Online Appendix
Reputation for Playing Mixed Actions: A Characterization Theorem

Harry PEI

February 14, 2022

A Proof of Statement 2 Theorem 1

A.1 Construction of T-Period Strategy

Given that $\lambda(\alpha_1^*)$ does not belong to the closure of $\underline{\Lambda}(\theta^*, \alpha_1^*)$, I construct $T \in \mathbb{N}$ that depends only on $\lambda(\alpha_1^*)$ and players' stage-game payoffs, as well as a T-period strategy for the strategic types such that:

- Every strategic type in $\Theta^b_{(\theta^*, \alpha_1^*)}$ plays an action that belongs to $\Delta(A_1^*)$ in each of the T periods.
- Player 2's posterior likelihood ratio vector in period T is bounded away from $\Lambda(\theta^*, \alpha_1^*)$ regardless of player 1's actions.

Let $d(\cdot, \cdot)$ be the Hausdorff distance. Let $A_1^* \equiv \text{supp}(\alpha_1^*)$ and let $\mathcal{H}_*^t$ be the set of period t histories such that player 1 has played actions in A_1^* in every period from 0 to $t-1$. Let $\lambda(h^t)$ be the likelihood ratio vector with respect to α_1^* induced by player 2's belief at history h^t. Let ext$(\cdot)$ denote the exterior of a set.

Proposition A.1. *For every $\varsigma > 0$, every $\alpha_1^* \in \mathcal{A}_1^*$ that is nontrivially mixed, and every prior likelihood ratio vector $\lambda(\alpha_1^*) \in ext\big(\underline{\Lambda}(\theta^*, \alpha_1^*)\big)$, there exist $T \in \mathbb{N}$ as well as strategies for strategic types other than type θ^* denoted by $\{\sigma_{\tilde{\theta}}\}_{\tilde{\theta} \neq \theta^*}$ such that there exists $\varsigma > 0$ such that $d\Big(\lambda(h^T), \Lambda(\theta^*, \alpha_1^*)\Big) > \varsigma$ for every $h^T \in \mathcal{H}_*^T$.*

First, note that the conclusion of this proposition is trivially true when the prior likelihood ratio vector $\lambda(\alpha_1^*)$ does not belong to the closure of $\Lambda(\theta^*, \alpha_1^*)$. Therefore, it is without loss of generality to focus on the case in which the prior likelihood ratio vector $\lambda(\alpha_1^*)$ belongs to the set $\text{cl}\Big(\Lambda(\theta^*, \alpha_1^*)\Big) \Big\backslash \underline{\Lambda}(\theta^*, \alpha_1^*)$.

Abusing notation, let $\lambda_\theta(h^t)$ be the posterior likelihood ratio at history h^t between strategic type θ and commitment type α_1^*, and let $\lambda_\theta(h^t, a_1)$ be this posterior likelihood ratio after observing action a_1 at h^t. Recall that we are constructing strategies where every strategic type in $\Theta^b_{(\theta^*, \alpha_1^*)}$ plays a (potentially mixed) action that belongs to $\Delta(A_1^*)$ in each period, $\{\lambda_\theta(h^t)\}$ is a martingale under the probability measure induced by commitment type α_1^*, so we have $\mathbb{E}_{\alpha_1^*}[\lambda_\theta(h^{t+1})|h^t] = \lambda_\theta(h^t)$ for every $h^t \in \mathcal{H}_*^t$ and $\theta \in \Theta^b_{(\theta^*,\alpha_1^*)}$. Pick an arbitrary $a_1 \in A_1^*$ and let $\beta \equiv \alpha_1^*(a_1)$ for every $a_1 \in A_1$. Since α_1^* is nontrivially mixed, we have $\beta \in (0,1)$. Consider strategies of player 1 that satisfy:

$$\lambda_\theta(h^t, a_1') = \lambda_\theta(h^t, a_1'') \text{ for every } a_1', a_1'' \in A_1^* \backslash \{a_1\},\ \theta \in \Theta^b_{(\theta^*,\alpha_1^*)}, \text{ and } h^t \in \mathcal{H}_*^t.$$

If the strategic types other than θ^* are playing mixed actions that belong to $\Delta(A_1^*)$, then for every $\theta \in \Theta^b_{(\theta^*,\alpha_1^*)}$, the likelihood ratio between type θ and type α_1^* is a martingale. This implies that:

$$\beta\lambda(h^t, a_1) + (1-\beta)\lambda(h^t, a_1') = \lambda(h^t) \text{ for every } a_1' \in A_1^* \backslash \{a_1\}. \tag{A.1}$$

Consider the following strategy for the strategic types that belong to $\Theta^b_{(\theta^*,\alpha_1^*)}$:

$$\sigma_{\tilde{\theta}}(h^t)[\tilde{a}_1] \equiv \begin{cases} \frac{\beta\lambda(h^t,a_1)[\tilde{\theta}]}{\lambda(h^t)[\tilde{\theta}]} & \text{if } \tilde{a}_1 = a_1 \\ \frac{\alpha_1^*(\tilde{a}_1)\lambda(h^t,\tilde{a}_1)[\tilde{\theta}]}{\lambda(h^t)[\tilde{\theta}]} & \text{if } \tilde{a}_1 \in A_1^* \backslash \{a_1\}. \end{cases}$$

