alpha_2 \delta_2^{2k+1}$ for player 2.
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Player i is rational with prob $1 - z_i$.

Player i is committed with prob z_i.

- a set of bargaining postures $C_i \equiv \{\alpha_i^1, \alpha_i^2, ..., \alpha_i^{k_i}\} \subset [0, 1]$
- with prob $z_i \pi_i(\alpha_i^j)$, always demands α_i^j, and accepts iff receives $\geq \alpha_i^j$.

$\pi_i(\alpha_i^1) + \pi_i(\alpha_i^2) + ... + \pi_i(\alpha_i^{k_i}) = 1.$

Question: How will players behave and what is the division of surplus?

- As the bargaining friction vanishes, can we say anything that applies to all (or a large class of) bargaining protocols?

Lesson from 80s: Bargaining is hard when informed party can make offers.

- Directly solving this game is hard.

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Abreu and Gul (2000)'s Approach

Three steps:

1. Continuous-time war-of-attrition with one commitment type for each player. Each player either mimics the commitment type or concedes.
2. Extend the results by allowing for multiple commitment types. Which commitment type will the rational type imitate?
3. In reputational bargaining games, when players can make offers frequently ($\Delta \to 0$), revealing rationality $\approx$ conceding to opponent.

   When offers are frequent, players' payoffs in the reputational bargaining game $\approx$ their payoffs in a war-of-attribution game.

Payoffs in the reputational bargaining game $\approx$ Rubinstein payoffs when

- offers are frequent,
- commitment types occur with low probability and players' commitment probabilities are comparable,
- the set of commitment types is rich enough.

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War-of-Attrition with One Commitment Type on Each Side

Two players decide how to divide a dollar.

- Time $t \in [0, +\infty)$. Players' interest rates r_1 and r_2.
- Player i's commitment demand: α_i^*, with $\alpha_1^* + \alpha_2^* > 1$.
- With prob z_i, player i is committed, demands α_i^*, and never concedes.
- With prob $1 - z_i$, player i is rational and chooses $\widetilde{t}_i \in [0, +\infty]$.
  - * $\widetilde{t}_i$ is the time at which player i concedes,
  - * commitment type chooses $\widetilde{t}_i = +\infty$.
- The game ends at $\widetilde{t} \equiv \min\{\widetilde{t}_1, \widetilde{t}_2\}$.
- The rational types' payoffs:
  - * if $\widetilde{t}_1 > \widetilde{t}_2$, then $\alpha_1^* e^{-r_1 \widetilde{t}}$ for P1 and $(1 - \alpha_1^*) e^{-r_2 \widetilde{t}}$ for P2.
  - * if $\widetilde{t}_1 < \widetilde{t}_2$, then $(1 - \alpha_2^*) e^{-r_1 \widetilde{t}}$ for P1 and $\alpha_2^* e^{-r_2 \widetilde{t}}$ for P2.
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- Player i's commitment demand: α_i^*, with $\alpha_1^* + \alpha_2^* > 1$.
- With prob z_i, player i is committed, demands α_i^*, and never concedes.
- With prob $1 - z_i$, player i is rational and chooses $\widetilde{t}_i \in [0, +\infty]$.
  - * $\widetilde{t}_i$ is the time at which player i concedes,
  - * commitment type chooses $\widetilde{t}_i = +\infty$.
- The game ends at $\widetilde{t} \equiv \min\{\widetilde{t}_1, \widetilde{t}_2\}$.
- The rational types' payoffs:
  - * if $\widetilde{t}_1 > \widetilde{t}_2$, then $\alpha_1^* e^{-r_1 \widetilde{t}}$ for P1 and $(1 - \alpha_1^*) e^{-r_2 \widetilde{t}}$ for P2.
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War-of-Attrition with One Commitment Type on Each Side

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Mixed Strategy in Continuous Time War-of-Attrition

Rational-type of player i's mixed action can be represented by:

- a distribution of their concession time $\widetilde{F}_i(\cdot) \in \Delta[0, +\infty]$.

We will work with $F_i(\cdot) \equiv (1 - z_i)\widetilde{F}_i(\cdot)$.

