Online Appendix
Reputation Effects under Interdependent Values

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A Proof of Statement 4 Theorem 1 and its Corollaries

Layout: In this Appendix, I show statement 4 of Theorem 1 in Pei (2018) as well as several of its corollaries. In subsection A.1, I state and prove Proposition A.1, which fully characterizes the set of distributional environments such that $\underline{\Lambda}(\alpha_1^*,\theta)$ and $\Lambda(\alpha_1^*,\theta)$ coincide. This result identifies situations in which attaining mixed commitment payoff requires strictly more demanding conditions than attaining a *nearby* pure commitment payoff.

Motivated by situations where $\underline{\Lambda}(\alpha_1^*,\theta) \subsetneq \Lambda(\alpha_1^*,\theta)$, I introduce the concept of '*mixed-ϵ elaboration*' in subsection A.2 and state a result saying that small perturbations of a pure commitment strategy can lead to a significant decrease in player 1's lowest limiting equilibrium payoff (Proposition A.2).

In subsections A.3-A.5, I develop the building blocks for the proofs of Proposition A.2 and statement 4 of Theorem 1. In subsections A.6 and A.7, I provide constructive proofs to these results, respectively. These proofs follow along the same spirit as the proof of statement 2 (see Appendix B in Pei 2018) while applying the conclusions in subsections A.3-A.5 to adjust for the differences between pure and non-trivially mixed commitment strategies.

A.1 Relationship Between $\underline{\Lambda}(\alpha_1^*,\theta)$ and $\Lambda(\alpha_1^*,\theta)$

Recall the definitions of $\Lambda(\alpha_1^*,\theta)$, $\underline{\Lambda}(\alpha_1^*,\theta)$ and $\psi(\theta')$. Motivated by Theorem 1 in Pei (2018), one might wonder when will $\Lambda(\alpha_1^*,\theta)$ and $\underline{\Lambda}(\alpha_1^*,\theta)$ coincide and when will the former be a strict subset of the latter. Proposition A.1 provides a full characterization.

Proposition A.1. $\Lambda(\alpha_1^*,\theta)=\underline{\Lambda}(\alpha_1^*,\theta)$ *if and only if there exists* $a_2'\neq a_2^*$ *such that for every* $\tilde{\theta}\in\Theta^b_{(\alpha_1^*,\theta)}$,

$$u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2^*)+\psi(\tilde{\theta})u_2(\tilde{\theta},\alpha_1^*,a_2^*)=u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2')+\psi(\tilde{\theta})u_2(\tilde{\theta},\alpha_1^*,a_2'). \tag{A.1}$$

According to Proposition A.1, $\Lambda(\alpha_1^*,\theta)=\underline{\Lambda}(\alpha_1^*,\theta)$ if and only if there exists an alternative action a_2' such that player 2 is indifferent between a_2^* and a_2' for every λ on the hyperplane $\sum_{\tilde{\theta}\in\Theta^b_{(\alpha_1^*,\theta)}}\lambda(\tilde{\theta})/\psi(\tilde{\theta})=1$. In another word, there exists a common binding incentive constraint for player 2 for all the bad states. This necessary and sufficient condition is violated in the $3\times 3\times 3$ game example in Appendix G.8, but is always satisfied when $|A_2|=2$ or when $k(\alpha_1^*,\theta)=1$.

PROOF OF PROPOSITION A.1: I start with the '*if*' statement. According to (A.1), for every $\tilde{\lambda}$ satisfying $\sum_{i=1}^{k(\alpha_1^*,\theta)}\tilde{\lambda}_i/\psi_i=1$, we have:

$$u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2^*)+\sum_{\theta'\in\Theta^b_{(\alpha_1^*,\theta)}}\tilde{\lambda}(\theta')u_2(\theta',\alpha_1^*,a_2^*)=\sum_{\theta'\in\Theta^b_{(\alpha_1^*,\theta)}}\frac{\tilde{\lambda}(\theta')}{\psi(\theta')}\Big(u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2^*)+\psi(\theta')u_2(\theta',\alpha_1^*,a_2^*)\Big)$$

$$=\sum_{\theta'\in\Theta^b_{(\alpha_1^*,\theta)}}\frac{\tilde{\lambda}(\theta')}{\psi(\theta')}\Big(u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2')+\psi(\theta')u_2(\theta',\alpha_1^*,a_2')\Big)=u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2')+\sum_{\theta'\in\Theta^b_{(\alpha_1^*,\theta)}}\tilde{\lambda}(\theta')u_2(\theta',\alpha_1^*,a_2').$$

Hence for every $\hat{\lambda}\notin\underline{\Lambda}(\alpha_1^*,\theta)$, i.e. $\sum_{i=1}^{k(\alpha_1^*,\theta)}\hat{\lambda}_i/\psi_i\geq 1$, there exists $\tilde{\lambda}$ such that $\mathbf{0}\ll\tilde{\lambda}\ll\hat{\lambda}$ and $\sum_{i=1}^{k(\alpha_1^*,\theta)}\tilde{\lambda}_i/\psi_i=1$. But this suggests that:

$$u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2^*)+\sum_{\theta'\in\Theta^b_{(\alpha_1^*,\theta)}}\tilde{\lambda}(\theta')u_2(\theta',\alpha_1^*,a_2^*)=u_2(\phi_{\alpha_1^*},\alpha_1^*,a_2')+\sum_{\theta'\in\Theta^b_{(\alpha_1^*,\theta)}}\tilde{\lambda}(\theta')u_2(\theta',\alpha_1^*,a_2').$$

This is to say, a_2^* is not a strict best reply at $\tilde{\lambda}$, and therefore, $\hat{\lambda}\notin\Lambda(\alpha_1^*,\theta)$.

Next, I show the '*only if*' direction. First, since $\underline{\Lambda}(\alpha_1^*,\theta)$ and $\Lambda(\alpha_1^*,\theta)$ are open at the boundaries intersecting their exteriors, which are denoted by $\underline{\Pi}(\alpha_1^*,\theta)$ and $\Pi(\alpha_1^*,\theta)$, respectively, so $\mathrm{int}\Big(\Lambda(\alpha_1^*,\theta)\backslash\underline{\Lambda}(\alpha_1^*,\theta)\Big)\neq\{\varnothing\}$ if and only if $\Lambda(\alpha_1^*,\theta)\backslash\underline{\Lambda}(\alpha_1^*,\theta)\neq\{\varnothing\}$. A key step is the following Lemma:

Lemma A.1. *If* $\Lambda(\alpha_1^*,\theta)\backslash\underline{\Lambda}(\alpha_1^*,\theta)=\{\varnothing\}$*, then* $\overline{\Lambda}(\alpha_1^*,\theta)=\Lambda(\alpha_1^*,\theta)$.

PROOF OF LEMMA A.1: Suppose towards a contradiction that $\overline{\Lambda}(\alpha_1^*,\theta)\supsetneqq\Lambda(\alpha_1^*,\theta)$ but

$$\Lambda(\alpha_1^*,\theta)\backslash\underline{\Lambda}(\alpha_1^*,\theta)=\{\varnothing\},$$

then there exists $\tilde{\lambda}\in\overline{\Lambda}(\alpha_1^*,\theta)$ and $\theta'\in\Theta^b_{(\alpha_1^*,\theta)}$ such that the projection of $\tilde{\lambda}$ on the $\tilde{\lambda}(\theta')$ coordinate is not in $\overline{\Lambda}(\alpha_1^*,\theta)$. This implies that $\tilde{\lambda}(\theta')/\psi(\theta')>1$. Let

$$\hat{\lambda}\equiv\Big(\min\{\tilde{\lambda}(\tilde{\theta}),\psi(\tilde{\theta})\}\Big)_{\tilde{\theta}\in\Theta^b_{(\alpha_1^*,\theta)}}\text{ and }\Psi\equiv\Big\{\Big(\phi(\tilde{\theta})\Big)_{\tilde{\theta}\in\Theta^b_{(\alpha_1^*,\theta)}}\Big|\phi(\tilde{\theta})\in\{0,\psi(\tilde{\theta})\}\Big\}.$$

Let $\text{ort}(\Psi)$ be the orthotope with Ψ being the set of vertices. By construction, $\hat{\lambda} \in \text{ort}(\Psi)$ and $\hat{\lambda} \in \text{co}(\{\tilde{\lambda}\} \cup \Psi)$.[1] Since $\overline{\Lambda}(\alpha_1^*, \theta)$ is convex and

$$\Lambda(\alpha_1^*, \theta) = \overline{\Lambda}(\alpha_1^*, \theta) \bigcap \text{co}\big(\{\tilde{\lambda}\} \cup \Psi\big),$$

we know that $\hat{\lambda}$ belongs to the closure of $\Lambda(\alpha_1^*, \theta)$. Therefore, for every $\epsilon > 0$, we have:

$$B(\hat{\lambda}, \epsilon) \bigcap \Lambda(\alpha_1^*, \theta) \neq \{\varnothing\}.$$