Let $M \equiv \sum_{\theta \in \Theta^b_{(\theta^*,\alpha_1^*)}} \frac{\lambda_\theta}{\psi_\theta^*} < 1$. The assumption that $\lambda(\alpha_1^*) \in \text{ext}(\underline{\Lambda}(\theta^*, \alpha_1^*))$ implies that $M > 1$. Let

$$\Lambda(M) \equiv \left\{ \widetilde{\lambda} \middle| \widetilde{\lambda} \gg \mathbf{0} \text{ and } \sum_{\theta \in \Theta^b_{(\theta^*,\alpha_1^*)}} \frac{\widetilde{\lambda}_\theta}{\psi_\theta^*} = M \right\} \tag{A.2}$$

and let $\Psi(M)$ be the set of intersections of $\Lambda(M)$ on the coordinates. We know that $\text{co}(\Psi(M)) = \Lambda(M)$. Let

$$\Lambda_0^\varsigma \equiv \text{int}\Big(\big\{\lambda \in \Lambda(M) \big| d\big(\lambda, \Lambda(\theta^*, \alpha_1^*)\big) > \varsigma\big\}\Big).$$

Given $M > 1$, when ς is small enough, we have $d\big(\Psi(M), \Lambda(\theta^*, \alpha_1^*)\big) < \varsigma/2$, in which case $\Psi(M) \subset \Lambda_0^\varsigma$. So there exists $\overline{\varsigma} > 0$ such that for every $\varsigma \in (0, \overline{\varsigma})$, we have $\text{co}(\Psi(M)) = \text{co}(\Lambda_0^\varsigma) = \Lambda(M)$. Take such a small enough $\varsigma > 0$. I define a sequence of sets $\{\Lambda_k^\varsigma\}_{k=1}^{+\infty}$ recursively according to

$$\Lambda_k^\varsigma \equiv \left\{ \lambda \middle| \text{ there exist } \lambda', \lambda'' \in \Lambda_{k-1}^\varsigma \text{ such that } \beta\lambda' + (1-\beta)\lambda'' = \lambda \right\}. \tag{A.3}$$

By definition, $\Lambda_{k-1}^\varsigma \subset \Lambda_k^\varsigma$ for all $k \in \mathbb{N}$. Let $\Lambda^\varsigma \equiv \bigcup_{k=0}^{+\infty} \Lambda_k^\varsigma$. I show the following lemma:

Lemma A.1. *We have* $\Lambda^\varsigma =$ *int*$(\Lambda(M))$.

PROOF OF LEMMA A.1: By definition, $\Lambda^{\varsigma} \subset \text{int}(\Lambda(M))$. In what follows, I show that $\Lambda^{\varsigma} \supset \text{int}(\Lambda(M))$. Let $k(\theta^*, \alpha_1^*) \equiv |\Theta^b_{(\theta^*, \alpha_1^*)}|$. For every $\lambda^* \in \text{int}(\Lambda(M))$, the Carathéodory Theorem (Eckhoff 1993) implies that there exists $X_0 \equiv \{\lambda^i\}_{i=1}^{k(\theta^*, \alpha_1^*)} \subset \Lambda_0^{\varsigma}$ such that $\lambda^* \in \text{int}(\text{co}(X_0))$. Define X_k with $k \in \mathbb{N}$ recursively according to:

$$X_k \equiv \left\{\lambda \middle| \text{ there exist } \lambda', \lambda'' \in X_{k-1} \text{ such that } \beta\lambda' + (1-\beta)\lambda'' = \lambda\right\}. \tag{A.4}$$

Let $X \equiv \cup_{k=0}^{\infty} X_k$. I show the following Lemma:

Lemma A.2. *We have X is a dense subset of $co(X_0)$.*

PROOF OF LEMMA A.2: Let $\Upsilon \equiv \text{int}\Big(\text{co}(X_0) \backslash X\Big)$. I show that $\Upsilon = \{\varnothing\}$. Suppose by way of contradiction that $\Upsilon \neq \{\varnothing\}$, then for every $\lambda \in \partial \Upsilon$ and every $\eta > 0$, there exists $\lambda^{\eta} \in \Lambda^{\varsigma} \cap B(\lambda, \eta)$. Since there exist $\lambda^* \in \Upsilon$ and $\lambda', \lambda'' \in \partial\Upsilon$ such that $\lambda^* = \beta\lambda' + (1-\beta)\lambda''$. We know that when η is small enough, there also exist $\lambda^{**} \in B(\lambda^*, \eta) \subset \Upsilon$ and $\hat{\lambda}, \tilde{\lambda} \in X$ such that $\lambda^{**} = \beta\hat{\lambda} + (1-\beta)\tilde{\lambda}$, leading to a contradiction. The above argument implies that $\Upsilon = \{\varnothing\}$, and therefore, $\text{co}(X_0) \backslash X$ is hallow, so X must be dense in $\text{co}(X_0)$. □

Back to the proof of Lemma A.1. If $\lambda^* \in X$, then since $X \subset \Lambda^{\varsigma}$, we have $\lambda^* \in \Lambda^{\varsigma}$. If $\lambda^* \notin X$, then for every $\eta > 0$, there exists λ' such that $\lambda' \in B(\lambda^*, \eta) \cap X$. According to the definition of X, there exists $K \in \mathbb{N}$ such that:

$$\lambda' = \sum_{i=1}^{k(\theta^*, \alpha_1^*)} \varrho(i) \lambda^i,$$

where every $\varrho(i)$ can be written as the sum of terms in the form of $(1-\beta)^m \beta^n$ with $0 \leq m+n \leq K$, $m, n \geq 0$. Pick $\eta > 0$ small enough such that $B(\lambda^i, \eta) \subset \Lambda_0^{\varsigma}$ for every $i \in \{1, 2, ..., k(\alpha_1^*, \theta)\}$, we have:

$$\lambda^* = \sum_{i=1}^{k(\theta^*, \alpha_1^*)} \varrho(i) \Big(\lambda^i + (\lambda^* - \lambda')\Big). \tag{A.5}$$

Since $\lambda^i + (\lambda^* - \lambda') \in \Lambda_0^{\varsigma}$ for every i. Equation (A.5) implies that $\lambda^* \in \Lambda_0^{\varsigma}$. □

Since $\Lambda^{\varsigma} = \text{int}(\Lambda(M))$, $\text{int}(\Lambda(M))$ is compact, and $\{\Lambda_k^{\varsigma}\}_{k=1}^{\infty}$ is an open cover, the Heine-Borel Finite Cover Theorem implies the existence of $T \in \mathbb{N}$ such that $\Lambda(M) \subset \bigcup_{k=0}^{T} \Lambda_k^{\varsigma}$.

