Lecture 5: Reputation under Interdependent Values

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Review

Lectures 1-4: Reputation models with two assumptions.

1. Private values, i.e., uninformed player's payoff (more precisely, best reply) does not depend on the informed player's private info.
2. Uninformed players are myopic.

Relax These Assumptions (assume perfect monitoring)

Lecture 5: Pei (2020, 2021).

- Relax the private value assumption.
- Uninformed players are myopic.

Lectures 6-7: Schmidt (1993), Cripps and Thomas (1997), Chan (2000), Cripps, Dekel and Pesendorfer (2005), etc.

- The uninformed player is forward-looking.
- Private values.

Takeaways:

- Commitment payoff theorem applies only to special classes of games.
- Unclear what the commitment payoff is, results are not clean.
- Lots of open questions, lots of room for future research.

Motivation

In many long-term relationships, economic agents

- build reputations for playing certain actions,
- persistent private info that affects their opponents' payoffs.

Reputation Builder	**Opponents**	**Reputation**	**Private Info**
incumbent firm	entrants	fight entrants	demand curve
seller	buyers	good service	durability/safety

Interaction between building reputations and signalling payoff relevant state

1. affect the value of reputations,
2. affect incentives to build reputations.

Model

Infinitely repeated game: $t = 0, 1, 2, \ldots$

- Long-lived player 1 (P1), discount factor $\delta \in (0, 1)$, *vs* an infinite sequence of short-lived player 2s (P2).
- Players' actions: $a_1 \in A_1$ and $a_2 \in A_2$.

P1 has *perfectly persistent* private info about two aspects:

1. State $\theta \in \Theta$. Stage game payoffs: $u_1(\theta, a_1, a_2)$, $u_2(\theta, a_1, a_2)$.
2. *Either* rational: maximizes $\sum_{t=0}^{+\infty}(1-\delta)\delta^t u_1(\theta, a_{1,t}, a_{2,t})$.

   Or committed: follows some commitment plan $\gamma : \Theta \rightarrow A_1$.

   Plays the same action in every period.

   The set of possible commitment plans: Γ.

Model

P2's prior $\mu \in \Delta\Big(\Theta \times (\Gamma \cup \{\gamma^*\})\Big)$.

P2's history in period t: $\{a_{1,s}, a_{2,s}\}_{s<t}$.

- Today: P2 only observes P1's past actions.

μ has full support. Θ, A_1, and A_2 are finite.

Example: Reputation Failure under Common Interests

$\theta = \theta_1$	h	l
H	$1, 1$	$0, 0$
L	$0, 0$	ϵ, ϵ

$\theta = \theta_2$	h	l
H	$0, 0$	ϵ, ϵ
L	$1, 1$	$0, 0$

One commitment plan: H in state θ_1 and L in state θ_2.

- $\epsilon \in (0, 1)$,
- distribution of two dimensions of private info are independent.

When the prob that P1 is committed is small enough, there exist equilibria in which P1's payoff is ϵ in both states regardless of δ.

- Player 2 plays l at every history.
- Rational P1 plays L in state θ_1, and plays H in state θ_2.

Example: Reputation Failure under Common Interests

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$\theta = \theta_2$	h	l
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L	$1, 1$	$0, 0$

Low payoff equilibria exist despite:

1. Players have common interests.
2. Conditional on {P1 is committed and plays H}, θ_1 occurs with prob 1.
3. Applies to all full support distributions of θ.

What goes wrong:

- P2s believe that rational P1 is more likely to play H when $\theta = \theta_2$.
- P1 faces a tradeoff between playing H and signalling $\theta = \theta_1$.

  P1 faces a tradeoff between playing L and signalling $\theta = \theta_2$.

General Negative Result

If no restrictions are made on u_1 and the prob of commitment is small,

- then for every u_2, we can find u_1 under which there exists equilibrium s.t. P1's payoff is less than his complete info commitment payoff.

Pei (2021): When commitment action is mixed, guaranteeing commitment payoff in all equilibria becomes harder.

Monotone Supermodular Games with $|A_2| = 2$

Assumption: Monotone-Supermodularity (MSM)

There exist a ranking on Θ, a ranking on A_1, and a ranking on A_2,

1. $u_1(\theta, a_1, a_2)$ *is strictly decreasing in* a_1*, and is strictly increasing in* a_2*.*
2. $u_1(\theta, a_1, a_2)$ *has strictly increasing differences in* θ *and* (a_1, a_2)*.*
3. $u_2(\theta, a_1, a_2)$ *has strictly increasing differences in* a_2 *and* (θ, a_1)*.*

Let $\overline{a}_i \equiv \max A_i$ and $\underline{a}_i \equiv \min A_i$. By definition, $A_2 = \{\overline{a}_2, \underline{a}_2\}$. Let

- $$\Theta^* \equiv \left\{\theta \in \Theta \middle| u_1(\theta, \overline{a}_1, \overline{a}_2) > u_1(\theta, \underline{a}_1, \underline{a}_2)\right\}$$
- $\mu(\overline{a}_1)$: prob of the event that P1 is committed and plays $\overline{a}_1$.
- $\phi_{\overline{a}_1} \in \Delta(\Theta)$: state distribution conditional on the above event.
- $\mu(\theta)$: prob of the event that P1 is rational and the state is θ.

