Lecture 5: Multiple Long-Run Players

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From Myopic Players to Patient Players

Previous lectures:

- Games with only one long-run player, private values.
- Reputation leads to a sharp prediction on the patient player's payoff.

Two ways to break the reputation result:

- Multiple long-run players (Schmidt 1993, Cripps and Thomas 1997).
- Interdependent values (Pei 2020, 2022).

Today: Games where the uninformed player is forward-looking.

- We assume that values are private and monitoring is perfect.

Example in Cripps and Thomas (1997)

Example: Players' stage-game payoffs:

-	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

Both players' discount factors are δ.

With prob π_0, P1 is committed and plays T at every history.

With prob $1 - \pi_0$, P1 is the rational type.

Theorem: Cripps and Thomas (1997)

For every $\varepsilon > 0$, there exist $\overline{\pi} > 0$ and $\underline{\delta} \in (0, 1)$ s.t. for all $\pi_0 < \overline{\pi}$ and $\delta > \underline{\delta}$, there exists a sequential equilibrium in which P1's payoff $< \varepsilon$.

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Comments

This is *not* a robustness exercise:

- Result breaks down when there is a commitment type that plays a completely mixed strategy.
- Result breaks down under a renegotiation proofness refinement.

Cripps and Thomas (1997)'s proof generalizes to every stage-game that

- has a strictly Pareto dominant pure action profile (a_1^*, a_2^*),
- there exists $a_2 \in A_2$ s.t. a_1^* is not a strict best reply.

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Constructive Proof: Overview of Equilibrium Strategies

Length of the learning phase $N \in \mathbb{N}$ and mixing prob $\{\phi_t\}_{t=0}^{N-1}$.

For every $t \in \{0, ..., N-1\}$, the rational type P1 plays $\phi_t T + (1-\phi_t)B$ if T has been played in all previous periods.

If P1 has played T from 0 to $N-1$, then

- play (T, L) forever starting from period N.

In period 0 to $N-1$, if P2 has not observed B, then she plays R.

If P1 plays B for the first time in period $t \leq N-1$ and $a_{2,t} = R$,

- Continuation play in period $t+1$ delivers payoff δ^{N-1-t}.

If P1 plays B for the first time in period $t \leq N-1$ and $a_{2,t} = L$,

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Constructive Proof: Idea

From P1's perspective:

- He needs to suffer for N periods in order to obtain the reward 1.
- He can end the suffering at any time by revealing rationality.
- The earlier he ends the suffering, the smaller reward he receives.
- In equilibrium, he is indifferent between sustaining his reputation and ending the suffering at any time from 0 to $N-1$.

From P2's perspective:

- She knew that L is optimal in the stage game.
- But why does she play R from period 0 to $N-1$?
- The fear of being punished in the future if she plays L while P1 plays B.

Tradeoff between Learning and Incentive Provision

Question: Do there exist mixing prob $\{\phi_t\}_{t=0}^{N-1}$ and N that work?

- We need N to be large enough s.t. players receive low payoff.
- P1's prob of playing B must be large enough to deter P2 to play L.
- P1's prob of playing B must be small enough to slow down learning.
- We need N to be small enough s.t. P1's reputation in period N does not exceed 1.

Key step of proof: Construct $\{\phi_t\}_{t=0}^{N-1}$ and N s.t.

1. ϕ_t is small enough s.t. P2 has an incentive to play R.
2. ϕ_t is large enough and N is small enough s.t. P2's belief about the commitment type is below 1 after observing T from period 0 to $N-1$.
3. T is large enough so that $1-\delta^T$ is close to 1.

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Incentive Constraints & Learning

For every $t \in \{0, 1, ..., N-1\}$

- π_t: Prob of commitment type after observing T from 0 to $t-1$.
- P2's payoff if he plays R: δ^{N-t}

  P2's payoff if he plays L: $(\pi_t + (1-\pi_t)\phi_t)(1-\delta+\delta^{N-t})$
- P2's incentive constraints implies:

$$\pi_t + (1-\pi_t)\phi_t \leq \frac{\delta^{N-t}}{1-\delta+\delta^{N-t}}.$$

- Bayes Rule suggests that:

$$\pi_{t+1} = \frac{\pi_t}{\pi_t + (1-\pi_t)\phi_t} \quad \Leftrightarrow \quad \underbrace{\pi_t + (1-\pi_t)\phi_t}_{\text{prob of } T \text{ in period } t} = \frac{\pi_t}{\pi_{t+1}}$$

- Suppose ϕ_t is just small enough s.t. IC binds, then $\pi_N < 1$ iff:

$$\Pi_{\tau=0}^{N-1} \frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}} = \frac{\pi_0}{\pi_N} > \pi_0.$$

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Existence of π_0 and N

Remaining task: Can we find $\pi_0 \in (0,1)$ such that for every δ close to 1, there exists N such that:

$$\Pi_{\tau=0}^{N-1} \frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}} > \pi_0 \tag{1}$$

and

$$\delta^N < \varepsilon. \tag{2}$$

This is not trivial since

- The first inequality requires N to be small enough.
- The second inequality requires N to be large enough.