A.2 Upper Bound on Continuation Payoff after the First Deviation

I construct strategies for strategic types other than θ^* at histories where player 2's posterior belief attaches zero probability to type θ^*, i.e., type θ^* can reach such histories only after he deviates from his equilibrium strategy. I derive a uniform upper bound on type θ^*'s continuation value after he has deviated from his equilibrium strategy for the first time, no matter when and how he deviates. My construction modifies the one in the proof of Statement 1 (in Subsection 3.3 of the main text) while allowing for commitment action α_1^* to be mixed. Throughout this section, I write a_2^* in short for $a_2^*(\theta^*, \alpha_1^*)$. Let $u_1(\theta, a_1, a_2) = 0$ as long as $\theta \neq \theta^*$ or $a_2 \neq a_2^*$. Let $A_1 \equiv \{a_1^1, ..., a_1^n\}$ and let $v^i \equiv u_1(\theta, a_1^i, a_2^*)$ for $i \in \{1, 2, ..., n\}$ with $v^i > 0$ for all i. Without loss of generality, I assume that $v^1 \geq v^2 \geq ... \geq v^n$. Let $\mathbf{v} \equiv (v^1, ..., v^n) \in \mathbb{R}^n$. I derive an upper bound on type θ^*'s payoff at histories where player 2's posterior belief $\widetilde{\mu}$ satisfies the following conditions:

- $\widetilde{\mu}$ attaches zero probability to strategic type θ^*.
- The distribution over strategic types is such that there exists $a_2' \neq a_2^*$ such that:

$$\widetilde{\mu}(\alpha_1^*)\Big(u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2') - u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2^*)\Big) + \sum_{\tilde{\theta} \in \Theta} \widetilde{\mu}(\tilde{\theta})\Big(u_2(\tilde{\theta}, \alpha_1^*, a_2') - u_2(\tilde{\theta}, \alpha_1^*, a_2^*)\Big) > \varsigma. \tag{A.6}$$

- There exists $\eta > 0$ such that for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*} \equiv \mathcal{A}_1^* \backslash \{\alpha_1^*\}$ such that $\widetilde{\mu}(\alpha_1) > 0$, we have:

$$v^* \equiv \alpha_1^* \cdot \mathbf{v} > \eta + \alpha_1 \cdot \mathbf{v} \tag{A.7}$$

  with "$\cdot$" denote the inner product between two vectors in $\mathbb{R}^n$.

Let μ^* be the prior probability of commitment types other than α_1^* and let l be the number of commitment types other than α_1^*.

Proposition A.2. *For every $\widetilde{\mu}$ that satisfies the above requirements, there exist $\sigma_1 \equiv \{\sigma_{\tilde{\theta}}\}_{\tilde{\theta} \neq \theta^*}$ and σ_2 such that:*

1. *σ_2 is optimal for player 2 given σ_1 and μ at every history.*
2. *No matter how type θ^* plays at belief $\widetilde{\mu}$, his continuation value is no more than*

$$v^* - \eta + \varrho(\delta, \mu^*, \varsigma), \tag{A.8}$$

   where $\lim_{\delta \to 1} \varrho(\delta, \mu^*, \varsigma) = 0$ for every $(\mu^*, \varsigma) \in [0, 1) \times (0, +\infty)$.

If $l = 0$, then all the strategic types other than θ^* play α_1^* in every period with probability 1, player 2's belief about player 1's type remains constant. Since $a_2^*(\theta^*, \alpha_1^*)$ is strictly suboptimal for player 2 under such a belief according to (A.6). We know that type θ^*'s continuation value is 0, which concludes the proof for the case where $l = 0$.

If $l \neq 0$, the proof is similar to the constructive proof in the main text. Let $p \in (0,1)$ be chosen such that:

$$\mu(\alpha_1^*)\Big(u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2') - u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2^*)\Big) + p\sum_{\tilde{\theta}\in\Theta}\mu(\tilde{\theta})\Big(u_2(\tilde{\theta},\alpha_1^*,a_2') - u_2(\tilde{\theta},\alpha_1^*,a_2^*)\Big) = \varsigma/2,$$

where a_2' is the same as (A.6). According to (A.6), such p exists. The strategic types play α_1^* in every period with probability p, and adopts non-stationary strategy $\sigma(\alpha_1)$ with probability $(1-p)/l$ for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$. I use $\theta(\alpha_1)$ to denote the strategic type who plays $\sigma(\alpha_1)$.

In what follows, I establish the existence of $\sigma(\alpha_1)$ under which type θ's payoff is bounded from above by (A.8). Let μ_t be the belief in period t with $\mu_0 \equiv \mu$. Let

$$\beta_t(\alpha_1) \equiv \mu_t\big(\theta(\alpha_1)\big)/\mu_t(\alpha_1) \text{ and } \beta(\alpha_1) \equiv \mu\big(\theta(\alpha_1)\big)/\mu(\alpha_1).$$

I will be keeping track of the l-dimensional likelihood ratio vector $\{\beta_t(\alpha_1)\}_{\alpha_1 \in \widetilde{\mathcal{A}_1^*}}$. First, for small enough $\varepsilon > 0$, there exists $\alpha_1^\varepsilon \in \Delta(A_1)$ such that $\alpha_1^\varepsilon(a_1) > \varepsilon$ for all $a_1 \in A_1$ and

$$\sum_{\widetilde{\theta}\in\Theta^b_{(\theta^*,\alpha_1^*)}}\mu(\widetilde{\theta})u_2(\widetilde{\theta},\alpha_1^\varepsilon,a_2') > \sum_{\widetilde{\theta}\in\Theta^b_{(\theta^*,\alpha_1^*)}}\mu(\widetilde{\theta})u_2(\widetilde{\theta},\alpha_1^\varepsilon,a_2^*).$$