- $F_i(\cdot)$ is the *unconditional distribution* of player i's concession time.

  $F_i(t)$ is the prob that player i concedes before or at time t.

  This is what their opponent cares about.
- $F_i(\cdot) \in [0, 1 - z_i]$.

  If $F_i(t) = 1 - z_i$, then player i has a perfect reputation at time t.

We construct an equilibrium, and then provide intuition for its uniqueness.

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Equilibrium Construction

We construct an equilibrium with the following features:

1. At most one player concedes with positive prob at time 0.
2. The rational type of both players finish conceding in finite time.
3. Both players finish conceding at the same time τ.
4. Both players concede at a constant rate when $t \in (0, \tau]$, i.e., both of them are indifferent from 0 to τ.

Let us pin down the values of:

1. Players' concession rates when $t \in (0, \tau]$
2. The time at which concession stops τ.
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Equilibrium Construction: Compute Concession Rates

Player i's concession rate at t:

$$\lambda_i(t) \equiv \left| \frac{d(1 - F_i(t))/dt}{1 - F_i(t)} \right|.$$

Player j is indifferent between conceding at $t \in (0, \tau)$ and conceding at the next time instant:

$$\lambda_i(t) \underbrace{\Big(\alpha_j^* - (1 - \alpha_i^*)\Big)}_{\text{player } j\text{'s gain if player } i \text{ concedes}} = \underbrace{r_j(1 - \alpha_i^*)}_{\text{player } j\text{'s cost of waiting}}.$$

This yields the expression for the equilibrium concession rate:

$$\lambda_i(t) = \frac{(1 - \alpha_i^*) r_j}{\alpha_i^* + \alpha_j^* - 1}.$$

Since the above expression is independent of t, we write λ_i instead of $\lambda_i(t)$.

For every $t \in [0, \tau]$,

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Compute τ & Who Concedes in At Time 0

Suppose nobody concedes with positive prob at time 0,

- Let T_i be the time it takes for player i to build a perfect reputation:

$$e^{-\lambda_i T_i} = z_i,$$

or equivalently,

$$T_i = -\frac{\log z_i}{\lambda_i}.$$

If $T_1 = T_2$, then nobody concedes with positive prob at 0.

- $\tau = T_1 = T_2$

If $T_i > T_j$, then $\tau = T_j$ and player i concedes with positive prob at time 0 s.t.

$$\Big(1 - \underbrace{F_i(0)}_{\text{concession prob at 0}}\Big) e^{-\lambda_i T_j} = z_i \quad \Rightarrow \quad F_i(0) = 1 - z_i z_j^{-\lambda_i/\lambda_j}$$

Both players finish conceding at the same time if player i concedes with probability $F_i(0)$ at time 0.

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Lessons from this equilibrium

Equilibrium payoffs when player i concedes with positive prob at $t = 0$:

- Player i's payoff is $1 - \alpha_j^*$.
- Player j's payoff is $(1 - \alpha_i^*)(1 - F_i(0)) + \alpha_j^* F_i(0)$.

The strength of player i increases in his rate of reputation building

$$\lambda_i \equiv \frac{r_j(1 - \alpha_i^*)}{\alpha_i^* + \alpha_j^* - 1},$$

and increases in his initial commitment probability z_i.

A player is *stronger* if:

- he is more patient than his opponent,
- his commitment demand is less greedy,
- and he is more likely to be the commitment type.

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The Uniqueness of Equilibrium

We establish some necessary conditions for equilibrium:

1. At most one player concedes with positive prob at time 0.

   Otherwise, one player strictly prefers to wait for another instant.

2. The rational type of every player concedes in finite time.

   If i doesn't concede at t, then i expects j to concede before $t + T$ with positive prob. If j does not concede, j's prob of committed goes up.

3. Both players stop conceding at the same time.

   No incentive to wait when the other player will never concede.

4. Both players concede at a constant rate when $t \in (0, \tau]$.

   Key step: F_1 and F_2 must be continuous and strictly increasing.

