Lecture 6: Long-Run Medium-Run Models

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Relax Two Assumptions: Myopia & Private Values

Lecture 5: Pei (2020, 2021).

- Relax the private value assumption.

Lectures 6-7: Schmidt (1993), Cripps and Thomas (1997), etc.

- The uninformed player is forward-looking.
- Assume private values and perfect monitoring.

Today: Uninformed player is impatient compared to informed player.

Thursday: Both players are equally patient.

This week: Reputation with two patient players

- Time: $t = 0, 1, 2, \ldots$.
- Informed player 1 (P1), discount factor $\delta_1 \in (0, 1)$,
  vs uninformed player 2 (P2), discount factor $\delta_2 \in (0, 1)$.

- Actions: $a_1 \in A_1$ and $a_2 \in A_2$.
- Stage-game payoffs: $u_1(a_1, a_2)$, $u_2(a_1, a_2)$.

- Public history: $h^t \equiv \{a_{1,s}, a_{2,s}\}_{s=0}^{t-1}$, with $h^t \in \mathcal{H}^t$ and $\mathcal{H} \equiv \cup_{t=0}^{\infty} \mathcal{H}^t$.
  - * Restricting attention to perfect monitoring.
- Player i's strategy: $\sigma_i : \mathcal{H} \to \Delta(A_i)$.

Types and Information

P1's type space $\Omega \equiv \{\omega^r\} \bigcup \Omega^m$.

1. ω^r is the *rational type*.

2. Each $\sigma_1^* \in \Omega^m$ represents a *commitment type*, with $\sigma_1^* : \mathcal{H} \to A_1$.

   commitment types playing pure strategies, potentially *nonstationary*.

P2's prior belief: $\pi \in \Delta(\Omega)$.

P1's history $=$ his type $+$ public history.

P2's history $=$ public history.

Assumptions:

1. A_1, A_2 and Ω^m are finite sets.
2. π has full support.

Two classes of models

1. Today: Long-run medium-run model.

$$\frac{1-\delta_1}{1-\delta_2} \to 0.$$

   Uninformed player is arbitrarily less patient than the informed player.

2. This Thursday: Long-run long-run model.

   $\delta = \delta_1 = \delta_2$ with $\delta \to 1$.

   Uninformed player is as patient as the informed player.

3. Lesson: When the uninformed player becomes more patient, it generates more equilibrium possibilities, making it harder for the informed player to build a reputation.

An Example of Reputation Failure (Schmidt 1993)

-	L	C	R
T	$10, 10$	$0, 0$	$-7, 9$
B	$0, 0$	$1, 1$	$1, 0$

Player 1 has three types:

1. Rational type (80 %).
2. Stackelberg commitment type (10 %): Plays T no matter what.
3. Another type called the punishment type (10 %):

   Plays T until one of the following happens:

   - P2 did not play L in an even period.
   - P2 did not play R in an odd period.

   and then plays B in every subsequent period.

Rational P1 can guarantee payoff 10 when P2 is short-lived.

A low-payoff equilibrium when P2 is forward-looking

-	L	C	R
T	$10, 10$	$0, 0$	$-7, 9$
B	$0, 0$	$1, 1$	$1, 0$

Equilibrium strategies:

- Rational P1: Plays T in every period on the equilibrium path.

  off-path: Plays T unless B has occurred before.

- P2: plays L in even periods and plays R in odd periods on path.

  off-path: Plays L if B has not occurred. Plays C after B has occurred.

Verify incentive constraints:

- Rational P1: on-path payoff ≈ 1.5, off-path payoff at most 1.
- P2: on-path payoff ≈ 9.5. off-path payoff: at most 1 when facing punishment type, and therefore, at most 9.1 in expectation.

Reconcile this with Fudenberg and Levine (1989, 1992)

Decompose FL's argument: $\forall$ equilibrium (σ_1, σ_2) and $\forall \gamma \in (0,1)$,

1. Under (a_1^*, σ_2), the expected number of periods s.t. P2 believes that a_1^* will be played with prob $< \gamma$ is bounded from above by $T(\gamma) \in \mathbb{N}$.

   In fact, $T(\gamma) = 0$ in the above equilibrium.