For every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$, let

$$\overline{\beta}(\alpha_1) \equiv \inf\Big\{\beta\in\mathbb{R}_+\Big|\mu(\alpha_1)u_2(\phi_{\alpha_1},\alpha_1,a_2') + \beta\sum_{\widetilde{\theta}\in\Theta^b_{(\theta^*,\alpha_1^*)}}\mu(\widetilde{\theta})u_2(\widetilde{\theta},\alpha_1^\varepsilon,a_2')$$

$$> \mu(\alpha_1)u_2(\phi_{\alpha_1},\alpha_1,a_2^*) + \beta\sum_{\widetilde{\theta}\in\Theta^b_{(\theta^*,\alpha_1^*)}}\mu(\widetilde{\theta})u_2(\widetilde{\theta},\alpha_1^\varepsilon,a_2^*)\Big\}. \tag{A.9}$$

By definition, $\overline{\beta}(\alpha_1) \in (0,\infty)$. Next, I describe strategy $\sigma(\alpha_1)$.

1. If $\beta_t(\alpha_1) > \overline{\beta}(\alpha_1)$ for all $\alpha_1 \in \widetilde{\mathcal{A}_1^*} \equiv \mathcal{A}_1^*\backslash\{\alpha_1^*\}$, then type $\theta(\alpha_1)$ plays α_1^ε for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$. Since a_2^* is strictly dominated by a_2' in period t, type θ^*'s stage-game payoff in this period is 0.

2. If $\beta_t(\alpha_1) \leq \overline{\beta}(\alpha_1)$ for some $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$, type $\theta(\alpha_1)$ plays mixed strategy $\breve{\alpha}_1(\alpha_1) \in \Delta(A_1)$ for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$, which will be described below. Abusing notation, I write $\breve{\alpha}_1$ instead of $\breve{\alpha}_1(\alpha_1)$.

Next, I specify $\breve{\alpha}_1(\alpha_1)$. For every constant $\kappa \in (0,1)$, let

$$G^\kappa \equiv \{i|v^i > v^* - \kappa\eta\} \text{ and } B^\kappa \equiv \{j|v^j \leq v^* - \kappa\eta\}.$$

By construction, G^κ and B^κ are non-empty, and $\{G^\kappa, B^\kappa\}$ is a partition of A_1. For every $i \in G^\kappa$ and $j \in B^\kappa$, let $\beta^\kappa(i,j) \in [0,1]$ be defined as:

$$\beta^\kappa(i,j)v^i + (1-\beta^\kappa(i,j))v^j = v^* - \kappa\eta. \tag{A.10}$$

I construct $\breve{\alpha}_1(\alpha_1)$ for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$ in the following lemma:

Lemma A.3. *For every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$, there exists $\breve{\alpha}_1 \in \Delta(A_1)$ such that for every $i \in G^\kappa$ and $j \in B^\kappa$, $\breve{\alpha}_1(a_1^i) > \alpha(a_1^i)$ and*

$$\Big(\frac{\breve{\alpha}_1(a_1^i)}{\alpha_1(a_1^i)}\Big)^{\beta^\kappa(i,j)}\Big(\frac{\breve{\alpha}_1(a_1^j)}{\alpha_1(a_1^j)}\Big)^{1-\beta^\kappa(i,j)} > 1. \tag{A.11}$$

PROOF OF LEMMA A.3: For every $\iota \in \mathbb{R}_+$ and $\alpha_1(i), \alpha_1(j) \in (0,1)$, define the following function of $\varepsilon > 0$:

$$f\big(\varepsilon\big|\iota, \alpha_1(i), \alpha_1(j)\big) \equiv \big(\alpha_1(i)+\varepsilon\big)^\beta\big(\alpha_1(j)-\iota\varepsilon\big)^{1-\beta}.$$

Expand f around $\varepsilon = 0$, we obtain:

$$f\big(\varepsilon\big|\iota, \alpha_1(i), \alpha_1(j)\big) = \alpha_1(i)^\beta\alpha_1(j)^{1-\beta} + \Big(\underbrace{\beta\alpha_1(i)^{\beta-1}\alpha_1(j)^{1-\beta} - \iota(1-\beta)\alpha_1(i)^\beta\alpha_1(j)^{-\beta}}\Big)\varepsilon + \mathcal{O}(\varepsilon^2).$$

The term in the curly bracket is strictly positive if and only if:

$$\iota < \frac{\beta}{1-\beta}\frac{\alpha_1(j)}{\alpha_1(i)}. \tag{A.12}$$

For every $i \in G^\kappa$ and $j \in B^\kappa$, replace β with $\beta^\kappa(i,j)$, and replace $\alpha_1(i), \alpha_1(j)$ with $\alpha_1(a_1^i)$ and $\alpha_1(a_1^j)$, we can define $\iota(i,j)$ analogously. According to (A.10), we have

$$\beta^\kappa(i,j) = \frac{v^* - \kappa\eta - v^j}{v^i - v^j}.$$

Plugging the above expression into (A.12), we have:

$$\iota(i,j) < \frac{\alpha_1(a_1^j)}{\alpha_1(a_1^i)}\frac{v^* - \kappa\eta - v^j}{v^i - (v^* - \kappa\eta)}.$$