But according to the definition of $\hat{\lambda}$, we have $\sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}} \hat{\lambda}(\tilde{\theta})/\psi(\tilde{\theta}) > 1$, in which case $B(\hat{\lambda}, \epsilon) \bigcap \underline{\Lambda}(\alpha_1^*, \theta) = \{\varnothing\}$ for small enough $\epsilon > 0$. This leads to a contradiction. $\square$

According to Lemma A.1, $\Lambda(\alpha_1^*, \theta) \backslash \underline{\Lambda}(\alpha_1^*, \theta) = \{\varnothing\}$ implies that

$$\overline{\Lambda}(\alpha_1^*, \theta) = \Lambda(\alpha_1^*, \theta) = \underline{\Lambda}(\alpha_1^*, \theta). \tag{A.2}$$

That is to say, for every $\tilde{\lambda}$ with $\sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}} \tilde{\lambda}(\tilde{\theta})/\psi(\tilde{\theta}) = 1$, there exists $a_2' \neq a_2^*$ such that:

$$u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2^*) + \sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}} \tilde{\lambda}(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^*, a_2^*) = u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2') + \sum_{\theta' \in \Theta^b_{(\alpha_1^*, \theta)}} \tilde{\lambda}(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^*, a_2'). \tag{A.3}$$

This is because otherwise, $\tilde{\lambda} \in \text{int}\Big(\overline{\Lambda}(\alpha_1^*, \theta)\Big)$, which contradicts (A.2). Pick $\tilde{\lambda}$ such that $\tilde{\lambda}(\tilde{\theta}) > 0$ for all $\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}$. Suppose towards a contradiction that there exists no such $a_2' \neq a_2^*$, then for every $a_2 \neq a_2^*$, we have:

$$u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2^*) + \sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}} \tilde{\lambda}(\tilde{\theta}) u_2(\theta', \alpha_1^*, a_2^*) > u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2) + \sum_{\theta' \in \Theta^b_{(\alpha_1^*, \theta)}} \tilde{\lambda}(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^*, a_2), \tag{A.4}$$

which contradicts (A.3). This finishes the proof of the '*only if*' direction. $\square$

A.2 Discontinuity in Equilibrium Payoffs Between Pure and Mixed Commitment Actions

Motivated by the differences between $\Lambda(\alpha_1^*, \theta)$ and $\underline{\Lambda}(\alpha_1^*, \theta)$, I formalize the idea that small perturbations to a pure commitment action can lead to a large decrease in player 1's lowest equilibrium payoff. Fixing Θ, A_1, A_2, u_2 and $(a_1^*, \theta) \in A_1 \times \Theta$. Let $\{a_2^*\} = \text{BR}_2(a_1^*, \theta)$ and let

$$u_1(\tilde{\theta}, a_1, a_2) \equiv \mathbf{1}\{\tilde{\theta} = \theta, a_1 = a_1^*, a_2 = a_2^*\}.$$

[1] I use $B(\lambda, \epsilon)$ to denote an open ball centered around λ with radius $\epsilon > 0$.

I start with defining my notion of *small perturbations*. Let $\Gamma \equiv (\Omega, \mu, \phi)$ be the '*original game*' satisfying:

1. $\Lambda(\alpha_1^*, \theta) \neq \underline{\Lambda}(\alpha_1^*, \theta)$,
2. The elements in Ω are convex independent.

An '*elaboration*' of Γ is denoted by $\Gamma^\epsilon \equiv (\Omega^\epsilon, \mu^\epsilon, \phi^\epsilon)$, where μ^ϵ is the new joint distribution between the state and player 1's characteristics, $\phi^\epsilon = (\phi^\epsilon_{\alpha_1'})_{\alpha_1' \in \Omega^\epsilon}$ with $\phi^\epsilon_{\alpha_1'} \in \Delta(\Theta)$ and Ω^ϵ is a finite subset of $\Delta(A_1)$. Next I introduce the definition of *mixed ϵ-elaboration*:

Definition 1. *For every $\epsilon > 0$, $(\Omega^\epsilon, \mu^\epsilon, \phi^\epsilon)$ is a 'mixed ϵ-elaboration' of Γ with respect to a_1^* if:*

1. *For every $\alpha_1' \in \Omega^\epsilon$, $\alpha_1'(a_1^*) < 1$.*
2. *For every $\theta \in \Theta$, $\big|\mu(\theta) - \mu^\epsilon(\theta)\big| < \epsilon$.*
3. *$|\Omega^\epsilon| = |\Omega|$ and for every $\alpha_1 \in \Omega$, there exists $\alpha_1' \in \Omega^\epsilon$ such that:*

$$\big\|\alpha_1' - \alpha_1\big\| < \epsilon, \ \big|\mu(\alpha_1) - \mu^\epsilon(\alpha_1')\big| < \epsilon \textit{ and } \big\|\phi_{\alpha_1} - \phi^\epsilon_{\alpha_1'}\big\| < \epsilon.$$

Intuitively, in a mixed ϵ-elaboration of the original game, the commitment actions, the probabilities of commitment and strategic types are close to the original game. To embody the idea that the elaboration is mixed, no commitment type in Ω^ϵ plays a_1^* with probability 1. Let $\underline{V}_\theta(\delta, \mu^\epsilon)$ be type θ's lowest equilibrium payoff in Γ^ϵ. The next Proposition uncovers the discontinuity in player 1's lowest equilibrium payoff once we perturb a pure commitment action into a non-trivially mixed one:

Proposition A.2. *For every $\lambda \in$ int$\Big(\Lambda(a_1^*, \theta) \backslash \underline{\Lambda}(a_1^*, \theta)\Big)$, there exist $\overline{\epsilon} > 0$ and $\tau > 0$ such that for every $\epsilon \in (0, \overline{\epsilon})$ and every mixed ϵ-elaboration of the original game Γ^ϵ, we have:*

$$\lim \inf_{\delta \to 1} \underline{V}_\theta(\delta, \mu) > \tau + \lim \sup_{\delta \to 1} \underline{V}_\theta(\delta, \mu^\epsilon). \tag{A.5}$$

According to Proposition A.2, when $\lambda \in \text{int}\Big(\Lambda(a_1^*, \theta) \Big\backslash \underline{\Lambda}(a_1^*, \theta)\Big)$, even small perturbations to a pure commitment action can lead to a substantial decrease in player 1's lowest equilibrium payoff. This is in sharp contrast with Fudenberg and Levine (1989,1992) in which there is no discontinuity between pure and mixed commitment actions. A sufficient condition for this discontinuity is given by Proposition A.1.[2]

[2] Albeit Proposition A.1 has completely characterized the set of environments in which $\Lambda(\alpha_1^*, \theta) \neq \underline{\Lambda}(\alpha_1^*, \theta)$, but given that Theorem 1 in Pei (2018) only provides a lower bound on player 1's equilibrium payoff, discontinuity could occur even when $\Lambda(\alpha_1^*, \theta) = \underline{\Lambda}(\alpha_1^*, \theta)$. This is why (A.1) is only sufficient but may not be necessary for such a discontinuity.

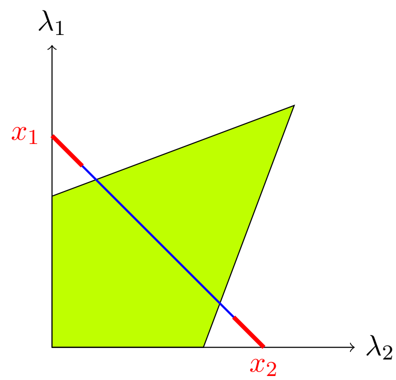

Figure 1: Set $\overline{\Lambda}(\alpha_1^*, \theta)$ in green. Set $\Lambda(M)$ is the solid blue line and Λ_0^ς is the red line. Proposition A.3 says that for any $\beta \in (0,1)$, the union of finite rounds of $(\beta, 1-\beta)$ convex combinations of Λ_0^ς can cover $\Lambda(M)$.

As a technical remark, Proposition A.2 only claims the discontinuity in the $\delta \to 1$ limit of type θ's lowest equilibrium payoff, i.e. the $\sup$ over all mixed ϵ-elaborations is taken *after* $\liminf_{\delta \to 1}$. Once we swap the order between $\sup$ and $\liminf_{\delta\to 1}$, the conclusion is no longer valid. This is because as $\epsilon \to 0$, the mixed commitment action places very low probability on actions other than a_1^* so that it will take a long time for μ_t to escape $\Lambda(a_1^*, \theta)$. Despite these periods have negligible impact on player 1's discounted average payoff when $\delta \to 1$, but the payoff consequences can be substantial for any fixed δ.

