Online Appendix for Trust and Betrayals

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A Proof of Proposition 4.1

Let $v^N \equiv (0, ..., 0)$, $v^H \equiv (1-\theta_1, ..., 1-\theta_m)$, and $v^L \equiv (1, ..., 1)$. The payoff that needs to be attained is:

$$v(\gamma) \equiv \frac{\theta_1(1-\gamma)}{1-\gamma\theta_1} v^N + \frac{(1-\theta_1)\gamma}{1-\gamma\theta_1} v^H + \frac{(1-\theta_1)(1-\gamma)}{1-\gamma\theta_1} v^L. \tag{A.1}$$

Defining Constants: There exists a rational number $\widehat{n}/\widehat{k} \in (\gamma^*, \gamma)$ with $\widehat{n}, \widehat{k} \in \mathbb{N}$. Hence, there exists an integer $j \in \mathbb{N}$ such that

$$\frac{\widehat{n}}{\widehat{k}} = \frac{\widehat{n}j}{\widehat{k}j} < \frac{\widehat{n}j}{\widehat{k}j - 1} < \gamma.$$

Let $n \equiv \widehat{n}j$ and $k \equiv \widehat{k}j$. Let $\delta \in (0, 1)$ be large enough such that:

$$\frac{\delta + \delta^2 + ... + \delta^n}{\delta + \delta^2 + ... + \delta^k} < \widetilde{\gamma} < \frac{\delta^{k-n-1}(\delta + \delta^2 + ... + \delta^n)}{\delta + \delta^2 + ... + \delta^{k-1}}. \tag{A.2}$$

Later on in the proof, I impose two other requirements on δ, given by (A.27). These are compatible with (A.2) since all of these requirements are satisfied when δ is above some cutoff. Let

$$\widetilde{\gamma} \equiv \frac{1}{2}\Big(\frac{n}{k} + \frac{n}{k-1}\Big) \text{ and } \widehat{\gamma} \equiv \frac{1}{2}\Big(\frac{n}{k} + \gamma^*\Big). \tag{A.3}$$

By construction, $\gamma^* < \widehat{\gamma} < \frac{n}{k} < \widetilde{\gamma} < \frac{n}{k-1} < \gamma$. Let π_j be the prior probability of type θ_j. For every $j \geq 3$, let $k_j \in \mathbb{N}$ be large enough such that:

$$(1-\gamma^*\pi_1)\frac{(\pi_j/k_j)}{\sum_{n=2}^{k} \pi_n + (\pi_j/k_j)} \leq 1 - \gamma^*. \tag{A.4}$$

Let $K \equiv \sum_{j=3}^{m} k_j$. Let $\eta^* \in [\gamma^*\pi_1, \pi_1)$ be large enough such that for every $\eta \in [\eta^*, \pi_1]$, we have:

$$\frac{\pi_1 - \eta}{\pi_1(1-\eta)} \leq \min_{j \in \{3,...,m\}} \Big\{ \frac{\pi_j/k_j}{\pi_2 + ... + \pi_j} \Big\} \tag{A.5}$$

Let $\lambda \in (0, \frac{1-\sqrt{\gamma^*}}{\gamma^*})$ be small enough such that:

$$\big(1-\lambda\gamma^*\big)^{1-\widehat{\gamma}}\big(1+\lambda(1-\gamma^*)\big)^{\widehat{\gamma}} > 1. \tag{A.6}$$

State Variables: The constructed equilibrium keeps track of three sets of state variables:

1. $\eta(h^t)$, which is the probability P2's belief attaches to type θ_1 at h^t.

2. $v(h^t) \equiv \{v_j(h^t)\}_{j=1}^m$, which is P1's continuation value at h^t. I verify in section A.4 that for every on-path h^t, $v(h^t)$ is a convex combination of v^N, v^H and v^L, namely $v(h^t) \equiv p^L(h^t)v^L + p^H(h^t)v^H + p^N(h^t)v^N$, and P1's continuation value is summarized by $p^L(h^t)$, $p^H(h^t)$ and $p^N(h^t)$.

3. $\overline{\theta}(h^t)$, which is the highest cost type in the support of P2's belief at h^t, and its probability.

The third state variable is implied by the first one when there are only two types in the support of buyers' prior belief. I describe players' actions and the evolution of P1's continuation value at *on-path histories*. At *off-path histories*, P2 plays N and every type of P1 plays L. I partition the set of on-path histories into three classes:

- **Class 1 Histories:** h^t is such that $p^L(h^t) \geq 1-\delta$.

- **Class 2 Histories:** h^t is such that $p^L(h^t) \in (0, 1-\delta)$.

- **Class 3 Histories:** h^t is such that $p^L(h^t) = 0$.

Play starts from a Class 1 history h^0 and eventually reaches some Class 3 histories. Class 3 histories are absorbing in the sense that if $p^L(h^t) = 0$, then $p^L(h^s) = 0$ for all $h^s \succeq h^t$. Active learning about P1's type happens at Class 1 and Class 2 histories, but stops after reaching Class 3 histories.

A.1 Class 1 Histories

Players' Actions: At every h^t that satisfies $p^L(h^t) \geq 1-\delta$:

- Player 2 plays T for sure.

- Type θ_1 plays H with probability:

$$\frac{\eta(h^t)-\eta(h^t,L)}{\eta(h^t,H)-\eta(h^t,L)} \cdot \frac{\eta(h^t,H)}{\eta(h^t)}, \tag{A.7}$$

and other types in the support of P2's belief play H with the same probability, equal to:

$$\frac{\eta(h^t)-\eta(h^t,L)}{\eta(h^t,H)-\eta(h^t,L)} \cdot \frac{1-\eta(h^t,H)}{1-\eta(h^t)}, \tag{A.8}$$

where the posterior beliefs $\eta(h^t, H)$ and $\eta(h^t, L)$ are functions of $\eta(h^t)$, given by:

$$\eta(h^t, H) = \eta^* + \min\Big\{1 - \eta^*, \big(1 + \lambda(1-\gamma^*)\big)\big(\eta(h^t) - \eta^*\big)\Big\}, \tag{A.9}$$

$$\eta(h^t, L) = \eta^* + (1 - \lambda\gamma^*)(\eta(h^t) - \eta^*), \tag{A.10}$$

with η^* a constant that satisfies (A.5), and λ is a constant that satisfies (A.6).

One can use (A.7), (A.8), (A.9), and (A.10) to write the probability that each type of player 1 playing H at h^t as a function of $\eta(h^t)$, i.e., P1's action at Class 1 histories only depends on P2's belief about her being type θ_1.