For some $\zeta > 0$ small enough, let

$$\breve{\alpha}_1(a_1^i) \equiv \alpha_1(a_1^i) + \zeta\alpha_1(a_1^i)\Big[v^i - (v^* - \kappa\eta)\Big] \text{ for every } i \in G^\kappa, \tag{A.13}$$

and let

$$\breve{\alpha}_1(a_1^j) \equiv \alpha_1(a_1^j) - \zeta\alpha_1(a_1^j)\Big[(v^* - \kappa\eta) - v^j\Big] + \zeta(1-\kappa)\eta \text{ for every } j \in B^\kappa. \tag{A.14}$$

We can verify that first,

$$\frac{\alpha_1(a_1^j) - \breve{\alpha}_1(a_1^j)}{\breve{\alpha}_1(a_1^i) - \alpha_1(a_1^i)} < \iota(i,j),$$

for all $i \in G^\kappa$ and $j \in B^\kappa$, and hence, inequality (A.11) holds when ζ is small enough. Second, $\breve{\alpha}_1(a_1^i) > \alpha_1(a_1^i)$ for all $i \in G^\kappa$. Third,

$$\sum_{i \in G^\kappa} \breve{\alpha}_1(a_1^i) + \sum_{j \in B^\kappa} \breve{\alpha}_1(a_1^j) = \sum_{i \in G^\kappa} \alpha_1(a_1^i) + \sum_{j \in B^\kappa} \alpha_1(a_1^j) = 1,$$

which guarantees that the constructed $\breve{\alpha}_1$ is indeed a probability measure. □

For some intuition behind the constructed $\breve{\alpha}_1$ in Lemma A.3, player 1's action is classified into *good* and *bad* actions. Strategic type θ^* can obtain a stage game payoff no less than $v^* - \kappa\eta$ if and only if he plays an action in G^κ and player 2 best responds by playing $a_2^*(\theta^*, \alpha_1^*)$.

- By definition of $\overline{\beta}(\alpha_1)$, a_2^* is not a best respond when $\beta_t(\alpha_1) > \overline{\beta}(\alpha_1)$ for all $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$.

- When $\beta_t(\alpha_1) \leq \overline{\beta}(\alpha_1)$ for some $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$, the constructed $\breve{\alpha}_1$ implies that $\beta_{t+1}(\alpha_1) > \beta_t(\alpha_1)$ if $a_{1,t} \in G^\kappa$. Moreover, there exists a constant $\chi > 0$ such that $\beta_{t+1}(\alpha_1) \geq \chi\beta_t(\alpha_1)$ for all $a_1 \in A_1$.

  In another word, in every period such that type θ obtains flow payoff no less than $v^* - \kappa\eta$, the likelihood ratio $\beta_t(\alpha_1)$ increases. Since $\breve{\alpha}_1(a_1)$ is bounded from below for every $a_1 \in A_1$, $\beta_t(\alpha_1)$ will not decline too fast even when actions in B^κ are being played. Once $\beta_t(\alpha_1) > \overline{\beta}(\alpha_1)$ for all $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$, a_2^* is strictly dominated by a_2' and type θ obtains a low stage game payoff in that period.

- Equation (A.11) ensures that when δ is close enough to 1, type θ can obtain payoff no more than $v^* - \kappa\eta$ while keeping at least one $\beta_t(\alpha_1)$ below its cutoff, $\overline{\beta}(\alpha_1)$.

  This is because for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$, let $r(a_1^i|\alpha_1) \equiv \frac{\breve{\alpha}_1(a_1^i)}{\alpha_1(a_1^i)}$. Consider the following constraint optimization problem:

$$\max_{\alpha_1 \in \Delta(A_1)} \sum_{i=1}^n \alpha(a_1^i)(v^i - v^* + \kappa\eta),$$

  subject to:

$$\min_{\alpha_1 \in \widetilde{\mathcal{A}_1^*}} \Big\{ \sum_{i=1}^n \alpha(a_1^i) \log r(a_1^i|\alpha_1) \Big\} \leq 0$$

If the value of this program is non-negative, then there exists at least one $\alpha_1 \in \tilde{\Omega}$ such that

$$\sum_{i=1}^{n} \alpha(a_1^i) \log r(a_1^i|\alpha_1) \leq 0$$

at the optimum. Focusing on a revised program with the same objective but just the above inequality constraint. This is a relaxed program of the original one. Since the objective function and the constraint both are linear, there exists an optimum in which there exists at most two a_1^i such that $\alpha(a_1^i) > 0$, i.e.

$$\arg\max_{i \in G^\kappa} \left| \frac{v^i - v^* + \kappa\eta}{\log r(a_1^i|\alpha_1)} \right| \quad \text{and} \quad \arg\min_{i \in B^\kappa} \left| \frac{v^i - v^* + \kappa\eta}{\log r(a_1^i|\alpha_1)} \right|.$$

The value of the above program is strictly negative.

- Let

$$K \equiv \left\lceil \frac{-\log \varepsilon}{\min_{a_1 \in G^\kappa} \log \frac{\breve{\alpha}_1(a_1)}{\alpha_1(a_1)}} \right\rceil + 1,$$

and when δ close enough to 1, choose M large enough such that

$$Kv^1 < (K+1)M. \tag{A.15}$$

The above inequality puts an upper bound on type θ's payoff and ensures that he cannot get more than $v^* - \kappa\eta$ by choosing actions in G^κ too frequently such that $\beta_t(\alpha_1)$ exceeds $\overline{\beta}(\alpha_1)$.

Therefore, under the constructed strategy, type θ's highest continuation payoff after he first deviates is bounded below $v^* - \kappa\eta$ when δ is large enough. Since κ can take any value between 0 and 1, the bound in (A.8) is established in the $\delta \to 1$ limit. Moreover, according to our construction, κ only depends on δ and $\{\beta(\alpha_1)\}_{\alpha_1 \in \widetilde{\mathcal{A}_1^*}}$, and the latter only depend on μ^* and ς.