A.3 Distinction Between Pure and Mixed Commitment Payoffs

In this subsection, I state and show a result which constructs a T-period strategy for the bad strategic types under which player 2's posterior in period T will be bounded away from $\overline{\Lambda}(\alpha_1^*, \theta)$ regardless of player 1's action choices. The motivation for this is twofold. First, it highlights the distinction between pure and mixed commitment payoffs in terms of the set of feasible belief updating processes. Second, I will use this to construct low payoff equilibria in the proofs of Proposition A.2 and statement 4 of Theorem 1. Let $d(\cdot,\cdot)$ be the Hausdorff distance between sets, let $A_1^* \equiv \text{supp}(\alpha_1^*)$ and let $\mathcal{H}_*^t$ be the set of period t histories such that player 1 has played actions in A_1^* in every period from 0 to $t-1$. Let $\lambda(h^t)$ be the likelihood ratio vector induced by player 2's belief at history h^t. The result is stated below:

Proposition A.3. *For every $\varsigma > 0$, $\alpha_1^* \in \Delta(A_1)$ and $\lambda \in \underline{\Pi}(\alpha_1^*, \theta)$, there exist $T \in \mathbb{N}$ and player 1's strategy $\{\sigma_{\tilde{\theta}}\}_{\tilde{\theta} \neq \theta}$ such that for every $h^T \in \mathcal{H}_*^T$, we have $d\Big(\lambda(h^T), \overline{\Lambda}(\alpha_1^*, \theta)\Big) > \varsigma$.*

The contents of Proposition A.3 is illustrated graphically in Figure 1. To prove this result, notice that the conclusion is trivially true when $\lambda \notin \overline{\Lambda}(\alpha_1^*, \theta)$. Therefore, it is without loss of generality to assume that:

$$\lambda \in \overline{\Lambda}(\alpha_1^*, \theta) \Big\backslash \underline{\Lambda}(\alpha_1^*, \theta).$$

I consider strategies in which every strategic type $\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}$ plays actions in A_1^* in every period. Therefore,

$$\mathbb{E}[\lambda(h^{t+1})|h^t] = \lambda(h^t) \text{ for every } h^t \in \mathcal{H}^t_*.$$

Pick an arbitrary $a_1 \in A_1^*$ and let $\beta \equiv \alpha_1^*(a_1)$ for every $a_1 \in A_1$. Since α_1^* is mixed, we have $\beta \in (0,1)$. Consider strategies of player 1 such that:

$$\lambda(h^t, a_1') = \lambda(h^t, a_1'') \text{ for every } a_1', a_1'' \in A_1^* \backslash \{a_1^*\} \text{ and } h^t \in \mathcal{H}^t_*.$$

The martingale property of likelihood ratio vectors implies that

$$\beta\lambda(h^t, a_1) + (1-\beta)\lambda(h^t, a_1') = \lambda(h^t) \text{ for every } a_1' \in A_1^* \backslash \{a_1\}. \tag{A.6}$$

This translates into the following behavior strategy for strategic types $\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}$:

$$\sigma_{\tilde{\theta}}(h^t)[\tilde{a}_1] \equiv \begin{cases} \beta\lambda(h^t, a_1)[\tilde{\theta}]/\lambda(h^t)[\tilde{\theta}] & \text{if } \tilde{a}_1 = a_1 \\ \alpha_1^*(\tilde{a}_1)\lambda(h^t, \tilde{a}_1)[\tilde{\theta}]/\lambda(h^t)[\tilde{\theta}] & \text{if } \tilde{a}_1 \in A_1^* \backslash \{a_1\}. \end{cases}$$

Let ψ_i be the intersection of $\overline{\Lambda}(\alpha_1^*, \theta)$ with the i-th coordinate and let $M \equiv \sum_{i=1}^{k(\alpha_1^*, \theta)} \lambda_i/\psi_i$. The assumption that $\lambda \in \underline{\Pi}(\alpha_1^*, \theta)$ implies that $M > 1$. Let

$$\Lambda(M) \equiv \left\{\tilde{\lambda} \Big| \tilde{\lambda} \gg \mathbf{0} \text{ and } \sum_{i=1}^{k(\alpha_1^*, \theta)} \tilde{\lambda}_i/\psi_i = M\right\} \tag{A.7}$$

and let $\Psi(M)$ be the set of intersections of $\Lambda(M)$ on the coordinates. We know that $\text{co}(\Psi(M)) = \Lambda(M)$. Let

$$\Lambda_0^{\varsigma} \equiv \text{int}\Big(\big\{\lambda \in \Lambda(M) \big| d\big(\lambda, \overline{\Lambda}(\alpha_1^*, \theta)\big) > \varsigma\big\}\Big).$$

where $\text{int}(\cdot)$ denotes the *relative interior* on the $k(\alpha_1^*, \theta) - 1$ dimensional manifold. Given $M > 1$, when ς is small enough, we have $d\big(\Psi(M), \overline{\Lambda}(\alpha_1^*, \theta)\big) < \varsigma/2$, in which case $\Psi(M) \subset \Lambda_0^{\varsigma}$. So there exists $\overline{\varsigma} > 0$ such that for every $\varsigma \in (0, \overline{\varsigma})$, we have $\text{co}(\Psi(M)) = \text{co}(\Lambda_0^{\varsigma}) = \Lambda(M)$.

Take such a small enough $\varsigma > 0$. I define a sequence of sets $\{\Lambda_k^\varsigma\}_{k=1}^{+\infty}$ recursively as follows:

$$\Lambda_k^\varsigma \equiv \Big\{\lambda \Big| \text{ there exist } \lambda', \lambda'' \in \Lambda_{k-1}^\varsigma \text{ such that } \beta\lambda' + (1-\beta)\lambda'' = \lambda\Big\}. \tag{A.8}$$

By definition, $\Lambda_{k-1}^\varsigma \subset \Lambda_k^\varsigma$ for all $k \in \mathbb{N}$. Let $\Lambda^\varsigma \equiv \bigcup_{k=0}^{+\infty} \Lambda_k^\varsigma$. I show the following Lemma:

Lemma A.2. $\Lambda^\varsigma = \mathit{int}(\Lambda(M))$.

PROOF OF LEMMA A.2: By definition, $\Lambda^\varsigma \subset \text{int}(\Lambda(M))$. In what follows, I show that $\Lambda^\varsigma \supset \text{int}(\Lambda(M))$. Note that for every $\lambda^* \in \text{int}(\Lambda(M))$, according to the Carathéodory Theorem (Eckhoff 1993), there exists $X_0 \equiv \{\lambda^i\}_{i=1}^{k(\alpha_1^*,\theta)} \subset \Lambda_0^\varsigma$ such that $\lambda^* \in \text{int}(\text{co}(X_0))$. Define X_k with $k \in \mathbb{N}$ recursively according to:

$$X_k \equiv \Big\{\lambda \Big| \text{ there exist } \lambda', \lambda'' \in X_{k-1} \text{ such that } \beta\lambda' + (1-\beta)\lambda'' = \lambda\Big\}. \tag{A.9}$$

and let $X \equiv \cup_{k=0}^{\infty} X_k$. I show the following Lemma:

Lemma A.3. *X is a dense subset of $co(X_0)$.*

PROOF OF LEMMA A.3: Let $\Upsilon \equiv \text{int}\Big(\text{co}(X_0)\backslash X\Big)$. I show that $\Upsilon = \{\varnothing\}$. Suppose towards a contradiction that $\Upsilon \neq \{\varnothing\}$, then for every $\lambda \in \partial\Upsilon$ and every $\eta > 0$, there exists $\lambda^\eta \in \Lambda^\varsigma \cap B(\lambda, \eta)$. Since there exist $\lambda^* \in \Upsilon$ and $\lambda', \lambda'' \in \partial\Upsilon$ such that $\lambda^* = \beta\lambda' + (1-\beta)\lambda''$. We know that when η is small enough, there also exist $\lambda^{**} \in B(\lambda^*, \eta) \subset \Upsilon$ and $\hat{\lambda}, \tilde{\lambda} \in X$ such that $\lambda^{**} = \beta\hat{\lambda} + (1-\beta)\tilde{\lambda}$, leading to a contradiction. The above argument implies that $\Upsilon = \{\varnothing\}$, and therefore, $\text{co}(X_0)\backslash X$ is hallow so X is dense in $\text{co}(X_0)$. □