P1's Continuation Value: For every h^t that satisfies $p^L(h^t) \geq 1 - \delta$:

1. If P1 plays L at h^t, then his continuation value is:

$$v(h^t, L) = \frac{p^N(h^t)}{\delta} v^N + \frac{p^L(h^t) - (1-\delta)}{\delta} v^L + \frac{p^H(h^t)}{\delta} v^H. \tag{A.11}$$

2. If h^t is such that $\eta(h^t, H) < 1$, P1's continuation value after playing H at h^t is:

$$v(h^t, H) = \frac{p^N(h^t)}{\delta} v^N + \frac{p^L(h^t)}{\delta} v^L + \frac{p^H(h^t) - (1-\delta)}{\delta} v^H. \tag{A.12}$$

If h^t is such that $\eta(h^t, H) = 1$, P1's continuation value after playing H at h^t is:

$$v(h^t, H) = \frac{v_1(h^t, H)}{1 - \theta_1} v^H + \Big(1 - \frac{v_1(h^t, H)}{1 - \theta_1}\Big) v^N \in \mathbb{R}^m, \tag{A.13}$$

$$\text{with } v_1(h^t, H) \equiv \frac{v_1(h^t) - (1-\delta)(1-\theta_1)}{\delta} \text{ and } v_1(h^t) \in \mathbb{R} \text{ is the first entry of } v(h^t). \tag{A.14}$$

Players' Incentives: I verify players' incentive constraints at Class 1 histories:

1. If $p^L(h^t) \geq 1 - \delta$ and $\eta(h^t, H) < 1$, then according to (A.11) and (A.12), all types of P1 are indifferent between playing H and L at h^t.

2. If $p^L(h^t) \geq 1 - \delta$ and $\eta(h^t, H) = 1$, then according to (A.11) and (A.13), type θ_1 is indifferent between H and L at h^t, and other types in the support of P2's belief strictly prefer to play L at h^t.

3. If P2's beliefs are updated according to (A.9) and (A.10), then H is played at h^t with probability at least γ^*, i.e., P2 has an incentive to play T at h^t. This is derived in section A.4.

Belief updating formulas (A.9) and (A.10), together with (A.6) lead to the following lemma:

Lemma A.1. *For every $\underline{\eta} \in (\eta^*, 1)$, there exist $T \in \mathbb{N}$ and $\underline{\delta} \in (0,1)$, s.t. when $\eta(h^r) \geq \underline{\eta}$ and $\delta > \underline{\delta}$, if $h^t \equiv (y_0, ..., y_{t-1}) \succ h^r$ and all histories between h^r and h^t belong to Class 1, then:*

$$\underbrace{(1-\delta)\sum_{s=r}^{t-1}\delta^{s-r}\mathbf{1}\{y_s = H\}}_{\text{weight of } (T,H) \text{ played from } r \text{ to } t} \leq \underbrace{(1-\delta^T)}_{\text{weight of initial } T \text{ periods}} + \underbrace{(1-\delta)\sum_{s=r}^{t-1}\delta^{s-r}\mathbf{1}\{y_s = L\}}_{\text{weight of } (T,L) \text{ played from } r \text{ to } t} \cdot \frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}. \tag{A.15}$$

The proof is in Online Appendix A.5. For some intuition, given the belief updating formulas (A.9) and (A.10), player 2's posterior belief at h^t depends only on her belief at h^r and the number of times H and L have been played from period r to t. Since the choice of λ satisfies the first inequality in (A.6), if player 1 plays H with (undiscounted) frequency above $\widehat{\gamma}$, then player 2's belief at h^t attaches higher probability to type θ_1 compared to her belief at h^r. When P2's belief at h^r attaches probability more than $\underline{\eta}$ to type θ_1, her posterior attaches probability 1 to type θ_1 before period $r+S$, where:

$$S \equiv \left\lceil \frac{\log \frac{1-\eta^*}{\underline{\eta}-\eta^*}}{\log\left\{(1-\lambda\gamma^*)^{1-\widehat{\gamma}}(1+\lambda(1-\gamma^*))^{\widehat{\gamma}}\right\}} \right\rceil, \tag{A.16}$$

after which P2's belief about type θ_1 reaches 1, and the convex weight of v^L equals 0 according to (A.13).

The requirement that all histories from h^r to h^t belonging to Class 1 not only leads to an upper bound on the *undiscounted* frequency with which (T,H) being played from r to t, but also imposes constraints on how frontloaded outcome (T,H) can be. For example, after P1 plays H in the first

$$T \equiv \left\lceil \frac{\log \frac{1}{\pi_1}}{\log\left(1+\lambda(1-\gamma^*)\right)} \right\rceil \tag{A.17}$$

periods, P2's belief about type θ_1 reaches 1, and the convex weight of v^L equals 0 according to (A.13). If δ is large enough, then the constraint on undiscounted frequency and the constraint on frontloadedness of outcome (T,H) lead to an upper bound on the *discounted frequency* with which outcome (T,H) occurs from r to t, with $\widehat{\gamma}$ being replaced by a larger $\widetilde{\gamma}$ to provide extra slack caused by the discount factor δ.

I apply Lemma A.1 by setting $h^r = h^0$ and $\underline{\eta} = \eta(h^0)$. If h^t and all its predecessors belong to Class 1, then:

$$(1-\delta)\sum_{s=0}^{t-1}\delta^s\mathbf{1}\{y_s = (T,L)\} \leq p^L(h^0) = \frac{(1-\theta_1)(1-\gamma)}{1-\gamma\theta_1}.$$

Lemma A.1 leads to an upper bound on $(1-\delta)\sum_{s=0}^{t-1}\delta^s\mathbf{1}\{y_s=(T,H)\}$, which implies that if δ is large enough, then

$$p^H(h^t) \geq Y \equiv \frac{1}{2}\underbrace{\left(\gamma-(1-\gamma)\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}\right)}_{>0}\frac{1-\theta_1}{1-\gamma\theta_1}, \tag{A.18}$$

for every h^t such that h^t and all its predecessors belonging to Class 1.

A.2 Class 2 Histories

Players' Actions: If h^t is such that $p^L(h^t) \in (0, 1-\delta)$, then at h^t,

1. Player 2 plays T for sure.
2. Types in the support of P2's belief at h^t *except for* type $\overline{\theta}(h^t)$ play H for sure. Type $\overline{\theta}(h^t)$ potentially mixes between H and L, with probabilities specified below.

Let

$$l(h^t) \equiv \#\left\{h^s \middle| h^s \prec h^t, h^s \text{ belongs to Class 2, and } \overline{\theta}(h^s)=\overline{\theta}(h^t)\right\} \tag{A.19}$$

be the number of histories that (1) strictly precede h^t, and (2) the highest-cost type in the support of P2's belief is also $\overline{\theta}(h^t)$. Consider two cases separately, depending on whether $\overline{\theta}(h^t)$ is θ_2 or not.

1. If $\overline{\theta}(h^t)=\theta_j$ with $j \geq 3$, then type $\overline{\theta}(h^t)$ plays L at h^t with probability

$$\frac{1}{k_j - l(h^t)}, \tag{A.20}$$

   in which k_j is the integer defined in (A.4).

2. If $\overline{\theta}(h^t)=\theta_2$, then type $\overline{\theta}(h^t)$ plays L at h^t with probability

$$\min\{1, \frac{1-\gamma^*}{1-\eta(h^t)}\}. \tag{A.21}$$

P1's Continuation Value: After player 1 plays L at h^t, P1's continuation value is

$$v(h^t, L) \equiv \frac{Q(h^t)}{\delta}v^H + \frac{\delta - Q(h^t)}{\delta}v^N, \tag{A.22}$$

where

$$Q(h^t) \equiv p^H(h^t) - \frac{1-\delta-p^L(h^t)}{1-\overline{\theta}(h^t)} \tag{A.23}$$

After player 1 plays H at h^t, his continuation value depends on whether $\eta(h^t, H)$ equals 1 or not, with $\eta(h^t, H)$ computed via Bayes Rule given P2's belief at h^t and type $\overline{\theta}(h^t)$'s mixing probability at h^t:

1. If $\eta(h^t, H) < 1$, then P1's continuation payoff at (h^t, H), denoted by $v(h^t, H)$, is given by (A.12).
2. If $\eta(h^t, H) = 1$, then P1's continuation payoff at (h^t, H), denoted by $v(h^t, H)$, is given by (A.13).