2. Under (a_1^*, σ_2), the expected number of periods s.t. P2 does not play a_2^* is at most $T(\gamma)$.

In Schmidt's model: $\forall$ equilibrium (σ_1, σ_2) and $\forall \gamma \in (0,1)$,

1. Under (a_1^*, σ_2), the expected number of periods s.t. P2 believes that a_1^* will be played with prob $< \gamma$ is bounded from above by $T(\gamma) \in \mathbb{N}$.
2. Under (a_1^*, σ_2), the expected number of periods s.t. P2 does not play a_2^* can be unbounded.

Why? a_2^* is a myopic best reply $\not\Rightarrow$ P2 has an incentive to play a_2^*.

When will this problem disappear?

Schmidt's idea: If a commitment action a_1^* minmaxes P2, then P2 has nothing to lose and will play his myopic best reply.

Action $a_1^* \in A_1$ minmaxes P2 if

$$\underline{v}_2 \equiv \max_{a_2 \in A_2} u_2(a_1^*, a_2) = \min_{\alpha_1 \in \Delta(A_1)} \max_{a_2 \in A_2} u_2(\alpha_1, a_2).$$

Commitment Payoff Theorem in Schmidt

Suppose $\pi(a_1^*) > 0$ for some a_1^* that minmaxes P2, then for every δ_2, there exists $K(\delta_2) \in \mathbb{N}$ such that rational P1's payoff in any NE is at least:

$$(1 - \delta_1^{K(\delta_2)}) \min_{a_2 \in A_2} u_1(a_1^*, a_2) + \delta_1^{K(\delta_2)} \min_{a_2 \in BR_2(a_1^*)} u_1(a_1^*, a_2).$$

As $\delta_1 \to 1$, the RHS converges to P1's commitment payoff from a_1^*.

Examples

1. Entry deterrence game with commitment action F:

-	Out	In
F	$1, 0$	$-1, -1$
A	$2, 0$	$0, 1$

Action F minmaxes player 2.

- If $F \in \Omega^m$, then P1 can guarantee payoff 1 in all equilibria.

2. Product choice game with commitment action H:

-	B	N
H	$1, 1$	$-1, 0$
L	$2, -1$	$0, 0$

Action L minmaxes player 2.

- Schmidt's theorem only implies that P1 can guarantee payoff 0.

Necessity of Conflicting Interests

Is this "conflicting interest" condition necessary?

- Yes, as long as P1's commitment payoff > his minmax payoff.

Necessity of Conflicting Interest

For every stage game $\mathcal{G}$ and $a_1^* \in A_1$. If a_1^* does not minmax player 2, and

$$\min_{a_2 \in BR_2(a_1^*)} u_1(a_1^*, a_2) > \min_{\alpha_2 \in \Delta(A_2)} \max_{\alpha_1 \in \Delta(A_1)} u_1(\alpha_1, \alpha_2),$$

then for every $\varepsilon > 0$, there exist $\eta > 0$, a type space s.t. $a_1^* \in \Omega^m$ and $\pi(\omega^r) \geq 1 - \varepsilon$, and a sequence of sequential equilibria such that in the limit where $\lim_{\delta_2 \to 1} \lim_{\delta_1 \to 1}$,

P1's equilibrium payoff is below $\min_{a_2 \in BR_2(a_1^*)} u_1(a_1^*, a_2) - \eta$.

Proof of Schmidt's Commitment Payoff Theorem

Commitment Payoff Theorem in Schmidt

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$$(1 - \delta_1^{K(\delta_2)}) \min_{a_2 \in A_2} u_1(a_1^*, a_2) + \delta_1^{K(\delta_2)} \min_{a_2 \in BR_2(a_1^*)} u_1(a_1^*, a_2).$$

Proof of Schmidt's Commitment Payoff Theorem

Let $\widehat{\Omega}$ be the event that P1 plays a_1^* at every history.

Lemma

Fix $\delta_2 < 1$ and $\eta > 0$,

there exist $T > 0$ and $\varepsilon > 0$, s.t.

for every BNE (σ_1, σ_2), a pure strategy $\widehat{\sigma}_2$ in the support of σ_2, and h^t that occurs with positive prob under $\widehat{\Omega}$ and $\widehat{\sigma}_2$.

If

$$\mathbb{E}[U_2(\sigma_1, \widehat{\sigma}_2)|\widehat{\Omega}, h^t] < \underline{v}_2 - \eta,$$

then there exists $\tau \in \{t, ..., t+T-1\}$ s.t.