Back to the proof of Lemma A.2. If $\lambda^* \in X$, then since $X \subset \Lambda^\varsigma$, we have $\lambda^* \in \Lambda^\varsigma$. If $\lambda^* \notin X$, then for every $\eta > 0$, there exists λ' such that $\lambda' \in B(\lambda^*, \eta) \cap X$. By definition of X, there exists $K \in \mathbb{N}$ such that:

$$\lambda' = \sum_{i=1}^{k(\alpha_1^*,\theta)} \varrho(i)\lambda^i$$

where every $\varrho(i)$ can be written as the sum of terms in the form of $(1-\beta)^m\beta^n$ with $0 \leq m+n \leq K$, $m, n \geq 0$. Pick $\eta > 0$ small enough such that $B(\lambda^i, \eta) \subset \Lambda_0^\varsigma$ for every $i \in \{1, 2, ..., k(\alpha_1^*, \theta)\}$, then

$$\lambda^* = \sum_{i=1}^{k(\alpha_1^*,\theta)} \varrho(i)\Big(\lambda^i + (\lambda^* - \lambda')\Big). \tag{A.10}$$

According to (A.10), we know that $\lambda^* \in \Lambda_0^\varsigma$ as by construction, $\lambda^i + (\lambda^* - \lambda') \in \Lambda_0^\varsigma$ for every i. □

After knowing that $\Lambda^\varsigma = \text{int}(\Lambda(M))$, since $\text{int}(\Lambda(M))$ is compact and $\{\Lambda_k^\varsigma\}_{k=1}^{\infty}$ is an open cover, the Heine-Borel Finite Cover Theorem implies the existence of $T \in \mathbb{N}$ such that $\Lambda(M) \subset \bigcup_{k=0}^{T} \Lambda_k^\varsigma$.

A.4 Continuity Properties of Sets Λ and $\underline{\Lambda}$

I start with introducing some notation. For every $\lambda \notin \text{clo}\Big(\overline{\Lambda}(\alpha_1^*, \theta)\Big)$, depict the line segment connecting λ to the origin. Let λ' be the intersection between the line segment and $\partial\overline{\Lambda}(\alpha_1^*, \theta)\backslash\{\mathbf{0}\}$. Let

$$r(\lambda) \equiv d(\lambda', \lambda)\big/d(0, \lambda'), \tag{A.11}$$

where $d(\cdot, \cdot)$ denotes the Hausdorff distance. By definition, $r(\lambda) > 0$. Let

$$M \equiv \max_{(a_1,a_2,a_2')} |u_2(\theta, a_1, a_2) - u_2(\theta, a_1, a_2')|. \tag{A.12}$$

Let $\Gamma^\epsilon = (\Omega^\epsilon, \mu^\epsilon, \phi^\epsilon)$ be a generic mixed ϵ-elaboration. Abusing notation, let $\overline{\Lambda}^\epsilon$, Λ^ϵ and $\underline{\Lambda}^\epsilon$ be its best response set, saturation set and strong saturation set, respectively. I start with the following proposition:

Proposition A.4. *For every $\{\epsilon_n\}_{n=1}^\infty$ with $\epsilon_n > 0$ and $\lim_{n\to\infty} \epsilon_n = 0$, and every $\{\Gamma^{\epsilon_n}\}_{n=1}^\infty$ with Γ^{ϵ_n} being a mixed-ϵ_n elaboration, we have:*

$$\lim_{n\to\infty} clo\Big(\underline{\Lambda}^{\epsilon_n}(\alpha_1^n, \theta)\Big) = clo\Big(\underline{\Lambda}(a_1^*, \theta)\Big), \tag{A.13}$$

and

$$\lim_{n\to\infty} \overline{\Lambda}^{\epsilon_n}(\alpha_1^n, \theta) \subset clo\Big(\overline{\Lambda}(a_1^*, \theta)\Big),^{3} \tag{A.14}$$

where $\alpha_1^n \in \Omega^{m,\epsilon_n}$ satisfies $1 - \epsilon_n < \alpha_1^n(a_1^*) < 1$ and $\max\Big\{||\alpha_1^n - a_1^*||, ||\phi_{a_1^*} - \phi_{\alpha_1^n}^{\epsilon_n}||\Big\} < \epsilon_n$.

PROOF OF PROPOSITION A.4: I start with (A.13). For every $\theta' \in \Theta^b_{(a_1^*,\theta)}$, let $\psi^n(\theta')$ be the intercept of $\overline{\Lambda}^{\epsilon_n}(\alpha_1^n, \theta)$ on the $\lambda(\theta')$ axis. According to the definition of $\underline{\Lambda}^{\epsilon_n}(\alpha_1^n, \theta)$, (A.13) is implied by:

- $\triangleright$ $\lim_{n\to\infty} \psi^n(\theta') = \psi(\theta')$ for every $\theta' \in \Theta^b_{(a_1^*,\theta)}$.

Since $\{a_2^*\} = \text{BR}_2(a_1^*, \phi_{a_1^*}) = \text{BR}_2(a_1^*, \theta)$, we know that $\text{BR}_2(\alpha_1^n, \phi^{\epsilon_n}_{\alpha_1^n}) = \{a_2^*\}$ when ϵ_n is small enough. Moreover, since $a_2^* \notin \text{BR}_2(\theta', a_1^*)$ for every $\theta' \in \Theta^b_{(a_1^*,\theta)}$, $a_2^* \notin \text{BR}_2(\theta', \alpha_1)$ for every $\theta' \in \Theta^b_{(a_1^*,\theta)}$ and α_1 satisfying $\alpha_1(a_1^*) \geq 1 - \epsilon_n$ when ϵ_n is small enough.

It is without loss of generality to focus on small enough $\epsilon_n > 0$ that meets the above two requirements. For every $a_2 \neq a_2^*$ and $\theta' \in \Theta^b_{(a_1^*,\theta)}$, let

$$\Delta_+^n(a_2) \equiv u_2(\phi^{\epsilon_n}_{\alpha_1^n}, \alpha_1^n, a_2^*) - u_2(\phi^{\epsilon_n}_{\alpha_1^n}, \alpha_1^n, a_2) \text{ and } \Delta_-^n(a_2, \theta') \equiv u_2(\theta', \alpha_1^n, a_2^*) - u_2(\theta', \alpha_1^n, a_2).$$

[3] The converse $\lim_{n\to\infty} \text{clo}\Big(\overline{\Lambda}^{\Gamma^{\epsilon_n}}(\alpha_1^n, \theta)\Big) \supset \overline{\Lambda}(a_1^*, \theta)$ is also true, albeit it is not needed for the proof of Proposition A.2.

Since $\lim_{n\to\infty} \epsilon_n = 0$, we have:

$$\lim_{n\to\infty} \Delta_+^n(a_2) = u_2(\phi_{a_1^*}, a_1^*, a_2^*) - u_2(\phi_{a_1^*}, a_1^*, a_2) \text{ and } \lim_{n\to\infty} \Delta_-^n(a_2, \theta') = u_2(\theta', a_1^*, a_2^*) - u_2(\theta', a_1^*, a_2).$$

Recall that $\psi^n(\theta')$ is the largest $\psi \in (0,\infty)$ such that:

$$a_2^* \in \arg\max_{a_2 \in A_2} \Big\{ u_2(\phi_{\alpha_1^n}^{\epsilon_n}, \alpha_1^n, a_2) + \psi u_2(\theta', \alpha_1^n, a_2) \Big\},$$

which implies that:

$$\psi^n(\theta') = \min_{a_2 \neq a_2^*} \frac{\Delta_+^n(a_2)}{\Big| \min\{\Delta_-^n(a_2, \theta'), 0\} \Big|}.$$

Since $\theta' \in \Theta^b_{(a_1^*, \theta)}$, we have $\min\{\Delta_-^n(a_2, \theta'), 0\} < 0$ for some $a_2 \neq a_2^*$, which implies that:

$$\lim_{n\to\infty} \psi^n(\theta') = \lim_{n\to\infty} \min_{a_2 \neq a_2^*} \frac{\Delta_+^n(a_2)}{\Big| \min\{\Delta_-^n(a_2, \theta'), 0\} \Big|} = \min_{a_2 \neq a_2^*} \frac{u_2(\phi_{a_1^*}, a_1^*, a_2^*) - u_2(\phi_{a_1^*}, a_1^*, a_2)}{|\min\{u_2(\theta', a_1^*, a_2^*) - u_2(\theta', a_1^*, a_2), 0\}|} = \psi(\theta'). \tag{A.15}$$

This establishes (A.13).