A.3 Equilibrium Construction

Let $n \equiv |A_1|$. Let $A_1 \equiv \{a_1^1, ..., a_1^n\}$ and let $A_1^* \equiv \text{supp}(\alpha_1^*) = \{a_1^1, ..., a_1^m\}$ with $2 \leq m \leq n$. Recall the definition of $a_2' \neq a_2^*(\theta^*, \alpha_1^*)$, which implies that there exists $1 \leq j \leq m$ such that $a_2' \notin \text{BR}_2(\theta^*, a_1^j)$. I write a_2^* for player 2's best reply to α_1^* at state θ^*. Every type other than θ^* has the same stage-game payoff function, which is constantly zero. Let type θ^*'s stage-game payoff function be given as:

$$u_1(\theta^*, a_1^i, a_2) = \begin{cases} v^i & \text{if } a_2 = a_2^* \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{v} = (v^1, v^2, ..., v^m, 0, 0..., 0) \in \mathbb{R}^n$ has the following properties:

- $v^i > 0$ for all $1 \leq i \leq m$.
- $v^* \equiv \alpha_1^* \cdot \mathbf{v} > \eta + \alpha_1 \cdot \mathbf{v}$ for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$.

The existence of such $\mathbf{v}$ and η follows directly from the separating hyperplane theorem using the fact that $\alpha_1^* \notin \text{co}(\widetilde{\mathcal{A}_1^*})$. Moreover, one can verify that $v^* - \eta/2 > \eta/2 > 0$. According to the above construction, type θ^*'s commitment payoff from α_1^* is strictly greater than his commitment payoff from any other commitment action in $\mathcal{A}_1^*$.

Since $\lambda \in \text{int}\big(\Lambda(\theta^*, \alpha_1^*) \backslash \underline{\Lambda}(\theta^*, \alpha_1^*)\big)$, there exists $M > 1$ such that $\lambda \in \Lambda(M)$. Let $\widetilde{M} \equiv \beta M + (1 - \beta)$ with $\beta \in (0, 1)$ specified later. There exists $\lambda^* \ll \lambda$ such that $\lambda^* \in \Lambda(\widetilde{M})$ and

$$\sum_{\widetilde{\theta} \in \Theta^b_{(\theta^*, \alpha_1^*)}} (\lambda(\widetilde{\theta}) - \lambda^*(\widetilde{\theta}))\Big(u_2(\widetilde{\theta}, \alpha_1^*, a_2') - u_2(\widetilde{\theta}, \alpha_1^*, a_2^*)\Big) > 0 \tag{A.16}$$

for some $a_2' \neq a_2^*$. Let β be close enough to 1 such that for every $\widetilde{\lambda} \in \Lambda(\widetilde{M})$ with $d(\widetilde{\lambda}, \Lambda(\theta^*, \alpha_1^*)) > \varsigma$, we have:

$$d\Big(\widetilde{\lambda} + (\lambda - \lambda^*), \Lambda(\theta^*, \alpha_1^*)\Big) > \varsigma/2.$$

Consider the following strategy profile. I start from describing player 1's strategies for types other than θ.

- **Strategic types that are not θ^* and do not belong to $\Theta^b_{(\theta^*, \alpha_1^*)}$:** Play any commitment strategy in $\mathcal{A}_1^*$ other than α_1^*.
- **Strategic types that belong to $\Theta^b_{(\theta^*, \alpha_1^*)}$:**
  - From period 0 to $T - 1$, type $\tilde{\theta}$ plays σ^* with probability $\lambda^*(\tilde{\theta})/\lambda(\tilde{\theta})$; plays $\hat{\sigma}(\alpha_1)$ with probability $\Big(\lambda(\tilde{\theta}) - \lambda^*(\tilde{\theta})\Big)\Big/ l\lambda(\tilde{\theta})$ for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*} \equiv \mathcal{A}_1^* \backslash \{\alpha_1^*\}$, where $\lambda^* \in \mathbb{R}^k$ is defined in (A.16).
  - In the beginning of period T, compute the likelihood ratio vector of all the bad strategic types and the commitment types, denoted by $\lambda(h^T)$ and plays $\breve{\sigma}(\lambda(h^T))$.[1]

In what follows, I describe strategies σ^*, $\hat{\sigma}(\alpha_1)$ and $\breve{\sigma}(\alpha_1)$.

- $\boldsymbol{\sigma^*}$: Consider prior belief $\widetilde{\lambda}$. Since $\widetilde{\lambda} \in \Lambda(\widetilde{M})$ with $\widetilde{M} > 1$, according to Proposition A.1, there exist $\{\sigma_{\widetilde{\theta}}\}_{\widetilde{\theta} \neq \theta^*}$ and $T \in \mathbb{N}$ such that $d\Big(\lambda(h^T), \Lambda(\theta^*, \alpha_1^*)\Big) > \varsigma$ for every h^T consisting of actions in $\text{supp}(\alpha_1^*)$. Under σ^*, type $\widetilde{\theta}$ plays according to strategy $\sigma_{\widetilde{\theta}}$ from period 0 to $T - 1$.

[1] If player 2 has ruled out commitment type α_1^* by period T, then let $\lambda(h^T) = (\infty, \infty, ..., \infty)$.

- $\widehat{\boldsymbol{\sigma}}(\boldsymbol{\alpha_1})$: Player 1 plays α_1 from period 0 to $T-1$.

- $\breve{\boldsymbol{\sigma}}(\boldsymbol{\lambda}(\boldsymbol{h^T}))$: Suppose an action $a_1 \notin A_1^*$ has occurred in h^T, then every strategic type in $\Theta^b_{(\theta^*,\alpha_1^*)}$ plays $\sigma(\alpha_1)$ starting from period T, where $\sigma(\alpha_1)$ is constructed in Proposition A.2.