   The indifference conditions for every $t \in (0, \tau]$ yield the unique rate.

Smooth & Positive Concession from 0 to τ

Lemma

$F_1(t)$ and $F_2(t)$ are continuous and strictly increasing when $t \in (0, \tau)$.

1. If F_1 jumps at t, then F_2 does not jump at t.

   This is because P2 can benefit from waiting at t.

2. If F_1 is constant on $[t', t'']$, then F_2 is also constant on $[t', t'']$.

   For P2, conceding at (t', t'') strictly dominated by conceding at t'.

3. $\nexists$ interval $[t', t''] \subset [0, \tau]$ s.t. both F_1 and F_2 are constants.

   Let t^* be the largest t'' s.t. F_1 and F_2 are constants on $[t', t'']$.

   Since F_1 and F_2 cannot both jump at t^*, either P1 or P2's payoff is continuous at t^*. Let's say P1's payoff is continuous at t^*.

   P1's payoff from conceding at $t' + \varepsilon >$ conceding at $t^* - \varepsilon$, by continuity at t^*, also $>$ conceding at $t^* + \varepsilon$, contradicting def of t^*.

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   P1's payoff from conceding at $t' + \varepsilon >$ conceding at $t^* - \varepsilon$, by continuity at t^*, also $>$ conceding at $t^* + \varepsilon$, contradicting def of t^*.

Smooth & Positive Concession from 0 to τ

Lemma

$F_1(t)$ and $F_2(t)$ are continuous and strictly increasing when $t \in (0, \tau)$.

1. If F_1 jumps at t, then F_2 does not jump at t.

   This is because P2 can benefit from waiting at t.

2. If F_1 is constant on $[t', t'']$, then F_2 is also constant on $[t', t'']$.

   For P2, conceding at (t', t'') strictly dominated by conceding at t'.

3. $\nexists$ interval $[t', t''] \subset [0, \tau]$ s.t. both F_1 and F_2 are constants.

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3. $\nexists$ interval $[t', t''] \subset [0, \tau]$ s.t. both F_1 and F_2 are constants.
4. 2 and 3 implies that F_1 and F_2 are strictly increasing on $[0, \tau]$.
5. Why are both F_1 and F_2 continuous?

   If F_1 jumps at t, then F_2 is constant on $(t - \varepsilon, t)$, contradicting 4.

Implication of this lemma:

- Both players are indifferent from 0 to τ.
- Their indifference conditions pin down their concession rates.

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Multiple commitment types

Let $C_i \subset [0, 1]$ be a finite set of commitment types.

- z_i: prob of player i is committed.
- $\pi_i(\alpha_i^*)$: Prob of committing to $\alpha_i^* \in C_i$ *conditional on* i is committed.

$t = -1$: players announce *which commitment types to imitate*.

Simplifying assumption: Transparent commitment types.

- can be relaxed when commitment types are stationary.
- important when commitment types are nonstationary (Wolitzky 11).

Players' Payoffs

There exists a unique equilibrium. Why?

- P1's incentive to take a bargaining posture becomes weaker when P2's belief about P1 taking that bargaining posture increases.

Interesting limit: Fix other parameters and take $(z_1, z_2) \to 0$.

- A sequence of commitment probabilities: $\{z_1^n, z_2^n\}_{n=1}^{\infty}$.
- v_i^n: Player i's equilibrium payoff in game (z_1^n, z_2^n).

Theorem: War-of-Attrition with Rich Set of Commitment Types

If $\lim z_1^n = \lim z_2^n = 0$ *and* $\liminf \frac{z_1^n}{z_1^n+z_2^n}, \limsup \frac{z_1^n}{z_1^n+z_2^n} \in (0,1)$, *then:*

$$\liminf_{n\to\infty} v_i^n \geq \max\left\{\alpha_i^* \in C_i \text{ s.t. } \alpha_i^* \leq \frac{r_j}{r_i + r_j}\right\}.$$

Implication: If C_i is sufficiently rich, then player i can approximately secure their Rubinstein bargaining payoff $\frac{r_j}{r_i+r_j}$.