P2's period t belief assigns prob less than $1-\varepsilon$ to P1 plays a_1^* in period τ.

Intuition Behind the Lemma

Lemma

Fix $\delta_2 < 1$ and $\eta > 0$, there exist $T > 0$ and $\varepsilon > 0$, s.t. for every BNE (σ_1, σ_2), a pure strategy $\widehat{\sigma}_2$ in the support of σ_2, and h^t that occurs with positive prob under $\widehat{\Omega}$, if

$$\mathbb{E}[U_2(\sigma_1, \widehat{\sigma}_2)|\widehat{\Omega}, h^t] < \underline{v}_2 - \eta,$$

then there exists $\tau \in \{t, ..., t + T - 1\}$ s.t. P2's period t belief assigns prob less than $1 - \varepsilon$ to P1 plays a_1^* in period τ.

Intuition:

- P2's continuation value at h^t must satisfy $\mathbb{E}[U_2(\sigma_1, \widehat{\sigma}_2)|h^t] \geq \underline{v}_2$.
- If P2's payoff is bounded below his minmax conditional on $\widehat{\Omega}$, then the prob P2's belief assigns to event $\widehat{\Omega}$ must be bounded away from 1.
- For any $\delta_2 \in (0, 1)$, this must be reflected in the next T periods.

Proof: Construct T and ε from η and δ_2

Pick $T \in \mathbb{N}$ to be large enough such that:

$$(1-\delta_2^T)(\underline{v}_2 - \eta/2) + \delta_2^T \min_{a \in A} u_2(a) > \underline{v}_2 - \eta$$

$$(1-\delta_2^T)(\underline{v}_2 - \eta/2) + \delta_2^T \max_{a \in A} u_2(a) < \underline{v}_2 - \eta/4$$

and then pick $\varepsilon > 0$ s.t. $(1-\varepsilon)^T$ is close to 1:

$$(1-\varepsilon)^T(\underline{v}_2 - \eta/4) + (1-(1-\varepsilon)^T) \max_{a \in A} u_2(a) < \underline{v}_2.$$

Suppose toward a contradiction that (σ_1, σ_2) is a BNE, $\widehat{\sigma}_2$ is a pure-strategy best reply to σ_1, with

$$\mathbb{E}[U_2(\sigma_1, \widehat{\sigma}_2) | \widehat{\Omega}, h^t] < \underline{v}_2 - \eta,$$

P2 believes that a_1^* is played with prob $\geq 1-\varepsilon$ in each of the next T periods.

Proof of Lemma

When P2 plays $\widehat{\sigma}_2$, let $v_2^{t,t+T}$ be her average payoff from period t to $t+T$ conditional on a_1^* being played from t to $t+T$, then:

$$(1-\delta_2^T)v_2^{t,t+T} + \delta_2^T \min_{a \in A} u_2(a) \leq \mathbb{E}[U_2(\sigma_1, \widehat{\sigma}_2)|\widehat{\Omega}, h^t] < \underline{v}_2 - \eta.$$

Given the requirement that

$$(1-\delta_2^T)(\underline{v}_2 - \eta/2) + \delta_2^T \min_{a \in A} u_2(a) > \underline{v}_2 - \eta$$

we have:

$$v_2^{t,t+T} \leq \underline{v}_2 - \eta/2.$$

Given the requirement that

$$(1-\delta_2^T)(\underline{v}_2 - \eta/2) + \delta_2^T \max_{a \in A} u_2(a) < \underline{v}_2 - \eta/4$$

P2's continuation value at h^t conditional on a_1^* being played from t to $t+T$ is at most $\underline{v}_2 - \eta/4$.

Proof of Lemma

From previous slide: P2's continuation value at h^t conditional on a_1^* being played from t to $t+T$ is at most $\underline{v}_2 - \eta/4$.

If P2 believes that a_1^* is played with prob $\geq 1-\varepsilon$ in each of the next T periods, then:

- The prob of the event a_1^* is played from t to $t+T$ is at least $(1-\varepsilon)^T$.

P2's (unconditional) continuation value at h^t by playing $\widehat{\sigma}_2$ is at most:

$$(1-\varepsilon)^T(\underline{v}_2 - \eta/4) + (1-(1-\varepsilon)^T) \max_{a \in A} u_2(a)$$

which is strictly less than his minmax payoff $\underline{v}_2$.