By construction of player 1's equilibrium actions, it is clear that $\eta(h^t, H) = 1$ at Class 2 history h^t only when $\overline{\theta}(h^t) = \theta_2$. This is because when $\overline{\theta}(h^t) > \theta_3$, type θ_2 plays H at h^t for sure, and $\eta(h^t, H) < 1$.

Players' Incentives: Lemma A.2 states that players' incentive constraints at Class 2 histories are satisfied.

Lemma A.2. *At every Class 2 history h^t,*

1. *P2 has an incentive to play T.*
2. *If h^t is such that $\eta(h^t, H) < 1$, then type $\overline{\theta}(h^t)$ is indifferent between H and L at h^t, and types that have strictly lower cost than $\overline{\theta}(h^t)$ strictly prefer to play H at h^t.*
3. *If h^t is such that $\eta(h^t, H) = 1$, then type $\overline{\theta}(h^t)$ strictly prefers to play L at h^t, and types that have strictly lower cost than $\overline{\theta}(h^t)$ strictly prefer to play H at h^t.*

Properties of Class 2 Histories: I state three properties of Class 2 histories, all of which are shown in section A.4. Lemma A.3 establishes a lower bound on P2's posterior belief after observing H at any Class 2 history.

Lemma A.3. *For any Class 2 history h^t.*

- *If $\overline{\theta}(h^t) \geq \theta_3$, then $\eta(h^t, H) \geq \eta(h^0)$ and $\eta(h^t, L) = 0$.*
- *If $\overline{\theta}(h^t) = \theta_2$, then $\eta(h^t, H) = \min\{1, \frac{\eta(h^t)}{\gamma^*}\}$ and $\eta(h^t, L) = 0$.*

Lemma A.4 establishes an upper bound on the number of Class 2 histories along every path of play.

Lemma A.4. *There exist $\underline{\delta} \in (0, 1)$ and $M \in \mathbb{N}$, such that when $\delta > \underline{\delta}$, the number of Class 2 histories along every path of equilibrium play is at most M.*

Lemma A.5 establishes a uniform lower bound on $p^H(h^t)$ for all Class 1 and Class 2 histories.

Lemma A.5. *There exist $\underline{\delta} \in (0, 1)$ and $\underline{Q} > 0$, such that when $\delta > \underline{\delta}$, we have $p^H(h^t) \geq \underline{Q}$ for all h^t belonging to Class 1 and Class 2.*

Lemma A.5 also implies a lower bound on $p^H(h^t)$ if h^t is the *first history* that reaches Class 3, i.e., h^t is such that $p^L(h^t) = 0$ and $p^L(h^s) > 0$ for all $h^s \prec h^t$.

A.3 Class 3 Histories

If h^t is such that $p^L(h^t) = 0$, then $v(h^t)$ is a convex combination of v^H and v^N. According to Lemma 3.7.2 of Mailath and Samuelson (2006, page 99), when δ is large enough, there exist $\{v^t\}_{t=0}^{\infty}$ with $v^t \in \{v^N, v^H\}$ such that (1) $v(h^t) = (1-\delta)\sum_{t=0}^{\infty} \delta^t v^t$, and (2) for every $s \in \mathbb{N}$, $(1-\delta)\sum_{t=s}^{\infty} \delta^{t-s} v^t$ is $\varepsilon-$close to $v(h^t)$. Players' continuation play following h^t is given by:

- For every $s \in \mathbb{N}$ such that $v^s = v^H$, P2 plays T and all types of P1 play H in period $t+s$.
- For every $s \in \mathbb{N}$ such that $v^s = v^N$, P2 plays N and all types of P1 play L in period $t+s$.

Players' Incentives: P2's incentive at Class 3 histories are trivially satisfied. For P1's incentives, pick ε in Lemma 3.7.2 of Mailath and Samuelson (2006) to be small enough. Lemma A.5 implies that P1's continuation value at every Class 3 history is no less than $(\underline{Q}/2)v^H + \Big(1 - \underline{Q}/2\Big)v^N$. When a patient P1 is asked to play H, she has a strict incentive to comply since (1) if she does not comply, then her continuation payoff is 0; (2) if she complies, then her continuation payoff is strictly bounded away from 0.

A.4 Incentive Constraints & Promise Keeping Constraints

First, I verify that at every on-path history, P1's continuation payoff is a convex combination of v^N, v^H, and v^L. Next, I show that P2 has an incentive to play T at every Class 1 history. Then, I show Lemmas A.2 to A.5, which together with Lemma A.1 imply the promise keeping condition, that the continuation play delivers every type of player 1 her promised continuation value at every on-path history.

A.4.1 P1's Continuation Value

P1's continuation value in the beginning $v(h^0)$ is a convex combination of v^N, v^H, and v^L. I show that:

- Suppose h^t is an on-path history and $v(h^t)$ is a convex combination of v^N, v^H, and v^L, then for every outcome $y_t \in \{N, H, L\}$ that occurs with positive probability at h^t, P1's continuation value after y_t, given by $v(h^t, y_t)$, is also a convex combination of v^N, v^H, and v^L.

First, consider the case in which h^t belongs to Class 3. Given that $p^L(h^t) = 0$, or equivalently, $v(h^t)$ is a convex combination of v^N and v^H, the only on-path outcomes at h^t are N and (T, H). As a result, the continuation payoffs $v(h^t, N)$ and $v(h^t, H)$ are both convex combinations of v^N and v^H.

Second, consider the case in which h^t belongs to Class 1. There are two possible outcomes at h^t: (T, H) and (T, L). If h^t is such that $\eta(h^t, H) \neq 1$, then according to (A.11) and (A.12), P1's continuation value remains to be a convex combination of v^N, v^H, and v^L. If h^t is such that $\eta(h^t, H) = 1$, then according to (A.11) and (A.13), P1's continuation value remains to be a convex combination of v^N, v^H, and v^L.

Third, consider the case in which h^t belongs to Class 2. There are two possible outcomes at h^t: (T, H) and (T, L). If player 1 plays L, then his continuation value is (A.22), which is a convex combination of v^N and v^H. If he plays H, then his continuation value is (A.12) if $\eta(h^t, H) \neq 1$, and is (A.13) if $\eta(h^t, H) = 1$. In both cases, $v(h^t, H)$ is a convex combination of v^N, v^L, and v^H.