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$$\Pi_{\tau=0}^{N-1} \frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}}.$$

Taking logs and use $\log x \geq 1 - 1/x$ for all $x \in (0,1)$, we have:

$$\sum_{\tau=0}^{N} \log \frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}} > \sum_{\tau=0}^{N} \left\{1 - \frac{1-\delta+\delta^{N-\tau}}{\delta^{N-\tau}}\right\} = -(1-\delta)\sum_{\tau=0}^{N} \delta^{\tau-N}.$$

$$= \delta - \delta^{-N}.$$

Hence,

$$\Pi_{\tau=0}^{N-1} \frac{\delta^{N-\tau}}{1-\delta+\delta^{N-\tau}} > \pi_0$$

is implied by $\delta - \delta^{-N} > \log \pi_0$.

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and

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Choose $\pi_0 \in (0, 1)$ to be small enough such that

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Chan (2000): Folk Theorem in Reputation Games

Let $\underline{v}_i$ be player i's minmax payoff, and let $\overline{v}_i$ be player i's highest feasible payoff conditional on player j's payoff is at least $\underline{v}_j$.

Failure of reputation effects besides two classes of games.

1. Dominant Action Games:

   If there exists $a_1^* \in A_1$ such that

   (a) a_1^* is a strictly dominant action for P1,
   (b) $u_1(a_1^*, a_2) = \overline{v}_1$ for every $a_2 \in \text{BR}_2(a_1^*)$.

2. Strictly Conflicting Interests:

   There exists $a_1^* \in A_1$ such that for every $a_2 \in \text{BR}_2(a_1^*)$,

   $$u_1(a_1^*, a_2) = \overline{v}_1 \text{ and } u_2(a_1^*, a_2) = \underline{v}_2.$$

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Chan (2000): Folk Theorem in Reputation Games

Let $\underline{v}_i$ be player i's minmax payoff, and let $\overline{v}_i$ be player i's highest feasible payoff conditional on player j's payoff is at least $\underline{v}_j$.

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Folk Theorem in Chan (2000)

Folk Theorem in Reputation Games (Chan 2000)

If the stage game belongs to none of these categories, then for every feasible and strictly individually rational payoff of P1, there exist $\overline{\pi} > 0$ *and* $\underline{\delta} \in (0, 1)$ *such that when the probability of commitment type is less than* $\overline{\pi}$ *and the discount factor is greater than* $\underline{\delta}$*, there exists a sequential equilibrium in which the rational-type player* 1 *obtains this payoff.*

Positive Result: Dominant Action Games

Chan (2000) establishes a reputation result for dominant action games.

(a) a_1^* is a strictly dominant action for P1,

(b) $u_1(a_1^*, a_2) = \overline{v}_1$ for every $a_2 \in \mathrm{BR}_2(a_1^*)$.

Rough intuition: Think about the last period in which rational type separates from commitment type.

- P1's continuation value from playing a_1^* is $\overline{v}_1$.
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Cripps, Dekel and Pesendorfer (2005) establish a reputation result for games with strictly conflicting interests:

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A leading example that satisfies their conditions:

-	L	R
T	$2, 0$	$0, -1$
B	$2, 0$	$1, 1$

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Intuition: Why Conflicting Interests not Common Interests?

Cripps and Thomas' equilibrium:

- Reputation forces: P1's reputation grows each time he plays T.
  Reputation grows faster if the ex ante prob of T is lower.
- Repeated game forces: P2 doesn't best reply against T.
  P2 expects to be rewarded in the future if she does not best reply today.
  P2's incentive is provided by the possibility of P1 playing B.
- Key: Difference in P2's continuation payoff after (B, L) and (B, R).
  This enables $N(\delta, \pi)$ to explode with δ.
- Thought experiment: *last period of the learning phase*.
  P1 mixes between T and B: Both give P1 continuation payoff 1.
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Speed of learning is bounded from below in conflicting interest games.

Thought experiment: last period of the learning phase.

- P1 is willing to mix between commitment action and another action.

  P1's continuation payoff after commitment action: $\overline{v}_1$.