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A Heuristic Explanation

Let $k \equiv \lim z_1^n / z_2^n$.

For every (α_1^*, α_2^*), one can compute T_1 and T_2.

- Player 1 is stronger when $T_1 < T_2$ and vice versa.

Recall that:

$$T_i \approx -\frac{(\alpha_i^* + \alpha_j^* - 1) \log z_i \pi_i(\alpha_i^*)}{r_j(1 - \alpha_i^*)}$$

Ratio between T_1 and T_2:

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- When $z \to 0$, it takes longer to build reputation, so T_1/T_2 depends only on the ratio between concession rates.

Fix (α_1^*, α_2^*), compute P2's concession prob at time 0 when n is large.

Using the formula we derived before, we have:

$$F_2(0) = 1 - z_2 z_1^{-\lambda_2/\lambda_1}$$

Compute the term $z_2 z_1^{-\lambda_2/\lambda_1}$ as $n \to \infty$.

- since $\lim z_1^n / z_2^n = k$, $\lim z_1^n = 0$ and $\lim z_2^n = 0$,

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Players' Guaranteed Payoffs and Equilibrium Payoffs

Recap: By committing to the Rubinstein bargaining payoff $\frac{r_2}{r_1+r_2}$,

- P1 guarantees payoff $\frac{r_2}{r_1+r_2}$ when $\alpha_2^* \leq \frac{r_1}{r_1+r_2}$.
- As $n \to \infty$, P1's payoff is approximately $\frac{r_2}{r_1+r_2}$ when $\alpha_2^* > \frac{r_1}{r_1+r_2}$ since P2's concession prob at time 0 is close to 1.

Similarly, P2 can guarantee payoff $\approx \frac{r_1}{r_1+r_2}$ by demanding $\frac{r_1}{r_1+r_2}$.

Since players' Rubinstein payoffs lie on the Pareto frontier:

- This approximately pins down both players' equilibrium payoffs.

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Similarly, P2 can guarantee payoff $\approx \frac{r_1}{r_1+r_2}$ by demanding $\frac{r_1}{r_1+r_2}$.

Since players' Rubinstein payoffs lie on the Pareto frontier:

- This approximately pins down both players' equilibrium payoffs.

Players' Guaranteed Payoffs and Equilibrium Payoffs

Recap: By committing to the Rubinstein bargaining payoff $\frac{r_2}{r_1+r_2}$,

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From War-of-Attrition to Bargaining

Each player picks a bargaining posture, and decides when to concede.

- Next: What if each player can flexibly choose what to offer in an alternating offer bargaining game?

Important insight: Reveal rationality ≈ concede when offers are frequent.

Lemma

$\forall\, \varepsilon > 0$, $\exists\, \overline{\Delta} > 0$, s.t. when $\Delta < \overline{\Delta}$, at every history h^t s.t.

- *P1 has revealed rationality*
- *P2 hasn't separated from commitment type α_2^*,*

then P1's payoff $\leq 1 - \alpha_2^* + \varepsilon$, and P2's payoff $\geq \alpha_2^* - \varepsilon$.

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A Heuristic Explanation

First, P1 will concede in finite time with prob 1.

Let T be the last time P1 concedes. We show that $T \to 0$ as $\Delta \to 0$.

- Suppose P1 has the option to concede at $T - \Delta$ but he does not.
- His incentive not to concede implies that P2 will accept his offer at $T - \Delta$ with positive prob, denoted by π.
- At time $T - \Delta$, P2 gets $\alpha_2^* e^{-r\Delta}$ by waiting, so she will not accept any offer that gives her less than $\alpha_2^* e^{-r\Delta}$.
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- Hence, P2's reputation is multiplied by $\frac{1}{1-\alpha_2^*}$ within Δ units of time.
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Reveal Rationality $\approx$ Concession as $\Delta \to 0$

P1's incentive not to concede:

- He expects P2 to accept his offer with positive prob in the near future.

If P2 does not accept P1's offer, then her reputation goes up.

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Robustness to Bargaining Protocols

Think about a general reputational bargaining game.