A.4.2 P2's Incentives at Class 1 Histories

I show that H is played with probability at least γ^* at every Class 1 history, which implies that P2 has an incentive to play T. Let $p_H(h^t)$ be the probability that P1 plays H at h^t *according to P2's belief*. Since P2's belief is a martingale, we have:

$$p_H(h^t)\eta(h^t, H) + (1 - p_H(h^t))\eta(h^t, L) = \eta(h^t).$$

The above equality is equivalent to:

$$p_H(h^t)\Big(\eta(h^t, H) - \eta(h^t)\Big) + (1 - p_H(h^t))\Big(\eta(h^t, L) - \eta(h^t)\Big) = 0$$

$$\Leftrightarrow \quad p_H(h^t)\Big(\eta(h^t, H) - \eta(h^t)\Big) = (1 - p_H(h^t))\Big(\eta(h^t) - \eta(h^t, L)\Big).$$

As long as $\eta(h^t, L) \neq \eta(h^t)$ and $p_H(h^t) \neq 0$, i.e., nontrivial learning happens at h^t, and H is played at h^t with positive probability, we have:

$$\frac{\eta(h^t, H) - \eta(h^t)}{\eta(h^t) - \eta(h^t, L)} = \frac{1 - p_H(h^t)}{p_H(h^t)}. \tag{A.24}$$

If P2 plays T with positive probability at h^t, then $p_H(h^t) \geq \gamma^*$. This implies that:

$$\frac{\eta(h^t, H) - \eta(h^t)}{\eta(h^t) - \eta(h^t, L)} = \frac{1 - p_H(h^t)}{p_H(h^t)} \leq \frac{1 - \gamma^*}{\gamma^*}. \tag{A.25}$$

The belief updating formulas in (A.9) and (A.10) satisfy (A.25), and therefore, P2 has an incentive to play T.

A.4.3 Proof of Lemma A.2

Let $\pi(h^t) \in \Delta(\Theta)$ be P2's belief at h^t. For every $\theta \in \Theta$, let $\pi(h^t)[\theta]$ be the probability it attaches to type θ. A useful observation from the constructed strategies is: for every Class 2 history h^t, and every $\theta_i < \theta_j$,

1. if θ_j belongs to the support of P2's belief at h^t, then θ_i also belongs to the support of that belief.

2. if $\overline{\theta}(h^t)=\theta_j$, then

$$\frac{\pi(h^t)[\theta_j]}{\pi(h^t)[\theta_i]}=\frac{\pi_j}{\pi_i}\cdot\frac{k_j-l(h^t)}{k_j}.$$

I start from verifying P2's incentives using the observation that at every history h^t belonging to Class 1 or Class 2,

$$\eta(h^t)\underbrace{\geq}_{\text{by induction on } t}\eta^*\underbrace{\geq}_{\text{according to (A.5)}}\gamma^*\eta(h^0).$$

Suppose $\overline{\theta}(h^t)=\theta_2$, then only types θ_1 and θ_2 can occur with positive probability at h^t. Since type θ_2 plays L at h^t with probability $\min\{1,\frac{1-\gamma^*}{1-\eta(h^t)}\}$, type θ_1 plays H for sure, and the probability of type θ_1 is $\eta(h^t)$, player 2 believes that L is played at h^t with probability at most $1-\gamma^*$. This implies her incentive to play T at h^t.

Next, I examine the case in which $\overline{\theta}(h^t)=\theta_j$ with $j\geq 3$. By definition, types with cost higher than θ_j occur with probability 0, and type θ_1 occurs with probability at least $\gamma^*\pi_1$. According to player 1's actions at Class 2 histories specified in section A.2, and using statement 2 in Claim 1, the probability with which L is played at h^t is at most:

$$(1-\gamma^*\pi_1)\frac{(\pi_j/k_j)}{\pi_2+...+\pi_{j-1}+((k_j-l(h^t))\pi_j/k_j)}\leq(1-\gamma^*\pi_1)\frac{(\pi_j/k_j)}{\pi_2+...+\pi_{j-1}+(\pi_j/k_j)}. \tag{A.26}$$

The RHS is no more than $1-\gamma^*$ according to the definition of k_j in (A.4). To verify P1's incentives, I consider two subcases:

1. If h^t is such that $\eta(h^t,H)<1$, then (A.12), (A.22) and (A.23) imply that type $\overline{\theta}(h^t)$ is indifferent between H and L at h^t, and types that are strictly lower than $\overline{\theta}(h^t)$ strictly prefer H to L.

2. If h^t is such that $\eta(h^t,H)=1$, then given that all types except for type $\overline{\theta}(h^t)$ play H with probability 1 at h^t, then we know that $\overline{\theta}(h^t)=\theta_2$. According to (A.12), (A.22) and (A.23), type θ_2 strictly prefers L at h^t, and type θ_1 strictly prefers H at h^t.

A.4.4 Proof of Lemma A.3

Case 1: Consider the case in which $\overline{\theta}(h^t)\geq\theta_3$. First, suppose $\eta(h^t)\geq\eta(h^0)$, then the conclusion of Lemma A.3 follows since $\eta(h^t,H)>\eta(h^t)\geq\eta(h^0)$. Second, suppose $\eta(h^t)<\eta(h^0)$, then given the value of $l(h^t)$ and the highest-cost type at h^t being θ_j, the posterior probability of type θ_1 is bounded from below by:

$$\frac{\eta(h^t)}{\eta(h^t)+(1-\eta(h^t))\frac{\pi_2+...+\pi_{j-1}+\frac{k_j-l(h^t)-1}{k_j}\pi_j}{\pi_2+...+\pi_{j-1}+\frac{k_j-l(h^t)}{k_j}\pi_j}}\geq\frac{\eta(h^t)}{\eta(h^t)+(1-\eta(h^t))\frac{\pi_2+...+\pi_{j-1}+\frac{k_j-1}{k_j}\pi_j}{\pi_2+...+\pi_{j-1}+\pi_j}}$$

Let

$$X \equiv 1 - \frac{\pi_2 + ... + \pi_{j-1} + \frac{k_j - 1}{k_j}\pi_j}{\pi_2 + ... + \pi_{j-1} + \pi_j} = \frac{\pi_j}{k_j(\pi_2 + ... + \pi_{j-1} + \pi_j)}.$$

The lower bound on posterior belief $\frac{\eta(h^t)}{\eta(h^t)+(1-\eta(h^t))(1-X)}$ is greater than π_1 if and only if:

$$X \geq 1 - \frac{(1-\pi_1)\eta(h^t)}{\pi_1(1-\eta(h^t))} = \frac{\pi_1 - \eta(h^t)}{\pi_1(1-\eta(h^t))}.$$

Given that $\eta(h^t) \geq \eta^*$ at every history h^t that belongs to Class 2, the above inequality is implied by (A.5).

Case 2: Consider the case in which $\overline{\theta}(h^t) = \theta_2$. If $\eta(h^t) \geq \gamma^*$, then type θ_2 plays L with probability $\min\{1, \frac{1-\gamma^*}{1-\eta(h^t)}\} = 1$, which implies that $\eta(h^t, H) = 1$. If $\eta(h^t) < \gamma^*$, then type θ_2 plays L with probability $\min\{1, \frac{1-\gamma^*}{1-\eta(h^t)}\} = \frac{1-\gamma^*}{1-\eta(h^t)}$, which implies that $\eta(h^t, H) = \eta(h^t)/\gamma^* \geq \gamma^*\eta(h^0)/\gamma^* = \eta(h^0)$.

A.4.5 Proof of Lemma A.4

Step 1: If h^t belongs to Class 2 and $\overline{\theta}(h^t) = \theta_j \geq \theta_3$, then according to (A.20), type $\overline{\theta}(h^t)$ plays L with probability 1 when $l(h^t) = k_j - 1$, after which play reaches a Class 3 history. Therefore, along every path of play, there are at most k_j Class 2 histories satisfying $\overline{\theta}(h^t) = \theta_j$, and there are at most $K \equiv k_3 + ... + k_m$ Class 2 histories that has $\overline{\theta}(h^t) \geq \theta_3$.