  Suppose all actions played from period 0 to $T-1$ belong to A_1^*, according to the construction of β or equivalently $\widetilde{M}$, we have $d(\lambda(h^T), \Lambda(\theta^*, \alpha_1^*)) > \varsigma/2$. There exists $a_2(h^T) \neq a_2^*$ such that:

$$\sum_{\widetilde{\theta} \in \Theta^b_{(\alpha_1^*,\theta)}} \lambda(h^T)(\widetilde{\theta})\Big(u_2(\widetilde{\theta}, \alpha_1^*, a_2(h^T)) - u_2(\widetilde{\theta}, \alpha_1^*, a_2^*)\Big) > 0$$

  Let $p(h^T) \in (0,1)$ be chosen such that

$$\Big(u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2(h^T)) - u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2^*)\Big) + p(h^T) \sum_{\widetilde{\theta} \in \Theta^b_{(\theta^*,\alpha_1^*)}} \lambda(h^T)(\widetilde{\theta})\Big(u_2(\widetilde{\theta}, \alpha_1^*, a_2(h^T)) - u_2(\widetilde{\theta}, \alpha_1^*, a_2^*)\Big) > 0$$

  Type $\widetilde{\theta}$ plays α_1^* in every period with probability $p(h^T)$ and with probability $(1-p(h^T))/l$, plays $\sigma^{a_2(h^T)}(\alpha_1)$ for every $\alpha_1 \in \widetilde{\mathcal{A}_1^*}$, where $\sigma^{a_2(h^T)}(\alpha_1)$ is the strategy $\sigma(\alpha_1)$ constructed in Proposition A.2 applying to $a_2(h^T)$ instead of a_2'.

Play belongs to the *normal phase* in period t if h^t occurs with positive probability under type θ^*'s equilibrium strategy. Play belongs to the *abnormal phase* in period t if h^t occurs with zero probability under type θ^*'s equilibrium strategy. I describe type θ^*'s strategy in the normal phase, i.e. histories at which he has never deviated. Later on, I will bound his continuation value after his first deviation. Type θ^*'s equilibrium strategy is pure starting from period 0 until period $M_\delta \in \mathbb{N}$, where M_δ an integer that I will specify later in the proof.

- In period $t \in \{0, 1, ..., m-1\}$, plays a_1^{t-1}.

- From period m to period M_δ, plays a_1^j with $j \leq m$ and $v^j < v^* - \eta/4$.

According to Proposition A.2, type θ^*'s continuation payoff after he deviates for the first time cannot exceed $v^* - 3\eta/4$ when δ is large enough.

Next I show how to compute M_δ for δ close enough to 1. For every $M \in \mathbb{N}$ and $\delta \in (0,1)$, let $V(M,\delta)$ be the set of continuation values (starting from period $M+1$) for type θ^* conditional on the following event:

- The period M history occurs with positive probability under type θ^*'s equilibrium strategy.

According to the folk theorem in Fudenberg and Maskin (1991), there exists $\underline{\delta} \in (0,1)$ such that for every $\delta > \underline{\delta}$, $V(M,\delta)$ is a convex subset of $[0, v^1]$ for every M. Under this assessment, player 2's posterior belief in period $M+1$ attaches probability 0 to all types in $\Theta^b_{(\theta^*,\alpha_1^*)}$.

For every $0 \leq t \leq M$, let u_t be type θ^*'s expected payoff in period t conditional on staying in the normal phase. By definition, $u_t \leq v^j < v^* - \kappa\eta$. When $M = 1$, there exists $\overline{\delta}$ such that for all $v \in V(1,\delta)$ and $\delta > \overline{\delta}$,

$$v > v^* - \frac{\eta}{4}$$

and given δ is large enough,

$$(1-\delta)v^n + \delta v > v^* - \frac{\eta}{4}.$$

Moreover, for any $\delta \in (0,1)$, there exists $\overline{M}_\delta \in \mathbb{N}$ such that for all $M > \overline{M}_\delta$,

$$(1-\delta)\sum_{t=0}^{M} \delta^t v^j + \delta^{T+1} v^1 < v^* - \frac{3\eta}{4}.$$

Therefore, for every $\delta > \overline{\delta}$, either one of the two circumstances will occur:

1. There exists $M_\delta \in [0, \overline{M}]$ such that $[v^* - 3\eta/4, v^* - \eta/4] \cap V(M_\delta, \delta) \neq \{\varnothing\}$,
2. Or there exists $M_\delta \in [0, \overline{M}]$ such that $v > v^* - \eta/4$ for every $v \in V(M_\delta, \delta)$, and $v' < v^* - 3\eta/4$ for every $v' \in V(M_\delta + 1, \delta)$.

The second situation cannot occur when δ is close enough to 1 since by definition, the difference between $V(M_\delta + 1, \delta)$ and $V(M_\delta, \delta)$ cannot be strictly greater than

$$\frac{1-\delta}{2}\Big|\max_{\theta,a_1,a_2} u_1(\theta,a_1,a_2) - \min_{\theta,a_1,a_2} u_1(\theta,a_1,a_2)\Big|,$$

which is strictly greater than $\frac{\eta}{2}$ when δ is sufficiently close to 1. Hence, there exists an on-path play such that type θ^*'s continuation value is at least $v^* - \frac{3\eta}{4}$ under such a strategy, and given that his continuation value is no more than $v^* - \frac{3\eta}{4}$ in the first period at which he reaches some off-path history. Hence, type θ^* has a strict incentive to play his equilibrium strategy, from which his expected payoff is $v^* - \frac{\eta}{4}$, i.e., it is strictly bounded below v^* his commitment payoff from α_1^*.

B Equilibrium Construction in the Example

In this appendix, I verify a claim made on page 8 of the main text that the likelihood ratio between type θ_2 and type α_1^* is no less than 5 at every on-path history where I has been played in at least one period from 0 to T.