  P1's continuation payoff after the other action: close to $\overline{v}_1$.

- P2's continuation value for not best replying: close to $\underline{v}_2$.

  P2's guaranteed continuation payoff by best replying: $\underline{v}_2$.

  i.e., P2's equilibrium continuation value is already at their minmax, they cannot be further punished.

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Stage Game with Perfect Information

Atakan and Ekmekci (2012) study stage games with *perfect info*, and establish reputation results in two classes of games:

1. games with strictly conflicting interests,
2. games with locally non-conflicting interests: for every u_2, u_2', such that $(\bar{v}_1, u_2), (\bar{v}_1, u_2') \in F \cap IR$, then $u_2 = u_2' > \underline{v}_2$.

Key: Unique payoff profile s.t. P1 attains $\bar{v}_1$.

Their solution concept:

- Perfect Bayesian Equilibrium with no signaling what you don't know (Fudenberg and Tirole 91)

Games with locally non-conflicting interests includes common interest games in Cripps and Thomas (1997).

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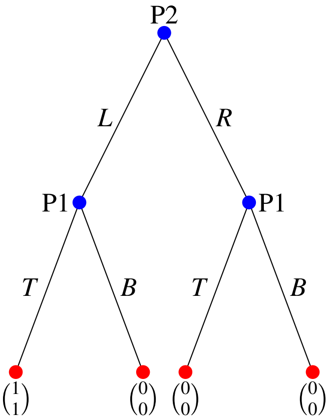
Example: Perfect Info Common Interest Game

Why P1 can secure payoff 1 when stage-game has perfect info, but not when players move simultaneously?



Intuition: Why Sequential Move?

Simultaneous-move stage game: *Last period of learning*.

- P2 plays R with probability 1.
- P2 does not play L since P1 plays B with positive prob, and outcome (B, L) triggers a grim punishment.

Sequential-move stage game: *Last period of learning*.

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- P1's willingness to punish *after observing L*
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What's special about the two classes of games

Atakan and Ekmekci (2012) study stage games with *perfect info*, and establish reputation results in two classes of games:

1. $\exists a_1^* \in A_1$ s.t. $\forall a_2 \in \text{BR}_2(a_1^*)$,

$$u_1(a_1^*, a_2) = \overline{v}_1 \text{ and } u_2(a_1^*, a_2) = \underline{v}_2.$$

   $\forall\, a \equiv (a_1, a_2) \in A_1 \times A_2$, if $u_1(a) = \overline{v}_1$, then $u_2(a) = \underline{v}_2$.

2. $\forall u_2, u_2'$, s.t. $(\overline{v}_1, u_2), (\overline{v}_1, u_2') \in F \cap IR$, then $u_2 = u_2' > \underline{v}_2$.

Key: Unique payoff profile s.t. P1 attains $\overline{v}_1$.

P1's continuation value $\approx \overline{v}_1$ by imitating commitment type.

$\Rightarrow$ approximately pins down P2's continuation value.

If P2 does not best reply, and given P2's continuation values are close, then it leads to a lower bound on the prob that P1 reveals rationality, and hence, a lower bound on the speed of learning.

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Summary

Games with unique payoff profile s.t. P1 attains $\overline{v}_1$. Last period of learning,

- P1's continuation value after playing a_1^* is close to $\overline{v}_1$.
- P1's continuation value after playing a_1' is also close to $\overline{v}_1$.

Both in simultaneous-move and in sequential-move games, the difference in P2's continuation value when she plays an on-path action is close.

Simultaneous-move games: P2's continuation value when P1 reveals rationality and P2 plays an off-path action can be low.

- This can motivate P2 not to best reply for a long time.

Sequential-move games: P1's sequential rationality implies that P1's continuation value following P2's off-path actions is also close to $\overline{v}_1$.

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Imperfect Monitoring of P2's Actions

Atakan and Ekmekci (2015): Simultaneous-move stage games

- P1 observes P2's actions with noise (full support).
- P2 perfectly observes P1's actions.

Reputation result when there exists $a_1^* \in A_1$ s.t. any best response of P2 against a_1^* gives player 1 payoff $\overline{v}_1$.

Idea is similar to Celentani et al. (1995).

- Imperfect monitoring of P2's actions $\Rightarrow$ P2 does not worry too much about the punishment when she plays a stage-game best reply.

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Next Lecture: Bargaining

Repeated bargaining:

- Schmidt (1993 JET).
- Lee and Liu (2013 ECMA).

Coasian bargaining:

- Gul, Sonnenstein, and Wilson (1986 JET).
- Gul (2001 ECMA).