Step 2: Let h^t be a Class 2 history with $\overline{\theta}(h^t) = \theta_2$. Let $N \equiv \lceil \frac{1}{1-\gamma} \rceil$, and recall T in Lemma A.1. In addition to the requirements on δ mentioned earlier, I also require δ to satisfy:

$$\delta^{T+1}(1 + \delta + ... + \delta^N) > N \text{ and } 2\delta^{T+N+2} > 1. \tag{A.27}$$

These are compatible given that all of them require δ to be sufficiently large.

First, I show that after P1 plays H at h^t, it takes at most $T + N$ periods for play to reach a history that belongs to *either Class 2 or Class 3*. According to the continuation value at (h^t, H), given by (A.12), we have:

$$p^L(h^t, H) = \frac{p^L(h^t)}{\delta} < \frac{1-\delta}{\delta}. \tag{A.28}$$

The last inequality comes from h^t belonging to Class 2, so that $p^L(h^t) < 1 - \delta$ by definition. According to Lemma A.1, for every Class 1 history h^s such that $h^s \succ (h^t, H)$ and all histories between (h^t, H) and h^s belong to Class 1,

$$(1-\delta)\sum_{r=t+1}^{s}\delta^{r-(t+1)}\mathbf{1}\{y_r=(T,H)\}\le(1-\delta^T)+(1-\delta)\sum_{r=t+1}^{s}\delta^{r-(t+1)}\mathbf{1}\{y_r=(T,L)\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}} \tag{A.29}$$

Moreover, (A.28) and the requirement that all histories between (h^t,H) and h^s belong to Class 1 imply that

$$(1-\delta)\sum_{r=t+1}^{s}\delta^{r-(t+1)}\mathbf{1}\{y_r=(T,L)\}<\frac{1-\delta}{\delta}. \tag{A.30}$$

Given that only outcomes (T,L) and (T,H) occur at Class 1 and Class 2 histories:

$$1-\delta^{s-(t+1)}=(1-\delta)\sum_{r=t+1}^{s}\delta^{r-(t+1)}\mathbf{1}\{y_r=(T,L)\}+(1-\delta)\sum_{r=t+1}^{s}\delta^{r-(t+1)}\mathbf{1}\{y_r=(T,H)\}$$

$$\le(1-\delta^T)+\frac{1-\delta}{\delta}+\frac{1-\delta}{\delta}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}\le(1-\delta^T)+\frac{1-\delta}{\delta}\frac{1}{1-\widetilde{\gamma}}\le(1-\delta^T)+\frac{1-\delta}{\delta}\frac{1}{1-\gamma} \tag{A.31}$$

To show that $s-(t+1)\le T+N$, suppose toward a contradiction that $s-(t+1)\ge T+N+1$, then

$$(1-\delta^T)+\frac{1-\delta}{\delta}N\ge(1-\delta^T)+\frac{1-\delta}{\delta}\frac{1}{1-\gamma}\ge 1-\delta^{s-(t+1)}\ge 1-\delta^{T+N+1},$$

which yields:

$$\frac{1-\delta}{\delta}N\ge\delta^T(1-\delta^{N+1}).$$

Dividing both sides by $\frac{1-\delta}{\delta}$, we have:

$$N\ge\delta^{T+1}(1+\delta+...+\delta^N),$$

which contradicts the first inequality of (A.27). The above contradiction implies that $s-(t+1)\le T+N$.

Second, I focus on history h^s that has the following two features:

1. h^s belongs to Class 2,
2. $h^s\succeq(h^t,H)$ and all histories between (h^t,H) and h^s, excluding h^s, belong to Class 1.

I show that there exists at most one period from (h^t,H) to h^s such that the stage-game outcome is (T,L). Suppose toward a contradiction that there exist two or more such periods, then

$$(1-\delta)\sum_{r=t+1}^{s}\delta^{r-(t+1)}\mathbf{1}\{y_r=(T,L)\}\ge 2(1-\delta)\delta^{T+N+1}.$$

The last inequality comes from the previous conclusion that $s-(t+1) \leq T+N$. This is because h^s belongs to Class 2 and h^{s-1} belongs to Class 1, and therefore, $(s-1)-(t+1) \leq T+N$, or equivalently, $s-(t+1) \leq T+N+1$. According to (A.30),

$$2(1-\delta)\delta^{T+N+1} < (1-\delta)\sum_{r=t+1}^{s}\delta^{r-(t+1)}\mathbf{1}\{y_r=(T,L)\} < \frac{1-\delta}{\delta}. \tag{A.32}$$

The above inequality contradicts the second inequality of (A.27) that $2\delta^{T+N+2} > 1$.

Let h^t be the first time play reaches a history that belongs to Class 2 with $\overline{\theta}(h^t) = \theta_2$. According to Lemma A.3, $\eta(h^t, H) \geq \frac{\eta^*}{\gamma^*} \geq \eta(h^0) = \pi_1$. Let h^s be the next history that belongs to Class 2 with $h^s \succ (h^t, H)$. Since we have shown that (T, L) occurs at most once between (h^t, H) and h^s, we know that

$$\eta(h^s, H) = \min\{1, \frac{\eta(h^s)}{\gamma^*}\} \geq \min\{1, \frac{\eta(h^t, H)}{\gamma^*}(1-\lambda\gamma^*)\}$$

Therefore, conditional on (h^s, H) is not a Class 3 history, player 2's belief at (h^s, H) attaches probability at least:

$$\eta(h^s, H) \geq \eta(h^t, H)\frac{1-\lambda\gamma^*}{\gamma^*} \geq \eta(h^t, H)\sqrt{\frac{1}{\gamma^*}} \tag{A.33}$$

to type θ_1, where the last inequality comes from $\lambda \in (0, \frac{1-\sqrt{\gamma^*}}{\gamma^*})$. Let

$$\widehat{M} \equiv \frac{\log(1/\pi_1)}{\log\sqrt{\frac{1}{\gamma^*}}} + 1.$$

Since $\eta(h^t, H) \geq \pi_1$ for the first Class 2 history h^t satisfying $\overline{\theta}(h^t) = \theta_2$, there can be at most $\widehat{M}$ Class 2 histories with θ_2 being the highest-cost type along every path of play. This is because otherwise, P2's posterior belief attaches probability greater than

$$\pi_1\Big(\frac{1}{\sqrt{\gamma^*}}\Big)^{\widehat{M}} > 1$$

at the $\widehat{M}+1$th such history, which leads to a contradiction. Summarizing the conclusions of the two parts, there exist at most $M \equiv K + \widehat{M}$ Class 2 histories along every path of equilibrium play.