Recall that T is the unique integer that satisfies $\frac{5}{2(1-\varepsilon)^T} \in [5(1-\varepsilon)^2, 5(1-\varepsilon))$. In the example, I also assume that $\varepsilon \in (0, \frac{1}{8}]$ and that player 2's prior belief μ satisfies $\frac{\mu(\theta_2)}{\mu(\alpha_1^*)} = \frac{5}{2}$, and for every $t \in \{0, ..., T-1\}$, if I has never been played before, then type θ_2 plays H with probability

$$P(\varepsilon, t) = \frac{2(1-\varepsilon)^{t+1} - 1}{2(1-\varepsilon)^t - 1}, \tag{B.1}$$

and plays I with probability

$$1 - P(\varepsilon, t) = \frac{2\varepsilon(1-\varepsilon)^t}{2(1-\varepsilon)^t - 1}.$$

First, I compute player 2's posterior belief at on-path histories where player 1 first plays I in period t^* for $t^* \in \{0, 1, ..., T-1\}$. According to Bayes rule, after observing H from period 0 to $t^* - 1$ and observing I in period t^*, the posterior likelihood ratio between type θ_2 and type α_1^* equals

$$\underbrace{\frac{\mu(\theta_2)}{\mu(\alpha_1^*)}}_{=5/2} \cdot \frac{P(\varepsilon, t^*)}{\varepsilon} \cdot \Pi_{t=0}^{t^*-1} \frac{P(\varepsilon, t)}{1-\varepsilon} = \frac{5}{2} \cdot \frac{2(1-\varepsilon)^{t^*}}{2(1-\varepsilon)^{t^*} - 1} \cdot \frac{2(1-\varepsilon)^{t^*} - 1}{(1-\varepsilon)^{t^*}} = 5.$$

Second, I compute player 2's posterior belief when player 1 plays H from period 0 to $T-1$ and plays I in period T. Given that type θ_2 plays I with probability 1 in period T if he plays H from period 0 to $T-1$, player 2's posterior likelihood ratio between type θ_2 and type α_1^* equals

$$\frac{\mu(\theta_2)}{\mu(\alpha_1^*)} \cdot \frac{1}{\varepsilon} \cdot \Pi_{t=0}^{T-1} \frac{P(\varepsilon, t)}{1-\varepsilon} = \frac{5}{2\varepsilon} \cdot \frac{2(1-\varepsilon)^T - 1}{(1-\varepsilon)^T}$$

Since $\frac{5}{2(1-\varepsilon)^T} \in [5(1-\varepsilon)^2, 5(1-\varepsilon))$, we have

$$2(1-\varepsilon) \geq \frac{1}{(1-\varepsilon)^T} \geq 2(1-\varepsilon)^2.$$

Hence,

$$\frac{5}{2\varepsilon} \cdot \frac{2(1-\varepsilon)^T - 1}{(1-\varepsilon)^T} = \frac{5}{2\varepsilon}\Big(2 - \frac{1}{(1-\varepsilon)^T}\Big) \geq \frac{5}{2\varepsilon}(2 - 2(1-\varepsilon)) = 5.$$

The two cases together finish our proof.

C Proof of Lemma 3.1

For every $n \in \mathbb{N}$, let $\widehat{X}_n \equiv \delta^n(X_n - \alpha_1^*(a_1))$. Define a triangular sequence of random variables $\{X_{k,n}\}_{0 \leq n \leq k, k, n \in \mathbb{N}}$, such that $X_{k,n} \equiv \xi_k \widehat{X}_n$, where $\xi_k \equiv \sqrt{\frac{1}{\sigma^2} \frac{1-\delta^2}{1-\delta^{2k}}}$. Let $Z_k \equiv \sum_{n=1}^k X_{k,n} = \xi_k \sum_{k=1}^n \widehat{X}_n$. According to the

Lindeberg-Feller Central Limit Theorem, Z_k converges in law to $N(0,1)$. By construction,

$$\frac{\sum_{n=1}^{k}\widehat{X}_n}{1+\delta+...+\delta^{k-1}}=\sigma\sqrt{\frac{1-\delta^{2k}}{1-\delta^2}}\frac{1-\delta}{1-\delta^k}Z_k.$$

The RHS of the above expression converges in law to a normal distribution with mean 0 and variance $\sigma^2\frac{1-\delta^{2k}}{1-\delta^2}\frac{(1-\delta)^2}{(1-\delta^k)^2}$. The variance term converges to $\mathcal{O}\Big((1-\delta)\Big)$ as $k\to\infty$. According to Theorem 7.4.1 in Chung (1974), we have:

$$\sup_{x\in\mathbb{R}}|F_k(x)-\Phi(x)|\leq C_0\sum_{n=1}^{k}|X_{k,n}|^3\sim C_1(1-\delta)^{\frac{3}{2}},$$

where C_0 and C_1 are constants, F_k is the empirical distribution of Z_k and $\Phi(\cdot)$ is the cdf of the standard normal distribution. Both the variance and the approximation error converge to 0 as $\delta\to 1$.

Therefore, for every $\eta>0$, there exists $\delta^*\in(0,1)$ such that for every $\delta>\delta^*$, there exists $K\in\mathbb{N}$, such that for all $k>K$,

$$\mathcal{P}^{(\alpha_1^*,\sigma_2)}\Big(\Big|\frac{\sum_{i=1}^{k}\widehat{X}_n}{1+\delta+...+\delta^{k-1}}\Big|\geq\eta\Big)<\frac{\eta}{n}.$$

The conclusion of Lemma 3.1 is obtained by taking $k\to\infty$.

References

[1] Chung, Kai-Lai (1974) *A Course in Probability Theory*, Third Edition, Elsevier.

[2] Pei, Harry (2021) "Reputation for Playing Mixed Actions: A Characterization Theorem," *Journal of Economic Theory*, forthcoming.