Next, I show (A.14). Suppose there exists a converging sequence $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \in \overline{\Lambda}^{\epsilon_n}(\alpha_1^n, \theta)$ but

$$\lambda \equiv \lim_{n\to\infty} \lambda_n \notin \mathrm{clo}\Big(\overline{\Lambda}(a_1^*, \theta)\Big). \tag{A.16}$$

According to (A.11), there exists $a_2 \neq a_2^*$ such that

$$u_2(\phi_{a_1^*}, a_1^*, a_2) + \sum_{\theta' \in \Theta^b_{(a_1^*, \theta)}} \lambda(\theta') u_2(\theta', a_1^*, a_2) - u_2(\phi_{a_1^*}, a_1^*, a_2^*) - \sum_{\theta' \in \Theta^b_{(a_1^*, \theta)}} \lambda(\theta') u_2(\theta', a_1^*, a_2^*)$$

$$\geq \underbrace{r(\lambda)\Big(u_2(\phi_{a_1^*}, a_1^*, a_2^*) - u_2(\phi_{a_1^*}, a_1^*, a_2)\Big)}_{\equiv \eta \text{ which is strictly positive}}. \tag{A.17}$$

For any $n \in \mathbb{N}$ large enough such that $\epsilon_n < \eta/6M$ and $d(\lambda_n, \lambda) < \eta/3M$, we have:

$$u_2(\phi_{\alpha_1^n}, \alpha_1^n, a_2) + \sum_{\theta' \in \Theta^b_{(a_1^*, \theta)}} \lambda(\theta') u_2(\theta', \alpha_1^n, a_2) - u_2(\phi_{\alpha_1^n}, \alpha_1^n, a_2^*) - \sum_{\theta' \in \Theta^b_{(a_1^*, \theta)}} \lambda(\theta') u_2(\theta', \alpha_1^n, a_2^*) \geq \eta/3$$

contradicting the assumption that $\lambda_n \in \overline{\Lambda}^{\epsilon_n}(\alpha_1^n, \theta)$. $\square$

Implication of Proposition A.4: For every $\epsilon > 0$, let $\mathcal{G}^\epsilon$ be the set of mixed ϵ-elaborations. Let

$$\underline{\Lambda}^\epsilon(a_1^*, \theta) \equiv \bigcup_{\mathcal{G}^\epsilon} \underline{\Lambda}^\epsilon(\alpha_1^\epsilon, \theta), \tag{A.18}$$

where every α_1^ϵ belongs to some $\Omega^{m,\Gamma^\epsilon}$ and satisfies: $1-\epsilon < \alpha_1^\epsilon(a_1^*) < 1$ and $\max\Big\{||\alpha_1^\epsilon - a_1^*||, ||\phi_{a_1^*} - \phi^\epsilon_{\alpha_1^\epsilon}||\Big\} < \epsilon$. Proposition A.4 implies that for every $\lambda \in \text{int}\Big(\Lambda(a_1^*, \theta)\Big\backslash \underline{\Lambda}(a_1^*, \theta)\Big)$, there exists $\epsilon > 0$ such that:

$$d\Big(\lambda, \underline{\Lambda}^\epsilon(a_1^*, \theta)\Big) > \epsilon. \tag{A.19}$$

For every $\theta' \in \Theta^b_{(a_1^*, \theta)}$, let $\psi^\epsilon(\theta')$ be the intercept of $\underline{\Lambda}^\epsilon(a_1^*, \theta)$ on the $\lambda(\theta')$ axis. According to (A.18), $\psi^\epsilon(\theta')$ converges to $\psi(\theta')$ as $\epsilon \to 0$. For every λ satisfying (A.19), there exists $\gamma \in \mathbb{R}_+$ such that:

$$\sum_{\theta' \in \Theta^b_{(a_1^*, \theta)}} \lambda(\theta')/\psi^\epsilon(\theta') \geq 1 + \epsilon\gamma. \tag{A.20}$$

The RHS of (A.20) provides a lower bound on $\sum_{\theta' \in \Theta^b_{(a_1^*, \theta)}} \lambda(\theta')/\psi^n(\theta')$ for every $\epsilon_n < \epsilon$. The above analysis implies the following Proposition:

Proposition A.5. *For every ϵ small enough, there exists $\xi > 0$ such that for every λ with $\lambda(\theta') < \xi$ for all $\theta' \in \Theta^b_{(a_1^*, \theta)}$ except for one, if λ satisfies (A.19) then $r(\lambda) > \epsilon\gamma/2$ under every mixed ϵ-elaboration Γ^ϵ.*

PROOF OF PROPOSITION A.5: Consider the set of likelihood ratio vectors λ satisfying (A.20). Suppose $\lambda(\tilde{\theta}) = 0$ for every $\tilde{\theta} \in \Theta^b_{(a_1^*, \theta)} \backslash \{\theta'\}$, then $r(\lambda) \geq \epsilon\gamma$. Since $r(\lambda)$ is locally continuous in λ, there exists $\xi > 0$ such that every λ satisfying:

- $\triangleright$ $\lambda(\tilde{\theta}) < \xi$ for every $\tilde{\theta} \in \Theta^b_{(a_1^*, \theta)} \backslash \{\theta'\}$.

Therefore, we have $r(\lambda) > \epsilon\gamma/2$. □

A.5 Upper Bound on Abnormal Phase Payoff

In this subsection, I derive a uniform upper bound on type θ's continuation payoff after he has deviated from his equilibrium strategy, no matter when and how he has deviated. The construction modifies the one in Appendix B while allowing for the target commitment action to be mixed. Throughout this subsection, let $u_1(\theta', a_1, a_2) = 0$ for all $\theta' \neq \theta$ or $a_2 \neq a_2^*$. Let $A_1 \equiv \{a_1^1, ..., a_1^n\}$ and let $v^i \equiv u_1(\theta, a_1^i, a_2^*)$ for $i \in \{1, 2, ..., n\}$ with $v^i > 0$ for all i. Without loss of generality, I assume that $v^1 \geq v^2 \geq ... \geq v^n$. Let $\mathbf{v} \equiv (v^1, ..., v^n) \in \mathbb{R}^n$.

The main result of this subsection is an upper bound on type θ's payoff at histories where player 2's posterior belief $\widetilde{\mu}$ satisfies the following conditions:

- $\triangleright$ $\widetilde{\mu}$ attaches zero probability to θ.

- $\triangleright$ The distribution over strategic types is such that there exists $a_2' \neq a_2^*$ such that:

$$\widetilde{\mu}(\alpha_1^*)\Big(u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2') - u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2^*)\Big) + \sum_{\tilde{\theta}} \widetilde{\mu}(\tilde{\theta})\Big(u_2(\tilde{\theta}, \alpha_1^*, a_2') - u_2(\tilde{\theta}, \alpha_1^*, a_2^*)\Big) > \varsigma. \tag{A.21}$$

- $\triangleright$ There exists $\eta > 0$ such that for every $\alpha_1 \in \tilde{\Omega} \equiv \Omega \backslash \{\alpha_1^*\}$ such that $\widetilde{\mu}(\alpha_1) > 0$, we have:

$$v^* \equiv \alpha_1^* \cdot \mathbf{v} > \eta + \alpha_1 \cdot \mathbf{v} \tag{A.22}$$

  with "$\cdot$" being the inner product between two vectors in $\mathbb{R}^n$.

Let $\mu^* \equiv \mu(\tilde{\Omega})$ and let $l \equiv |\tilde{\Omega}|$. The result is stated below:

Proposition A.6. *For every μ satisfying the above requirement, there exist $\sigma_1 \equiv \{\sigma_{\tilde{\theta}}\}_{\tilde{\theta} \neq \theta}$ and σ_2 such that:*

1. *σ_2 is optimal for player 2 given σ_1 and μ at every on-path history.*

2. *Type θ's continuation payoff at such belief is no more than:*

$$v^* - \eta + \varrho(\delta, \mu^*, \varsigma), \tag{A.23}$$

   such that for every fixed $(\mu^*, \varsigma) \in [0, 1) \times (0, +\infty)$, we have $\lim_{\delta \to 1} \varrho(\delta, \mu^*, \varsigma) = 0$.

If $l = 0$, then all the strategic types play α_1^* in every period with probability 1, player 2's belief about player 1's type remains constant along the equilibrium path and moreover, a_2^* is strictly dominated in every period according to (A.21). Therefore, type θ's equilibrium payoff is 0.