A.4.6 Proof of Lemma A.5

To start with, consider Class 2 history h^t such that no predecessor of h^t belongs to Class 2, in another word, all predecessors of h^t belong to Class 1. According to (A.18), $p^H(h^{t-1}) \geq Y$, which implies that $p^H(h^t) \geq$

$Y-(1-\delta)$. As a result

$$Q(h^t)=p^H(h^t)-\frac{1-\delta-p^L(h^t)}{1-\overline{\theta}(h^t)}\geq Y-(1-\delta)\big(1+\frac{1}{1-\theta_m}\big)>0.$$

If play remains at Class 1 or Class 2 history after h^t, then player 1 must be playing H at h^t, after which

$$p^H(h^t,H)\geq p^H(h^t)-(1-\delta)\geq Y-2(1-\delta) \text{ and } p^L(h^t,H)\leq\frac{1-\delta}{\delta}.$$

According to Lemma A.3, $\eta(h^t,H)\geq\eta(h^0)=\pi_1$. One can then apply Lemma A.1 again, which implies that at every Class 1 history h^s such that only one predecessor of h^s belongs to Class 2, we have:

$$p^H(h^s)\geq Z\equiv Y-2(1-\delta)-\frac{1-\delta}{\delta}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}-(1-\delta^T),$$

with T and $\widetilde{\gamma}$ being the same as in the previous step. When δ is large enough, $Z\geq Y/2$. One can then show that for every Class 2 history h^s such that there is only one strict predecessor history belongs to Class 2,

$$Q(h^s)=p^H(h^s)-\frac{1-\delta-p^L(h^s)}{1-\overline{\theta}(h^s)}\geq Z-(1-\delta)\big(1+\frac{1}{1-\theta_m}\big)>0.$$

Iteratively apply this process. Since

1. the number of Class 2 histories along every path of play is bounded from above by M (Lemma A.4),
2. for every Class 2 history h^t, $p^L(h^t,H)=\frac{1-\delta}{\delta}$ and $\eta(h^t,H)\geq\eta(h^0)$,

there exist $\underline{\delta}\in(0,1)$ and $\underline{Q}>0$ such that when $\delta>\underline{\delta}$, $p^H(h^t)\geq\underline{Q}$ for every Class 1 or Class 2 history h^t.

A.5 Proof of Lemma A.1

For every h^t, let $\Delta(h^t)\equiv\eta(h^t)-\eta^*$. For every $t\in\mathbb{N}$, let $N_{L,t}$ and $N_{H,t}$ be the number of periods in which L and H are played from period 0 to $t-1$, respectively. The proof is done by induction on $N_{L,t}$.

When $N_{L,t}\leq 2(k-n)$, then the conclusion holds as $N_{H,t}\geq 2n+X$. Moreover, $\Delta(h^T)$ reaches $1-\eta^*$ before period T, after which play reaches a Class 3 history.

Suppose the conclusion holds for when $N_{L,t}\leq N$ with $N\geq 2(k-n)$, and suppose toward a contradiction that there exists h^T with $T\geq k+X$ and $N_{L,T}=N+1$, such that every $h^t\preceq h^T$ belongs to Class 1, but

$$(1-\delta)\sum_{t=0}^{T-1}\delta^t\mathbf{1}\{y_t=H\}-(1-\delta^X)>(1-\delta)\sum_{t=0}^{T-1}\delta^t\mathbf{1}\{y_t=L\}\cdot\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}},\tag{A.34}$$

I obtain a contradiction in three steps.

Step 1: I show that for every $s < T$,

$$(1-\delta)\sum_{t=s}^{T-1}\delta^t \mathbf{1}\{y_t = H\} \geq (1-\delta)\sum_{t=s}^{T-1}\delta^t \mathbf{1}\{y_t = L\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}. \tag{A.35}$$

Suppose toward a contradiction that (A.35) fails. Then together with (A.34), we have:

$$(1-\delta)\sum_{t=0}^{s-1}\delta^t \mathbf{1}\{y_t = H\} - (1-\delta^X) > (1-\delta)\sum_{t=0}^{s-1}\delta^t \mathbf{1}\{y_t = L\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}} \tag{A.36}$$

and

$$(1-\delta)\sum_{t=s}^{T-1}\delta^t \mathbf{1}\{y_t = L\} > 0. \tag{A.37}$$

According to (A.37), $N_{L,s} < N_{L,T}$. Since $N_{L,T} = N + 1$, we have $N_{L,s} \leq N$. Applying the induction hypothesis and (A.36), we know that play reaches a Class 3 history before h^s, leading to a contradiction.

Step 2: I show that for every k consecutive periods

$$\{y_r, y_{r+1}, ..., y_{r+k-1}\} \subset h^T,$$

the number of outcome (T, H) in this sequence is at least $n+1$. According to (A.35) shown in the previous step, outcome (T, H) occurs at least $n+1$ times in the last k periods, namely, in the set $\{y_{T-k+1}, ..., y_T\}$.

Suppose toward a contradiction that there exists k consecutive periods in which outcome (T, H) occurs no more than n times, then the conclusion above that outcome (T, H) occurs at least $n+1$ times in the last k periods implies that there exists k consecutive periods $\{y_r, ..., y_{r+k-1}\}$ in which (T, H) occurs exactly n times and (T, L) occurs exactly $k - n$ times. According to (A.2), we have

$$(1-\delta)\sum_{t=r}^{r+k-1}\delta^t \mathbf{1}\{y_t = H\} < (1-\delta)\sum_{t=r}^{r+k-1}\delta^t \mathbf{1}\{y_t = L\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}, \tag{A.38}$$

but then

$$\Delta(h^{r+k}) > \Delta(h^{r+1}). \tag{A.39}$$

Next, let us consider the following new sequence with length $T - k$:

$$\widetilde{h}^{T-k} \equiv \{\widetilde{y}_0, \widetilde{y}_1, ..., \widetilde{y}_{T-k-1}\} \equiv \{y_0, y_1, ..., y_{r-1}, y_{r+k}, ..., y_{T-1}\}$$

which is obtained by removing $\{y_r, ..., y_{r+k-1}\}$ from the original sequence and front-loading the subsequent play $\{y_{r+k}, ..., y_{T-1}\}$. The number of (T, L) in this new sequence is at most $N + 1 - (n - k)$, which is no more than N. According to the conclusion in Step 1:

$$(1-\delta)\sum_{t=r+k}^{T-1}\delta^t \mathbf{1}\{y_t = H\} > (1-\delta)\sum_{t=r+k}^{T-1}\delta^t \mathbf{1}\{y_t = L\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}. \tag{A.40}$$

This together with (A.38) and (A.34) imply that

$$(1-\delta)\sum_{t=0}^{T-k-1}\delta^t \mathbf{1}\{\widetilde{y}_t = H\} - (1-\delta^X) > (1-\delta)\sum_{t=0}^{T-k-1}\delta^t \mathbf{1}\{\widetilde{y}_t = L\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}.$$

According to the induction hypothesis, play will reach a Class 3 history before period $T - k$ if player 1 plays according to $\{\widetilde{y}_0, \widetilde{y}_1, ..., \widetilde{y}_{T-k-1}\}$.

1. Suppose $\widetilde{h}^{T-k}$ reaches a Class 3 history before period r, then play will also reach a Class 3 history before period r according to the original sequence.
2. Suppose $\widetilde{h}^{T-k}$ reaches a Class 3 history in period s, with $s > t$, then according to (A.39), we have $\Delta(\widetilde{h}^s) \leq \Delta(h^{s+k})$. This implies that play will reach a Class 3 history in period $s + k$ according to the original sequence.

This contradicts the hypothesis that play has never reached a Class 3 history before h^T.