Result in Pei (2020)

Let $D(\theta) \equiv u_2(\theta, \overline{a}_1, \overline{a}_2) - u_2(\theta, \overline{a}_1, \underline{a}_2)$.

Theorem

Suppose payoffs are MSM, $|A_2| = 2$, and $\sum_{\theta\in\Theta} \mu(\overline{a}_1)\phi_{\overline{a}_1}(\theta)D(\theta) > 0$.

1. *If $\sum_{\theta\in\Theta} \mu(\overline{a}_1)\phi_{\overline{a}_1}(\theta)D(\theta) + \sum_{\theta\in\Theta^*} \mu(\theta)D(\theta) > 0$, then as $\delta \to 1$, P1's payoff in state θ is at least $\max\{u_1(\theta, \overline{a}_1, \overline{a}_2), u_1(\theta, \underline{a}_1, \underline{a}_2)\}$.*
2. *If $\sum_{\theta\in\Theta} \mu(\overline{a}_1)\phi_{\overline{a}_1}(\theta)D(\theta) + \sum_{\theta\in\Theta^*} \mu(\theta)D(\theta) \leq 0$, then there exists $\theta^* \in \Theta^*$ such that in every equilibrium when δ is large enough,*
   - *P1's payoff in state $\theta \leq \theta^*$ is $u_1(\theta, \underline{a}_1, \underline{a}_2)$.*
   - *P1's payoff in state $\theta > \theta^*$ is $ru_1(\theta, \overline{a}_1, \overline{a}_2) + (1-r)u_1(\theta, \overline{a}_1, \underline{a}_2)$ where $r \in [0, 1]$ is pinned down by:*

$$ru_1(\theta^*, \overline{a}_1, \overline{a}_2) + (1-r)u_1(\theta^*, \overline{a}_1, \underline{a}_2) = u_1(\theta^*, \underline{a}_1, \underline{a}_2).$$

   - *Rational P1 plays $\overline{a}_1$ in every period when $\theta > \theta^*$, plays $\underline{a}_1$ in every period when $\theta < \theta^*$, and mixes between playing $\overline{a}_1$ in every period and playing $\underline{a}_1$ in every period when $\theta = \theta^*$.*

Interpretation

If $\sum_{\theta\in\Theta}\mu(\overline{a}_1)\phi_{\overline{a}_1}(\theta)D(\theta)+\sum_{\theta\in\Theta^*}\mu(\theta)D(\theta)>0$, then

- lower bound on P1's payoff, no robust prediction on P1's behavior.

If $\sum_{\theta\in\Theta}\mu(\overline{a}_1)\phi_{\overline{a}_1}(\theta)D(\theta)+\sum_{\theta\in\Theta^*}\mu(\theta)D(\theta)\leq 0$, then

- unique equilibrium payoff and unique on-path behavior.
- stands in contrast to the private value benchmark.

Important things to understand:

1. Where are these payoffs coming from?
2. Where are the conditions on the state distribution coming from?
3. Why are the restrictions on stage-game payoff not redundant?

Example: Product Choice Game

$\theta = \theta_h$	T	N
H	$1, 1$	$-1, 0$
L	$2, -1$	$0, 0$

$\theta = \theta_l$	T	N
H	$1 - \eta, -1$	$-1 - \phi, 0$
L	$2, -2$	$0, 0$

One commitment plan $\Gamma = \{\gamma\}$, with

$$\gamma(\theta) \equiv \begin{cases} H & \text{if } \theta = \theta_h \\ L & \text{if } \theta = \theta_l. \end{cases}$$

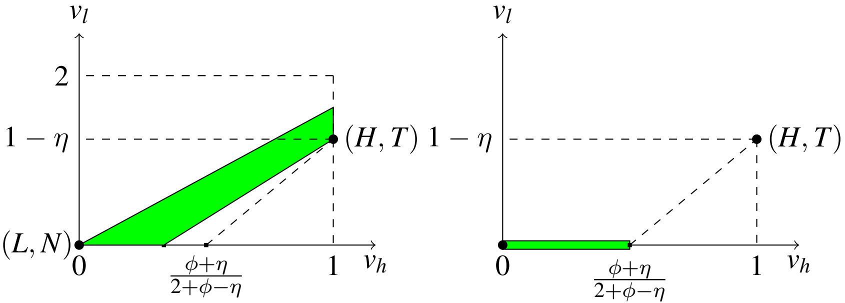
Whether P1 is committed or rational is independent of θ.