If $l \neq 0$, the proof follows along the same line as the abnormal phase construction in the proof of statement 2 (Appendix B in Pei 2018). Let $p \in (0, 1)$ be chosen such that:

$$\mu(\alpha_1^*)\Big(u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2') - u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2^*)\Big) + p \sum_{\tilde{\theta}} \mu(\tilde{\theta})\Big(u_2(\tilde{\theta}, \alpha_1^*, a_2') - u_2(\tilde{\theta}, \alpha_1^*, a_2^*)\Big) = \varsigma/2,$$

where a_2' is the same as (A.21). According to (A.21), such p exists. The strategic types play α_1^* in every period with probability p, and adopts non-stationary strategy $\sigma(\alpha_1)$ with probability $(1-p)/l$ for every $\alpha_1 \in \tilde{\Omega}$. I use $\theta(\alpha_1)$ to denote the strategic type who plays $\sigma(\alpha_1)$.

In what follows, I establish the existence of $\sigma(\alpha_1)$ under which type θ's payoff is bounded from above by (A.23). Let μ_t be the belief in period t with $\mu_0 \equiv \mu$. Let

$$\beta_t(\alpha_1) \equiv \mu_t\big(\theta(\alpha_1)\big)/\mu_t(\alpha_1) \text{ and } \beta(\alpha_1) \equiv \mu\big(\theta(\alpha_1)\big)/\mu(\alpha_1).$$

I will be keeping track of the l-dimensional likelihood ratio vector $\{\beta_t(\alpha_1)\}_{\alpha_1 \in \tilde{\Omega}}$. First, for small enough $\varepsilon > 0$, there exists $\alpha_1^\varepsilon \in \Delta(A_1)$ such that $\alpha_1^\varepsilon(a_1) > \varepsilon$ for all $a_1 \in A_1$ and

$$\sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*,\theta)}} \mu(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^\varepsilon, a_2') > \sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*,\theta)}} \mu(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^\varepsilon, a_2^*).$$

For every α_1, let

$$\overline{\beta}(\alpha_1) \equiv \inf\Big\{\beta \in \mathbb{R}_+ \Big| \mu(\alpha_1) u_2(\phi_{\alpha_1}, \alpha_1, a_2') + \beta \sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*,\theta)}} \mu(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^\varepsilon, a_2')$$

$$> \mu(\alpha_1) u_2(\phi_{\alpha_1}, \alpha_1, a_2^*) + \beta \sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*,\theta)}} \mu(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^\varepsilon, a_2^*)\Big\}. \quad \text{(A.24)}$$

By definition, $\overline{\beta}(\alpha_1) \in (0, \infty)$. Next, I describe $\sigma(\alpha_1)$.

1. If $\beta_t(\alpha_1) > \overline{\beta}(\alpha_1)$ for all $\alpha_1 \in \tilde{\Omega}$, then type $\theta(\alpha_1)$ plays α_1^ε for every $\alpha_1 \in \tilde{\Omega}$. Apparently, a_2^* is strictly dominated by a_2' in period t. Therefore, type θ's flow payoff is 0 in this period.
2. If $\beta_t(\alpha_1) \leq \overline{\beta}(\alpha_1)$ for some $\alpha_1 \in \tilde{\Omega}$, type $\theta(\alpha_1)$ plays mixed strategy $\breve{\alpha}_1(\alpha_1) \in \Delta(A_1)$ for every $\alpha_1 \in \tilde{\Omega}$, which will be described below. Abusing notation, I write $\breve{\alpha}_1$ instead of $\breve{\alpha}_1(\alpha_1)$ for simplicity.

Next, I specify $\breve{\alpha}_1(\alpha_1)$. For every constant $\kappa \in (0,1)$, let

$$G^\kappa \equiv \{i | v^i > v^* - \kappa\eta\} \text{ and } B^\kappa \equiv \{j | v^j \leq v^* - \kappa\eta\}.$$

By construction, G^κ and B^κ are non-empty, and $\{G^\kappa, B^\kappa\}$ is a partition of A_1. For every $i \in G^\kappa$ and $j \in B^\kappa$, let $\beta^\kappa(i,j) \in [0,1]$ be defined as:

$$\beta^\kappa(i,j) v^i + (1 - \beta^\kappa(i,j)) v^j = v^* - \kappa\eta. \quad \text{(A.25)}$$

I construct $\breve{\alpha}_1$ for every $\alpha_1 \in \tilde{\Omega}$ in the next Lemma:

Lemma A.4. *For every $\alpha_1 \in \tilde{\Omega}$, there exists $\breve{\alpha}_1 \in \Delta(A_1)$ such that for every $i \in G^\kappa$ and $j \in B^\kappa$, $\breve{\alpha}_1(a_1^i) >$*

$\alpha(a_1^i)$ *and*

$$\Big(\frac{\breve{\alpha}_1(a_1^i)}{\alpha_1(a_1^i)}\Big)^{\beta^\kappa(i,j)}\Big(\frac{\breve{\alpha}_1(a_1^j)}{\alpha_1(a_1^j)}\Big)^{1-\beta^\kappa(i,j)} > 1. \tag{A.26}$$

PROOF OF LEMMA A.4: For every $\iota \in \mathbb{R}_+$ and $\alpha_1(i), \alpha_1(j) \in (0,1)$, define the following function of $\varepsilon > 0$:

$$f\big(\varepsilon\big|\iota, \alpha_1(i), \alpha_1(j)\big) \equiv \big(\alpha_1(i)+\varepsilon\big)^\beta\big(\alpha_1(j)-\iota\varepsilon\big)^{1-\beta}.$$

Expanding f around $\varepsilon = 0$, we obtain:

$$f\big(\varepsilon\big|\iota, \alpha_1(i), \alpha_1(j)\big) = \alpha_1(i)^\beta\alpha_1(j)^{1-\beta} + \Big(\underbrace{\beta\alpha_1(i)^{\beta-1}\alpha_1(j)^{1-\beta} - \iota(1-\beta)\alpha_1(i)^\beta\alpha_1(j)^{-\beta}}\Big)\varepsilon + \mathcal{O}(\varepsilon^2)$$

The term in the bracket is strictly positive if and only if:

$$\iota < \frac{\beta}{1-\beta}\frac{\alpha_1(j)}{\alpha_1(i)}. \tag{A.27}$$

For every $i \in G^\kappa$ and $j \in B^\kappa$, replace β with $\beta^\kappa(i,j)$, and $\alpha_1(i), \alpha_1(j)$ with $\alpha_1(a_1^i)$ and $\alpha_1(a_1^j)$, we can define $\iota(i,j)$ analogously. According to (A.25),

$$\beta^\kappa(i,j) = \frac{v^* - \kappa\eta - v^j}{v^i - v^j}$$

Plugging the above expression into (A.27), we have:

$$\iota(i,j) < \frac{\alpha_1(a_1^j)}{\alpha_1(a_1^i)}\frac{v^* - \kappa\eta - v^j}{v^i - (v^* - \kappa\eta)}.$$

For some $\zeta > 0$ small enough, let

$$\breve{\alpha}_1(a_1^i) \equiv \alpha_1(a_1^i) + \zeta\alpha_1(a_1^i)\Big[v^i - (v^* - \kappa\eta)\Big] \tag{A.28}$$

for every $i \in G^\kappa$, and let

$$\breve{\alpha}_1(a_1^j) \equiv \alpha_1(a_1^j) - \zeta\alpha_1(a_1^j)\Big[(v^* - \kappa\eta) - v^j\Big] + \zeta(1-\kappa)\eta \tag{A.29}$$

for every $j \in B^\kappa$. We can verify that first,

$$\frac{\alpha_1(a_1^j) - \breve{\alpha}_1(a_1^j)}{\breve{\alpha}_1(a_1^i) - \alpha_1(a_1^i)} < \iota(i,j)$$

for all $i \in G^\kappa$ and $j \in B^\kappa$, and hence, inequality (A.26) holds when ζ is small enough. Second, $\breve{\alpha}_1(a_1^i) > \alpha_1(a_1^i)$ for all $i \in G^\kappa$. Third,

$$\sum_{i\in G^\kappa} \breve{\alpha}_1(a_1^i) + \sum_{j\in B^\kappa} \breve{\alpha}_1(a_1^j) = \sum_{i\in G^\kappa} \alpha_1(a_1^i) + \sum_{j\in B^\kappa} \alpha_1(a_1^j) = 1,$$

which guarantees that the constructed $\breve{\alpha}_1$ is indeed a probability measure. $\square$

To understand the intuition behind the constructed $\breve{\alpha}_1$ in Lemma A.4. Player 1's action is classified into '*good*' and '*bad*' actions. Type θ can obtain a stage game payoff no less than $v^* - \kappa\eta$ if and only if he plays an action in G^κ and player 2 best responds by playing a_2^*.