Step 3: For every history $h^T \equiv \{y_0, y_1, ..., y_{T-1}\} \in \{H, L\}^T$ and $t \in \{1, ..., T-1\}$, define the operator $\Omega_t : \{H, L\}^T \to \{H, L\}^T$ as:

$$\Omega_t(h^T) = (y_0, ..., y_{t-2}, y_t, y_{t-1}, y_{t+1}, ..., y_{T-1}), \tag{A.41}$$

in another word, swapping the order between y_{t-1} and y_t. Recall the belief updating formula in Class 1 histories and let

$$\mathcal{H}^{T,*} \equiv \Big\{h^T \Big| \Delta(h^t) < 1 - \eta^* \text{ for all } h^t \prec h^T\Big\}. \tag{A.42}$$

If $h^T \in \mathcal{H}^{T,*}$, then $\Omega_t(h^T) \in \mathcal{H}^{T,*}$ unless:

- $y_{t-1} = L, y_t = H$.
- *and*, $\Big(1 + \lambda(1-\gamma^*)\Big)\Delta(h^{t-1}) \geq 1 - \eta^*$.

Next, I show that the above situation cannot occur besides in the last k periods. Suppose toward a contradiction that there exists $t \leq T - k$ such that $h^T \in \mathcal{H}^{T,*}$ but $\Omega_t(h^T) \notin \mathcal{H}^{T,*}$. Based on the conclusion in Step 2, outcome

(T,H) occurs at least $n+1$ times in the sequence $\{y_t, ..., y_{t+k-1}\}$. Consider another sequence $\{y_{t-1}, ..., y_{t+k-1}\}$, in which outcome (T,H) occurs at least $n+1$ times and outcome (T,L) occurs at most $k-n$ times. This implies that:

$$\begin{aligned}
\Delta(h^{t+k}) &\geq \Delta(h^{t-1})\Big(1+\lambda(1-\gamma^*)\Big)^{n+1}\Big(1-\lambda\gamma^*\Big)^{k-n} \\
&= \Delta(h^{t-1})\underbrace{\Big(1+\lambda(1-\gamma^*)\Big)^{n}\Big(1-\lambda\gamma^*\Big)^{k-n}}_{\geq 1}\Big(1+\lambda(1-\gamma^*)\Big) \\
&\geq \Delta(h^{t-1})\Big(1+\lambda(1-\gamma^*)\Big) \\
&\geq 1-\eta^*, \qquad \text{(A.43)}
\end{aligned}$$

where second inequality follows from $n/k > \widehat{\gamma}$, and the 3rd inequality follows from the hypothesis that $\Omega_t(h^T) \notin \mathcal{H}^{T,*}$. Inequality (A.43) implies that play reaches the high phase before period $t+k \leq T$, contradicting the hypothesis that $h^T \in \mathcal{H}^{T,*}$.

To summarize, for every $t \leq T-k$, if $h^T \in \mathcal{H}^{T,*}$, then $\Omega_t(h^T) \in \mathcal{H}^{T,*}$. For every $t > T-k$, if $h^T \in \mathcal{H}^{T,*}$, then $\Omega_t(h^T) \in \mathcal{H}^{T,*}$ unless $y_{t-1} = L$ and $y_t = H$. Therefore, one can freely front-load outcome (T,H) from period 0 to $T-k-1$ and obtain the following revised sequence:

$$\{H, H, ...H, L, ..., L, y_{T-k}, ..., y_{T-1}\}, \qquad \text{(A.44)}$$

which meets the following two requirements. First, the revised sequence (A.44) still belongs to set $\mathcal{H}^{T,*}$. Second, the sequence in (A.44) satisfies (A.34).

According to the conclusion in Step 2, the number of outcome (T,L) from period 0 to $T-k-1$ cannot exceed $k-n-1$, and the number of outcome (T,L) from period $T-k$ to $T-1$ cannot exceed $k-n-1$. This is because otherwise, there exists a sequence of length k that has at most n periods of outcome (T,H), contradicting the two conditions that the revised sequence in (A.44) must satisfy. Therefore, the total number of outcome (T,L) in this sequence is at most $2(k-n-1)$. This contradicts the induction hypothesis that the number of outcome (T,L) exceeds $2(k-n)$.

B Proof of Corollary 2

I show the result by induction on the number of types in the support of the belief. When $|\Theta| = 2$, recall the definitions of $\overline{a}_1(\cdot)$ as well as $\overline{\mathcal{H}}$ in Appendix B.1. Let $\overline{h}_1^t$ be the first history in $\overline{\mathcal{H}}$ such that type θ_2 has a strict incentive to play L but $\overline{a}_1(\overline{h}_1^t) = H$. This history exists since type θ_2's equilibrium payoff is strictly greater than $1-\theta_2$. Type θ_1 plays H with positive probability at $\overline{h}^t$, after which she fully reveals her private information.

One can similarly define $\overline{a}_1(\cdot)$ and $\overline{\mathcal{H}}$ in the continuation game starting from history $(\overline{h}_1^t, L)$. When δ is large enough, both types occur with positive probability at $(\overline{h}_1^t, L)$. Let $\overline{h}_2^t$ be the first history in $\overline{\mathcal{H}}$ that succeeds $(\overline{h}_1^t, L)$ such that type θ_2 has a strict incentive to play L but $\overline{a}_1(\overline{h}_2^t) = H$. The existence of this history also comes from the requirement that type θ_2's payoff is greater than $1-\theta_2$. Type θ_1 plays H with positive probability at $\overline{h}_2^t$, after which she fully reveals her private information. Similarly, one can define $\overline{h}_3^t, \overline{h}_4^t, \ldots$. As $\delta \to 1$, the length of this sequence goes to infinity, and at every such history, type θ_1 fully reveals her information after playing H.

Next, suppose the conclusion holds for all posterior beliefs that have at most k elements in the support. When there are $k+1$ types, consider the incentives of type θ_2. After reaching history $\overline{h}_n^t$ for a given $n \in \mathbb{N}$, type θ_1 needs to play H with positive probability at $\overline{h}_n^t$. This is because otherwise, there exists type θ_j with $j > 2$ that cannot extract information rent in the future, leading to a contradiction. After type θ_1 plays H at $\overline{h}_n^t$, there are at most k types in the support of the posterior belief. The conclusion is then implied by the induction hypothesis.

C Proof of Theorem 2

Notation: For every $\sigma_\theta : \mathcal{H} \to \Delta(A_1)$ and $\sigma_2 : \mathcal{H} \to \Delta(A_2)$, let $\mathcal{H}(\sigma_\theta, \sigma_2)$ be the set of histories that occur with positive probability under the probability measure induced by $(\sigma_\theta, \sigma_2)$. Let $\overline{\sigma}_\theta : \mathcal{H} \to \Delta(A_1)$ be such that $\overline{\sigma}_\theta(h^t) = H$ for every $h^t \in \mathcal{H}$. Let $\underline{\sigma}_\theta : \mathcal{H} \to \Delta(A_1)$ be such that $\underline{\sigma}_\theta(h^t) = L$ for every $h^t \in \mathcal{H}$.

Completely Mixed Strategies: Suppose toward a contradiction that there exists a BNE $\sigma = \Big((\sigma_\theta)_{\theta\in\Theta}, \sigma_2\Big)$ that attains payoff within ε of v^*, and there exists a type $\widehat{\theta}$ that has a completely mixed best reply to σ_2. Then both $\overline{\sigma}_\theta$ and $\underline{\sigma}_\theta$ are type $\widehat{\theta}$'s best replies to σ_2. Since the stage game payoff is monotone-supermodular according to the orders $T \succ N$, $H \succ L$ and $\theta_1 \succ \theta_2 \succ \ldots \succ \theta_m$, Lemma C.1 in Pei (2018a) implies that:

1. For every $\theta_j \succ \widehat{\theta}$, type θ_j plays H with probability 1 at every $h^t \in \mathcal{H}(\overline{\sigma}_\theta, \sigma_2)$.