Stage-game payoffs satisfy MSM if $\eta \in (0, 1)$ and $\phi > 0$.

1. If the prob of θ_h is greater than $1/2$, then P1's payoff in state θ_h is at least 1 and his payoff in state θ_l is at least $1 - \eta$.
2. If the prob of θ_h is less than $1/2$ and the prob of commitment is small, then when δ is large enough, P1's payoff in state θ_h is $\frac{\phi+\eta}{2+\phi-\eta}$ and his payoff in state θ_l is 0.

Reputation as Equilibrium Selection

Patient P1's equilibrium payoff set (in green) in the benchmark repeated incomplete information game s.t. P1 is rational for sure.



prob of θ_h is greater than $1/2$ prob of θ_h is less than $1/2$

Proof: Partition the set of equilibria

$\theta = \theta_h$	T	N
H	$1, 1$	$-1, 0$
L	$2, -1$	$0, 0$

$\theta = \theta_l$	T	N
H	$1 - \eta, -1$	$-1 - \phi, 0$
L	$2, -2$	$0, 0$

Partition the set of equilibria into two subsets:

- Regular equilibria: Playing H in every period is type θ_l's best reply to P2's equilibrium strategy.
- Irregular equilibria: Playing H in every period is *not* type θ_l's best reply to P2's equilibrium strategy.

Irregular Equilibria

Let h_H^t be the period t history s.t. P1 has played H from 0 to $t-1$.

- $q_t \equiv \Pr\left(\text{player 1 is rational, } \theta = \theta_l \text{, and } h^t = h_H^t\right)$.
- $p_t \equiv \Pr\left(\text{player 1 is rational, } \theta = \theta_h \text{, and } h^t = h_H^t\right)$.

Lemma

For any $t \geq 1$, if $p_t \geq q_t$ but T is not P2's strict best reply at h_H^t, then:

$$(p_t+q_t)-(p_{t+1}+q_{t+1}) \geq \frac{1}{2} \underbrace{\Pr(\textit{P1 is committed and plays H in every period})}_{\equiv Q}$$

Proof: Suppose $t \geq 1$ and T is not a strict best reply at h_H^t, then

$$Q + p_{t+1} - (p_t - p_{t+1}) - q_{t+1} - 2(q_t - q_{t+1}) \leq 0,$$

$$\Rightarrow \quad Q \leq Q + p_{t+1} - q_{t+1} + (p_t - p_{t+1}) \leq 2(q_t - q_{t+1}) + 2(p_t - p_{t+1})$$

Irregular Equilibria

Lemma

In every irregular equilibrium, $p_t \geq q_t$ for every $t \in \mathbb{N}$.

Proof: Suppose $p_t < q_t$ for some $t \in \mathbb{N}$.

- Recall the definition of irregular equilibria: Playing H in every period is *not* type θ_l's best reply to P2's equilibrium strategy.
- There exists $N \in \mathbb{N}$ s.t. type θ_l has a strict incentive to play L at h_H^N.
- By definition, $q_N = 0$ so $\exists$ the largest $t^* \in \mathbb{N}$ such that $p_{t^*} < q_{t^*}$.
- Type θ_l's incentive at $h_H^{t^*}$.

  In equilibrium, he plays L with positive prob, after which his continuation value is 0 (**Why?**).

  If he plays H, then $p_t \geq q_t$ for every $t > t^*$.

  When $p_t \geq q_t$, $p_t + q_t - p_{t+1} - q_{t+1} \geq \frac{Q}{2}$ in every period s.t. P2 doesn't play $T \Rightarrow$ at most $\frac{2}{Q}$ periods s.t. P2 doesn't play T.

Irregular Equilibria

Lemma

For any $t \geq 1$, if $p_t \geq q_t$ but T is not P2's strict best reply at h_H^t, then:

$$p_t + q_t - p_{t+1} - q_{t+1} \geq \frac{Q}{2}.$$

Lemma

In every irregular equilibrium, $p_t \geq q_t$ for every $t \in \mathbb{N}$.

Recall that

- $q_t \equiv \Pr\left(\text{player 1 is rational, } \theta = \theta_l \text{, and } h^t = h_H^t\right)$.
- $p_t \equiv \Pr\left(\text{player 1 is rational, } \theta = \theta_h \text{, and } h^t = h_H^t\right)$.

Irregular equilibria can only exist when $p(\theta_h) \geq 1/2$.

- When δ is close to 1, type θ_h's payoff is at least 1 and type θ_l's payoff is at least $1 - \eta$.