- $t \in [0, +\infty)$.
- Bargaining protocol $g : [0, +\infty) \to \{0, 1, 2, 3\}$,

  $g(t) = 0$: no one can make offer at t.

  $g(t) = 1$: only P1 can make offer at t.

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  $g(t) = 3$: both players offer simultaneously at t.
- Assumptions:
  1. each player makes infinitely many offers from 0 to $+\infty$.
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Convergence Result

Definition: Convergence to Continuous Time

$\{g_n\}_{n=1}^{\infty}$ *converges to continuous time if for every* $\varepsilon > 0$*, there exists* $\bar{n}$ *s.t. for all* $n \geq \bar{n}$*,* $t \geq 0$*, and* $i \in \{1, 2\}$*, there exists* $\hat{t} \in [t, t+\varepsilon]$ *such that* $i = g_n(\hat{t})$.

Only requires each player can make at least one offer in any ε-interval.

- Allows for many ways to approach continuous time.

Payoff Convergence Theorem

Suppose $\{g_n\}_{n=1}^{\infty}$ *converges to continuous time. Let* σ_n *be a sequential equilibrium in* g_n*, and* $(v_{1,n}, v_{2,n})$ *be players' payoffs in* σ_n*, then* $\lim_{n\to\infty} v_{i,n}$ *is player i's payoff in continuous-time war-of-attrition.*

Continuous-time war-of-attrition captures what happens when players can make offers frequently.

- Not sensitive to the ways of approaching continuous time.

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- Time $t \in [0, +\infty)$. Two players with discount rates r_1 and r_2.
- Before time 0, players simultaneously announce their demands $\alpha_1^*, \alpha_2^* \in [0, 1]$.
- If $\alpha_1^* + \alpha_2^* \leq 1$, then the game ends at 0 where player i receives $\alpha_i^* + \frac{1}{2}(1 - \alpha_1^* - \alpha_2^*)$.
- If $\alpha_1^* + \alpha_2^* > 1$, then play enters a *war of attrition phase.*

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Player i becomes committed at time 0 with prob $\varepsilon_i > 0$ (is player i's private info and is independent of whether player $-i$ is committed).

At every $t \in [0, +\infty)$, the flexible type of every player decides whether to concede.

Player i chooses α_i^* to maximize their expected payoff.

Kambe (1999)

- Time $t \in [0, +\infty)$. Two players with discount rates r_1 and r_2.
- Before time 0, players simultaneously announce their demands $\alpha_1^*, \alpha_2^* \in [0, 1]$.
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Result

Theorem 1 in Kambe (1999)

When $\varepsilon_1, \varepsilon_2 \to 0$ while keeping $\frac{\varepsilon_1}{\varepsilon_2}$ fixed, every equilibrium converges to the following limit point.

- *Players' initial demands are their Rubinstein payoffs $\left(\frac{r_2}{r_1+r_2}, \frac{r_1}{r_1+r_2}\right)$.*
- *Players will reach a deal without any delay.*

Intuition: Player i secures payoff close to $\frac{r_{-i}}{r_i+r_{-i}}$ by demanding $\frac{r_{-i}}{r_i+r_{-i}}$.

- Player $-i$ has an incentive to make a compatible offer in order to avoid the loss from being committed.

Kambe (1999) also considers the case in which whether player i is committed is known to player i before choosing α_i^*.

- Kambe characterizes equilibria where both players use pure strategies in the announcement stage.
- Sankjohanser (2019) allows for mixed strategies.

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Kambe (1999) *vs* Abreu and Gul (2000)

Advantages of Kambe's formulation.

- The commitment types' demands are endogenous.
- Avoid requirements on rich type spaces.
- Convenient in context with incomplete info about values/costs/quality, or when players can make complicated commitments.
- Examples: Wolitzky (2012).

Disadvantages of Kambe's formulation:

- Why players do not know whether they are committed or not when choosing their initial demands? Stories?

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Next Lecture

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- Kambe (1999): Alternative formulation of reputational bargaining.
- Compte and Jehiel (2002): The role of outside options.
- Abreu and Pearce (2007): Repeated games with contracts.