2. For every $\theta_k \prec \widehat{\theta}$, type θ_k plays L with probability 1 at every $h^t \in \mathcal{H}(\underline{\sigma}_\theta, \sigma_2)$.

I consider the above two cases separately. First, suppose $\widehat{\theta} \neq \theta_m$, then type θ_m will play L with probability 1 at every $h^t \in \mathcal{H}(\underline{\sigma}_\theta, \sigma_2)$, from which she is supposed to receive payoff no less than $v_m^* - \epsilon$. On the other hand, the argument in Fudenberg and Levine (1992) implies that in every Nash equilibrium, there are at most

$$T_{\theta_m} \equiv \log \pi_0(\theta_m) \Big/ \log(1-\gamma^*) \tag{C.1}$$

periods in which player 2 plays T. That is to say, for every $\varepsilon > 0$, there exists $\overline{\delta} \in (0,1)$ such that when $\delta > \overline{\delta}$, type θ_m's payoff is less than ε in every BNE. Pick ε to be small enough such that $\epsilon < v_m^*/2$, we obtain a contradiction.

Second, suppose $\widehat{\theta} = \theta_m$. Then types θ_1, θ_2,...,θ_{m-1} play H with probability 1 at every $h^t \in \mathcal{H}(\overline{\sigma}_\theta, \sigma_2)$. Therefore, after playing L for the first time, type θ_m reveals her type so her continuation payoff is at most $1-\theta_m$. Hence, her discounted average payoff in the repeated game cannot exceed $(1-\delta)+\delta(1-\theta_m)$. Let ε be small enough such that $(1-\delta)+\delta(1-\theta_m) < v_m^* - \varepsilon$, we have a contradiction.

Stationary Strategies: The above argument rules out completely mixed strategies. To rule out stationary strategies, one only needs to show that no type plays stationary pure strategies. First, suppose toward a contradiction that type $\widehat{\theta}$ plays L in every period, then in every BNE, there are at most

$$T_{\widehat{\theta}} \equiv \log \pi_0(\widehat{\theta}) \Big/ \log(1-\gamma^*) \tag{C.2}$$

periods in which player 2 plays T. Therefore, her equilibrium payoff vanishes to 0 as δ approaches 1, contradicting the fact that $v_{\widehat{\theta}} \geq 1-\widehat{\theta} > 0$.

Second, suppose toward a contradiction that type $\widehat{\theta}$ plays H in every period. If $\widehat{\theta} \neq \theta_1$, then her equilibrium payoff is at most $1-\widehat{\theta}$, which is strictly less than $v_{\widehat{\theta}}^*$, leading to a contradiction. If $\widehat{\theta} = \theta_1$, then type θ_2 is separated from type θ_1 the first time she plays L, after which her continuation payoff is no more than $1-\theta_2$. Therefore, type θ_2's equilibrium payoff is at most $(1-\delta)+\delta(1-\theta_2)$, which is strictly less than v_2^* when δ is close enough to 1. This leads to a contradiction.

D Miscellaneous

D.1 Counterexample to Theorem 2 under Complete Information

I provide a counterexample to Theorem 2 when $|\Theta| = 1$, in the sense that there exists a sequential equilibrium in which player 1 plays a stationary mixed strategy and attains payoff v^*, equals $1-\theta$.

The long-run player plays H with probability γ^* at every history. Player 2 who arrives in period 0 plays T. Player 2's action in period t (≥ 1) depends on the game's outcome in period $t-1$, denoted by y_{t-1}:

$$\text{Player 2's action in period } t = \begin{cases} N & \text{if } y_{t-1} = N \\ T & \text{if } y_{t-1} = H \\ p^*T + (1-p^*)N & \text{if } y_{t-1} = L \end{cases} \tag{D.1}$$

where

$$p^* \equiv \frac{1-\theta/\delta}{1-\theta}, \tag{D.2}$$

which is strictly between 0 and 1 when δ is close enough to 1.

D.2 Stackelberg Strategies

First, I construct a Bayesian Nash equilibrium in which some type of player 1 plays a stationary strategy with stage-game action ε-close to her Stackelberg action, but nevertheless, her equilibrium payoff equals 0. Consider the following strategy profile:

- Player 2 plays N at every history.
- Type θ_1 player 1 plays H with probability $\gamma^* + \varepsilon$ at every history.
- Types other than θ_1 plays L at every history.

Let $\varepsilon > 0$ be small enough such that:

$$\varepsilon < \frac{c}{b+c} \frac{1-\pi(\theta_1)}{\pi(\theta_1)}, \tag{D.3}$$

the above strategy profile is a Bayesian Nash equilibrium for every $\delta \in (0,1)$.

Next, I show that no type will play any stationary strategy with stage-game actions ε-close to the Stackelberg action in any sequential equilibrium when ε is small enough.

Proposition D.1. *For every ε small enough, there exists no sequential equilibrium in which some type of player 1 plays a stationary strategy with stage-game action ε-close to her Stackelberg action.*

Proof. Let ε be small enough so that every stationary ε-Stackelberg strategy is completely mixed. Suppose toward a contradiction that there exists a sequential equilibrium (σ, π) with $\sigma = \Big((\sigma_\theta)_{\theta \in \Theta}, \sigma_2\Big)$ and $\pi : \mathcal{H} \to \Delta(\Theta)$ such that $\sigma_{\widehat{\theta}}$ is a stationary strategy with stage-game action ε-close to her Stackelberg action. Consider the history after player 2 playing T in period $t \in \mathbb{N}$. Both $\overline{\sigma}_\theta$ and $\underline{\sigma}_\theta$ are type $\widehat{\theta}$'s best replies against σ_2 at that history. Lemma C.1 in Pei (2018) implies that:

1. For every $\theta < \widehat{\theta}$, type θ plays H with probability 1 in period t.
2. For every $\theta > \widehat{\theta}$, type θ plays L with probability 1 in period t.

Therefore, after observing H in period 0, player 2's posterior attaches probability 1 to the event that $\theta \leq \widehat{\theta}$. For every $\theta \leq \widehat{\theta}$, we have shown before that she will play H with probability strictly greater than γ^* at every history where T is played with positive probability. Hence, T is the myopic players' strict best reply after they observe H in period 0 and regardless of player 1's action choices after period 0. As a result, at the history following player 2 plays T in period 0, type $\widehat{\theta}$ can obtain continuation payoff $(1-\delta)(1-\widehat{\theta}) + \delta$ by playing H in period 0 and playing L in all subsequent periods, which is strictly more than her payoff by playing H in every period, which is $1 - \widehat{\theta}$. This contradicts the previous claim that $\overline{\sigma}_\theta$ is her best reply. $\square$

References

[1] Fudenberg, Drew and David Levine (1992) "Maintaining a Reputation when Strategies are Imperfectly Observed," *Review of Economic Studies*, 59(3), 561-579.

[2] Liu, Shuo and Harry Pei (2018) "Monotone Equilibria in Signalling Games," Working Paper.

[3] Mailath, George and Larry Samuelson (2006) "Repeated Games and Reputations: Long-Run Relationships," Princeton University Press.

[4] Pei, Harry (2018a) "Reputation Effects under Interdependent Values," Working Paper.

[5] Pei, Harry (2018b) "Trust and Betrayals: Reputational Payoffs and Behaviors without Commitment," Working Paper.