Regular Equilibrium

$\theta = \theta_h$	T	N
H	$1, 1$	$-1, 0$
L	$2, -1$	$0, 0$

$\theta = \theta_l$	T	N
H	$1 - \eta, -1$	$-1 - \phi, 0$
L	$2, -2$	$0, 0$

Each $\{a_{1,t}\}_{t \in \mathbb{N}}$ induces a sequence of P2's actions $\{\alpha_{2,t}\}_{t \in \mathbb{N}}$.

- This is similar to a 1-shot signalling game.
- When $\eta, \phi > 0$, it is less costly to choose H when $\theta = \theta_h$.

Lemma

If playing H in every period is type θ_l's best reply to P2's equilibrium strategy, then type θ_h plays H with probability 1 at every on-path history.

Does it follow from the Spence-Mirrlees theorem?

- No! Payoffs are not necessarily separable in the sender's type.

One-Shot Signalling Game

Why is the Spence-Mirrlees theorem not applicable?

high type	h	l
H	4, 4	2, 2
L	3, 3	1, 1

low type	h	l
H	1, 1	3, 3
L	2, 2	4, 4

What saves the day?

- A monotonicity condition: u_1 increases in a_2 and decreases in a_1.

$\theta = \theta_h$	T	N
H	$1, 1$	$-1, 0$
L	$2, -1$	$0, 0$

$\theta = \theta_l$	T	N
H	$1 - \eta, -1$	$-1 - \phi, 0$
L	$2, -2$	$0, 0$

- Liu and Pei (2020): If payoffs are MSM, and $|A_2| = 2$, then the sender's equilibrium strategy must be non-decreasing in his type.
- The previous lemma is a direct corollary of the above theorem.

Regular Equilibria

Lemma

If playing H in every period is type θ_l's best reply to P2's equilibrium strategy, then type θ_h plays H with probability 1 at every on-path history.

Definition of regular equilibrium:

- Playing H in every period is type θ_l's best reply to P2's equilibrium strategy.

Therefore, type θ_h plays H with prob 1 at every on-path history.

- Type θ_l's continuation value after playing L is 0.

Regular Equilibria

Lemma

If playing H in every period is type θ_l's best reply to P2's equilibrium strategy, then type θ_h plays H with probability 1 at every on-path history.

$\theta = \theta_h$	T	N
H	$1, 1$	$-1, 0$
L	$2, -1$	$0, 0$

$\theta = \theta_l$	T	N
H	$1-\eta, -1$	$-1-\phi, 0$
L	$2, -2$	$0, 0$

Type θ_h plays H with prob 1 at every on-path history. Type θ_l's continuation value after playing L is 0.

- If θ_h occurs with prob more than $1/2$, then $p_t \geq q_t$ for every $t \in \mathbb{N}$,

  type θ_l's payoff is at least $1-\eta$ by playing H in every period, so he has no incentive to play L.
- Type θ_h's equilibrium payoff is 1, type θ_l's equilibrium payoff is $1-\eta$.

Regular Equilibria

Lemma

If playing H in every period is type θ_l's best reply to P2's equilibrium strategy, then type θ_h plays H with probability 1 at every on-path history.

$\theta = \theta_h$	T	N
H	$1, 1$	$-1, 0$
L	$2, -1$	$0, 0$

$\theta = \theta_l$	T	N
H	$1 - \eta, -1$	$-1 - \phi, 0$
L	$2, -2$	$0, 0$

Type θ_h plays H with prob 1 at every on-path history. Type θ_l's continuation value after playing L is 0.

- If θ_h occurs with prob less than $1/2$ and the prob of commitment is small, then P2 has no incentive to play T until type θ_l plays L with positive prob.
- Type θ_l either plays L in period 0 or never plays L (**Why?**).
- Prob that type θ_l plays L in period 0 is such that after observing H in period 0, P2 is indifferent in period 1.

Why Equilibrium Behavior is Unique?

Private values, commitment type, and perfect monitoring:

- Li and Pei (2021): Lots of plausible behaviors.

Interdependent values, pessimistic belief, and commitment type:

- Unique behavioral prediction.
- Presence of commitment type: P1's payoff is high by playing H in every period. (also present under private values)
- Presence of interdependent values: P1's payoff is low after playing L. (missing under private values)

What is hard about interdependent values?

1. What should be the right benchmark for high payoff?
   - What happens in a repeated game without commitment type?
   - I know the answer when payoffs are MSM.
2. How to exploit properties of players' stage-game payoffs to study repeated signaling games?
   - a repeated supermodular game is not supermodular.
3. What happens to MSM games when commitment action is mixed?
   - e.g., committed P1 plays H with prob $1 - \varepsilon$ in every period.
4. Sustainability of reputation under interdependent values.
   - Important assumption of CMS: P2 has a unique best reply to α_1^*.
   - This is generically satisfied under private values.
   - What about interdependent values?