- ▷ By definition of $\overline{\beta}(\alpha_1)$, a_2^* is not a best respond when $\beta_t(\alpha_1) > \overline{\beta}(\alpha_1)$ for all $\alpha_1 \in \tilde{\Omega}$.

- ▷ When $\beta_t(\alpha_1) \leq \overline{\beta}(\alpha_1)$ for some $\alpha_1 \in \tilde{\Omega}$, the constructed $\breve{\alpha}_1$ implies that $\beta_{t+1}(\alpha_1) > \beta_t(\alpha_1)$ if $a_{1,t} \in G^\kappa$. Moreover, there exists a constant $\chi > 0$ such that $\beta_{t+1}(\alpha_1) \geq \chi\beta_t(\alpha_1)$ for all $a_1 \in A_1$.

  In another word, in every period such that type θ obtains flow payoff no less than $v^* - \kappa\eta$, the likelihood ratio $\beta_t(\alpha_1)$ increases. Since $\breve{\alpha}_1(a_1)$ is bounded from below for every $a_1 \in A_1$, $\beta_t(\alpha_1)$ will not decline too fast even when actions in B^κ are being played. Once $\beta_t(\alpha_1) > \overline{\beta}(\alpha_1)$ for all $\alpha_1 \in \tilde{\Omega}$, a_2^* is strictly dominated by a_2' and type θ will obtain a low stage game payoff in that period.

- ▷ Equation (A.26) ensures that when δ is close enough to 1, type θ can obtain payoff no more than $v^* - \kappa\eta$ while keeping at least one $\beta_t(\alpha_1)$ below its cutoff, $\overline{\beta}(\alpha_1)$.

  To see why, for every $\alpha_1 \in \tilde{\Omega}$, let
  $$r(a_1^i|\alpha_1) \equiv \frac{\breve{\alpha}_1(a_1^i)}{\alpha_1(a_1^i)}.$$

  Consider the following constraint optimization problem:

  $$\max_{\alpha_1\in\Delta(A_1)} \sum_{i=1}^{n} \alpha(a_1^i)(v^i - v^* + \kappa\eta),$$

  subject to:
  $$\min_{\alpha_1\in\tilde{\Omega}} \Big\{ \sum_{i=1}^{n} \alpha(a_1^i) \log r(a_1^i|\alpha_1) \Big\} \leq 0$$

  If the value of this program is non-negative, then there exists at least one $\alpha_1 \in \tilde{\Omega}$ such that

  $$\sum_{i=1}^{n} \alpha(a_1^i) \log r(a_1^i|\alpha_1) \leq 0$$

at the optimum. Focusing on a revised program with the same objective but just the above inequality constraint. This is a relaxed program of the original one. Since the objective function and the constraint both are linear, there exists an optimum in which there exists at most two a_1^i such that $\alpha(a_1^i) > 0$, i.e.

$$\arg\max_{i \in G^\kappa} \left| \frac{v^i - v^* + \kappa\eta}{\log r(a_1^i|\alpha_1)} \right| \quad \text{and} \quad \arg\min_{i \in B^\kappa} \left| \frac{v^i - v^* + \kappa\eta}{\log r(a_1^i|\alpha_1)} \right|.$$

Applying the duality theorem, inequality (A.26) then implies that the value of this program must be strictly negative.

▷ Let

$$K \equiv \left\lceil \frac{-\log \varepsilon}{\min_{a_1 \in G^\kappa} \log \frac{\breve{\alpha}_1(a_1)}{\alpha_1(a_1)}} \right\rceil + 1,$$

and when δ close enough to 1, choose M large enough such that

$$Kv^1 < (K+1)M. \tag{A.30}$$

The above inequality puts an upper bound on type θ's payoff and ensures that he cannot get more than $v^* - \kappa\eta$ by choosing actions in G^κ too frequently such that $\beta_t(\alpha_1)$ exceeds $\overline{\beta}(\alpha_1)$.

Therefore, under the constructed σ, type θ's highest continuation payoff in the abnormal phase is bounded below $v^* - \kappa\eta$ when δ is large enough. Since κ can take any value between 0 and 1, the bound in (A.23) is established in the $\delta \to 1$ limit. Moreover, according to our construction, κ only depends on δ and $\{\beta(\alpha_1)\}_{\alpha_1 \in \tilde{\Omega}}$, and the latter only depend on μ^* and ς.

A.6 Proof of Proposition A.2

Proposition A.2 is only meaningful when $\lambda \in \text{int}\Big(\Lambda(a_1^*, \theta) \Big\backslash \underline{\Lambda}(a_1^*, \theta)\Big)$, which I assume throughout this subsection. For ϵ small enough, I construct sequential equilibrium for every mixed ϵ-elaboration under which type θ player 1 receives significantly lower payoff than his payoff lower bound in the original game. Throughout this subsection, let u_1 be given as in the proof of statement 2, Theorem 1. I construct sequential equilibria for high enough δ, in which type θ's equilibrium payoff is strictly bounded below $u_1(\theta, a_1^*, a_2^*)$, which equals to 1.

Since $\lambda \in \text{int}\Big(\Lambda(a_1^*, \theta) \Big\backslash \underline{\Lambda}(a_1^*, \theta)\Big)$, there exists $M > 1$ such that $\lambda \in \Lambda(M)$. For every mixed ϵ-elaboration Γ^ϵ, according to Proposition A.4, there exists $\nu : (0, +\infty) \to (0, +\infty)$ with $\lim_{\epsilon \to 0} \nu(\epsilon) = 0$ such that $\lambda^\epsilon \in \Lambda^\epsilon(M^\epsilon)$ with $|M^\epsilon - M| \leq \nu(\epsilon)$. Pick ϵ small enough such that $\nu(\epsilon) < (M-1)/2$. Let $\tilde{M}^\epsilon \equiv \beta M^\epsilon + (1-\beta)$

with $\beta \in (0,1)$ specified later. There exists $\lambda^* \ll \lambda^\epsilon$ such that $\lambda^* \in \Lambda^\epsilon(\tilde{M}^\epsilon)$ and

$$\sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*,\theta)}} (\lambda^\epsilon(\tilde{\theta}) - \lambda^*(\tilde{\theta}))\Big(u_2(\tilde{\theta}, \alpha_1^*, a_2') - u_2(\tilde{\theta}, \alpha_1^*, a_2^*)\Big) > 0 \tag{A.31}$$

for some $a_2' \neq a_2^*$. Let β be close enough to 1 such that for every $\tilde{\lambda} \in \Lambda^\epsilon(\tilde{M}^\epsilon)$ with $d(\tilde{\lambda}, \overline{\Lambda}^\epsilon) > \varsigma$, we have:

$$d\Big(\tilde{\lambda} + (\lambda^\epsilon - \lambda^*), \overline{\Lambda}^\epsilon\Big) > \varsigma/2.$$

Next, let us consider the following strategy profile in game Γ^ϵ:

- $\triangleright$ **Strategic Type θ:** Plays a_1^* from period 0 to $T-1$. Starting from period T, plays a_1^* with probability $1-\eta/2$ and plays $a_1 \neq a_1^*$ with probability $\eta/2$, depending on the realization of the public randomization device.[4]
- $\triangleright$ **Other Good Strategic Types:** Play a commitment strategy other than α_1^*, as in the proof of statement 2, Theorem 1.
- $\triangleright$ **Bad Strategic Types:**
  - From period 0 to $T-1$, type $\tilde{\theta}$ plays σ^* with probability $\lambda^*(\tilde{\theta})/\lambda^\epsilon(\tilde{\theta})$; plays $\hat{\sigma}(\alpha_1)$ with probability $\Big(\lambda^\epsilon(\tilde{\theta}) - \lambda^*(\tilde{\theta})\Big)\Big/ l\lambda^\epsilon(\tilde{\theta})$ for every $\alpha_1 \in \tilde{\Omega}$, where $\lambda^* \in \mathbb{R}^k$ is defined in (A.31).
  - In the beginning of period T, compute the likelihood ratio vector of all the bad strategic types and the commitment types, denoted by $\lambda(h^T)$ and plays $\breve{\sigma}(\lambda(h^T))$.[5]

In what follows, I describe σ^*, $\hat{\sigma}(\alpha_1)$ and $\breve{\sigma}(\alpha_1)$.