This leads to a contradiction.

Using this lemma to prove Schmidt's theorem

- Suppose when P2 follows $\widehat{\sigma}_2$, he does not play a_2^* at h^t.
- There exists $\eta > 0$ such that: $\mathbb{E}[U_2(\widehat{\sigma}_2)|\widehat{\Omega}, h^t] < \underline{v}_2 - \eta$.

  (why this step requires $\widehat{\sigma}_2$ to be pure?)
- Find $T \in \mathbb{N}$ and $\varepsilon > 0$ according to the previous lemma.
- If P1 plays a_1^* in every period, then significant learning occurs at most K times.

$$K \equiv \left\lceil \frac{\log \pi(a_1^*)}{\log(1-\varepsilon)} \right\rceil.$$

- If P1 plays a_1^* in every period and P2 plays $\widehat{\sigma}_2$, then there exist at most TK periods such that P2 does not play a_2^*.
- As $\delta_1 \to 1$, TK periods have negligible payoff consequences for P1.

Why Each Component is Indispensable?

Where did we use the *conflicting interest assumption*?

- Suppose when P2 follows $\widehat{\sigma}_2$, he does not play a_2^* at h^t, there exists $\eta > 0$ such that: $\mathbb{E}[U_2(\sigma_1, \widehat{\sigma}_2)|\widehat{\Omega}, h^t] < \underline{v}_2 - \eta$.
- Not true when P1's commitment action does not minmax P2. You'll face an order of limit problem if $\widehat{\sigma}_2$ is mixed.

Where did we use the order of limits?

- Fix $\delta_2 \in (0, 1)$,
- T is chosen s.t. $1 - \delta_2^T$ is close to 1,
- ε is chosen such that $(1 - \varepsilon)^T$ is close to 1,
- δ_1 is chosen such that $1 - \delta_1^{TK}$ is close to 0.

Weaker Payoff Lower Bounds

Cripps, Schmidt and Thomas (1996) develops a weaker payoff lower bound when a_1^* does not minmax P2.

- For every $a_1^* \in A_1$, let

$$D(a_1^*) \equiv \{\alpha_2 \in \Delta(A_2) | u_2(a_1^*, \alpha_2) \geq \underline{v}_2\}.$$

- They show that a patient P1's payoff is bounded from below by:

$$\min_{\alpha_2 \in D(a_1^*)} u_1(a_1^*, \alpha_2).$$

- The proof is a straightforward extension of Schmidt (1993).

When will this problem disappear?

Back to Schmidt's low-payoff equilibrium:

- Even if P1 can convince P2 that a_1^* will be played with high prob in the near future when P2 plays their equilibrium strategy, P2 may not want to best reply to a_1^* since P2 is afraid of being punished in the future.
- This hinges on perfect monitoring of P2's actions.
- P2 plays a myopic best response to a_1^* triggers an off-path event.
- P2 can't learn what happens off-path $\Rightarrow$ justifies adverse beliefs off the equilibrium path (P1 not playing commitment action in many periods).

Celentani, Fudenberg, Levine and Pesendorfer (1995)

Commitment payoff theorem when P2's actions are imperfectly monitored.

- Players can't be sure whether their opponents have deviated or not.

Their assumptions on the monitoring structure:

1. Support of $\rho(\cdot|\alpha_1, a_2)$ is independent of a_2 for every $\alpha_1 \in \Delta(A_1)$.
2. P1's actions are statistically identified.
3. P1 observes a_1 and y. P2 observes a_2 and y.

They establish the commitment payoff theorem under a mild assumption on the payoff structure:

- Exists $(a_1, a_2) \in A_1 \times A_2$ such that $u_2(a_1, a_2) > \underline{v}_2$.

My Favorite Intuition

$\forall$ equilibrium (σ_1, σ_2) and $\forall\, \gamma \in (0,1)$,

- Under (a_1^*, σ_2), the expected number of periods s.t.

  P2 believes that a_1^* is played in the next T periods with prob less than $1-\varepsilon$ is uniformly bounded from above.

- What about under (a_1^*, σ_2') for any σ_2'?

  When P2's actions are perfectly monitored, (a_1^*, σ_2') may not be absolutely continuous with respect to (a_1^*, σ_2).

  When P2's actions does not affect the support of signals, (a_1^*, σ_2') is absolutely continuous with respect to (a_1^*, σ_2).

- Imperfect monitoring blurs the distinction between on and off-path.

Caveats

In terms of the theory,

- with two patient players, the informed player can get more than his complete info commitment payoff (think about prisoner's dilemma).
- payoff lower bound is not tight.

Applications: P2's actions are imperfectly monitored,

- Reasonable in competition between firms.
- Unreasonable in buyer-seller applications.

Another Response: Rich Set of Commitment Types

Evans and Thomas (1997):

- Schmidt's converse result require particular type spaces.
- What if there is a rich set of commitment types?

Perfect monitoring and all commitment types play pure strategies.

Let a_1^* be a commitment action, and let a_1' be P1's pure minmax action.

- Assumption: $\max_{a_2 \in A_2} u_2(a_1^*, a_2) > \max_{a_2 \in A_2} u_2(a_1', a_2)$.

Assume that a_2^* is P2's unique best reply to a_1^*.

Constructing a Dynamic Commitment Type

Let σ_1^* be the following automaton strategy:

- Phase 0: Play a_1^* forever.
- Phase k: Play a_1' for k periods, and then play a_1^* forever.

  ...

- Play starts from phase 0. Play goes from phase k to phase $k+1$ if P2 fails to play a_2^* after the kth period in phase k.

Commitment Payoff Theorem: Rich Set of Commitment Types

Suppose P2's prior attaches positive prob to commitment type σ_1^*.

For every $\varepsilon > 0$, there exists $\underline{\delta}_2 < 1$ such that for all $\delta_2 > \underline{\delta}_2$,

there exists $\underline{\delta}_1 < 1$ such that for all $\delta_1 > \underline{\delta}_1$,

rational P1's payoff in any BNE is at least $u_1(a_1^*, a_2^*) - \varepsilon$.

Requires P2 to be patient and the existence of a particular commitment type.

Proof Sketch

Observation:

- For every $K \in \mathbb{N}$ and $\eta > 0$, there exists $T(K, \eta) \in \mathbb{N}$ s.t.

  regardless of P2's strategy, if P1 deviates and plays σ_1^*, then

  there exists at most $T(K, \eta)$ periods s.t. P2 attaches prob less than $1 - \eta$ to the event that P1 will follow σ_1^* in the next K periods.

This follows from Fudenberg and Levine (1989). In fact, $T(K, \eta)$ can equal

$$K \frac{\log \pi(\sigma_1^*)}{\log(1 - \eta)}$$

In what follows, we show that if rational P1 deviates and plays σ_1^*, then P2 triggers punishment for at most a bounded number of periods.

Proof Sketch

Fix δ_2 large enough such that:

$$(1-\delta_2)\max u_2+\delta_2\underline{v}_2 < \underbrace{\pi(\sigma_1^*)u_2(a_1^*,a_2^*) + (1-\pi(\sigma_1^*))[(1-\delta_2)\min u_2 + \delta_2\underline{v}_2]}_{\text{P2's minimal payoff by playing } a_2^*}.$$

This implies the existence of $K \in \mathbb{N}$ and $\eta > 0$ such that:

$$\underbrace{\eta \max u_2 + (1-\eta)[(1-\delta_2)\max u_2 + (\delta_2 - \delta_2^K)\underline{v}_2 + \delta_2^K \max u_2]}_{\text{P2's maximal payoff by triggering punishment in phase } K}$$

$$< \underbrace{\pi(\sigma_1^*)u_2(a_1^*,a_2^*) + (1-\pi(\sigma_1^*))[(1-\delta_2)\min u_2 + \delta_2\underline{v}_2]}_{\text{P2's minimal payoff by playing } a_2^*}.$$

If P2 believes that P1 follows σ_1^* in the next K periods with prob $> 1-\eta$, and the current play in phase $k \geq K$, then P2 has a strict incentive to play a_2^*.

- P2 can trigger at most $T(K,\eta) + K$ punishments if P1 plays σ_1^*.

Discussion

Under mild conditions on payoffs, the issues raised by Schmidt (1993):

- Disappears when P2's actions are imperfectly monitored.
- Disappears when P1 has a rich set of commitment types and P2 is patient.

Thursday:

- Negative results: Cripps and Thomas (1997) and Chan (2000).
- Positive result: Cripps, Dekel and Pesendorfer (2005), Atakan and Ekmekci (2012).