- $\triangleright$ $\boldsymbol{\sigma^*}$: Consider prior belief $\tilde{\lambda}^\epsilon$. Since $\tilde{\lambda}^\epsilon \in \Lambda^\epsilon(\tilde{M}^\epsilon)$ with $\tilde{M}^\epsilon > 1$, according to Proposition A.3, there exists $\{\sigma_{\tilde{\theta}}\}_{\tilde{\theta} \neq \theta}$ and $T \in \mathbb{N}$ such that $d\Big(\lambda(h^T), \overline{\Lambda}^\epsilon(\alpha_1^*, \theta)\Big) > \varsigma$ for every h^T consisting of actions in $\text{supp}(\alpha_1^*)$. Under σ^*, type $\tilde{\theta}$ plays according to $\sigma_{\tilde{\theta}}$ from period 0 to $T-1$.
- $\triangleright$ $\boldsymbol{\hat{\sigma}(\alpha_1)}$: Player 1 plays α_1 from period 0 to $T-1$.
- $\triangleright$ $\boldsymbol{\breve{\sigma}(\lambda(h^T))}$: Suppose an action $a_1 \notin A_1^*$ has occurred in h^T, then every strategic type in $\Theta^b_{(\alpha_1^*,\theta)}$ plays $\sigma(\alpha_1)$ starting from period T, where $\sigma(\alpha_1)$ is constructed in Proposition A.6.

[4]This step uses the definition of mixed ϵ-elaboration, i.e. there is no pure strategy commitment type that is always playing a_1^* in Γ^ϵ. Therefore, if play remains in the normal phase in period T, player 2 has ruled out all pure strategy commitment types.

[5]If player 2 has ruled out commitment type α_1^* by period T, then let $\lambda(h^T) = (\infty, \infty, ..., \infty)$.

Suppose all actions played from period 0 to $T-1$ are in A_1^*, according to the construction of β or equivalently $\tilde{M}^\epsilon$, we have $d(\lambda(h^T), \overline{\Lambda}^\epsilon) > \varsigma/2$. There exists $a_2(h^T) \neq a_2^*$ such that:

$$\sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}} \lambda(h^T)(\tilde{\theta}) \Big(u_2(\tilde{\theta}, \alpha_1^*, a_2(h^T)) - u_2(\tilde{\theta}, \alpha_1^*, a_2^*) \Big) > 0$$

Let $p(h^T) \in (0,1)$ be chosen such that

$$\Big(u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2(h^T)) - u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2^*) \Big) + p(h^T) \sum_{\tilde{\theta} \in \Theta^b_{(\alpha_1^*, \theta)}} \lambda(h^T)(\tilde{\theta}) \Big(u_2(\tilde{\theta}, \alpha_1^*, a_2(h^T)) - u_2(\tilde{\theta}, \alpha_1^*, a_2^*) \Big) > 0$$

Type $\tilde{\theta}$ plays α_1^* in every period with probability $p(h^T)$ and with probability $(1-p(h^T))/l$, plays $\sigma^{a_2(h^T)}(\alpha_1)$ for every $\alpha_1 \in \tilde{\Omega}$, where $\sigma^{a_2(h^T)}(\alpha_1)$ is the strategy $\sigma(\alpha_1)$ constructed in Proposition A.6 applying to $a_2(h^T)$ instead of a_2'.

Play is in the *normal phase* in period t if h^t is consistent with type θ's equilibrium play and in the *abnormal phase* otherwise. According to Proposition A.6, type θ's continuation payoff at the start of the abnormal phase is bounded from above by $(1-\delta^T) + \delta^T \Big(v^* - \eta + \varrho(\delta, \mu^*, \varsigma) \Big)$. By construction, μ^* is bounded from above and ς is bounded from below, so therefore, when δ is large enough, his continuation payoff at the start of the abnormal phase cannot exceed $v^* - 3\eta/4$. Type θ's continuation payoff at the normal phase is at least $\delta^T(1-\eta/2)$, which is strictly greater than $v^* - 3\eta/4$ when δ is large enough. This verifies incentive compatibility for type θ. The other types' incentive constraints are trivial, and therefore, the constructed strategy profile is a Nash Equilibrium. One can also verify that this strategy profile together with the belief system it induces constitute a sequential equilibrium.

In the above equilibrium, type θ's equilibrium payoff is at most $(1-\delta^T) + \delta^T(1-\eta/2)$, which is strictly lower than $1 - \eta/4$ when δ is close enough to 1. Since η is uniformly bounded from below across all mixed ϵ-elaborations, this establishes Proposition A.2.

A.7 Proof of Statement 4 Theorem 1

I modify the construction in the previous subsection to show statement 4. The *abnormal phase* construction remains the same as before. The *normal phase* is different because for a generic $\alpha_1^* \in \Delta(A_1)$, there is no guarantee that there exists $a_1 \in A_1$ such that

$$\text{BR}_2(\theta, a_1) = \text{BR}_2(\theta, \alpha_1^*) = \{a_2^*\}.$$

Therefore, we need to modify the *normal phase* construction and type θ's payoff function in order to make sure that type θ receives strictly higher payoff than the upper bound on the continuation payoff in the beginning of the *abnormal phase*. The main idea is to choose $M \in \mathbb{N}$ for every δ such that all bad strategic types are separated from type θ in period M, thus giving type θ a high continuation payoff.

Player 1's Payoff Function: Let $A_1 \equiv \{a_1^1, ..., a_1^n\}$ and let $A_1^* \equiv \text{supp}(\alpha_1^*) = \{a_1^1, ..., a_1^m\}$ with $2 \leq m \leq n$. Recall the definition of a_2'. Since $\{a_2^*\} \equiv \text{BR}_2(\theta, \alpha_1^*)$, there exists $1 \leq j \leq m$ such that $a_2' \notin \text{BR}_2(\theta, a_1^j)$. Let type θ's payoff function be given as:

$$u_1(\theta, a_1^i, a_2) = \begin{cases} v^i & \text{if } a_2 = a_2^* \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbf{v} = (v^1, v^2, ..., v^m, 0, 0..., 0) \in \mathbb{R}^n$ has the following properties:

- $v^i > 0$ for all $1 \leq i \leq m$.
- $v^* \equiv \alpha_1^* \cdot \mathbf{v} > \eta + \alpha_1 \cdot \mathbf{v}$ for every $\alpha_1 \in \tilde{\Omega}$.

The existence of such $\mathbf{v}$ and η follows directly from the separating hyperplane theorem using the fact that $\alpha_1^* \notin \text{co}(\tilde{\Omega})$. Moreover, one can verify that $v^* - \eta/2 > \eta/2 > 0$.

Modified Normal Phase Construction: In what follows, I describe type θ's strategy in the normal phase, i.e. one in which he has not deviated from his equilibrium strategy before. His equilibrium strategy is pure from period 0 to $M_\delta \in \mathbb{N}$, which is a large integer to be specified later.

- In period $t \in \{0, 1, ..., m-1\}$, plays a_1^{t-1}.
- From period m to period M_δ, plays a_1^j with $j \leq m$ and $v^j < v^* - \eta/4$.
- Starting from period $M_\delta + 1$, he plays any equilibrium strategy in the continuation game.

The strategic types in $\Theta^b_{(\alpha_1^*, \theta)}$ plays according to the construction in the previous subsection from period 0 to $M_\delta - 1$. They play $a_1^i \neq a_1^j$ for sure in period M_δ. Play starts from the normal phase, and remains in it as long as player 1's history of actions being consistent with type θ's equilibrium strategy up to period M_δ, and transits to the abnormal phase otherwise. Players' abnormal phase strategy is described as in the previous subsection, and according to Proposition A.6, type θ's continuation payoff at the beginning of the abnormal phase cannot exceed $v^* - 3\eta/4$ when δ is large enough.

Next I show how to compute M_δ for δ close enough to 1. For every $M \in \mathbb{N}$ and $\delta \in (0, 1)$, let $V(M, \delta)$ be the *set* of continuation payoffs (starting from period $M + 1$) for type θ conditional on:

- The period M history is consistent with type θ's equilibrium strategy.

Since there is a public randomization device, $V(M,\delta)$ is a convex subset of $[0,v^1]$ for every M. Under this assessment, player 2's posterior belief in period $M+1$ attaches probability 0 to all types in $\Theta^b_{(\alpha_1^*,\theta)}$.

For every $0 \leq t \leq M$, let u_t be type θ's expected payoff in period t conditional on staying in the normal phase. By definition, $u_t \leq v^j < v^* - \kappa\eta$. According to statement 3 of theorem 1, when $M=1$, there exists $\overline{\delta}$ such that for all $v \in V(1,\delta)$ and $\delta > \overline{\delta}$,

$$v > v^* - \frac{\eta}{4}$$

and given δ is large enough,

$$(1-\delta)v^n + \delta v > v^* - \frac{\eta}{4}.$$

Moreover, for any δ, there exists $\overline{M}_\delta$ such that for all $M > \overline{M}$,

$$(1-\delta)\sum_{t=0}^{M}\delta^t v^j + \delta^{T+1}v^1 < v^* - \frac{3\eta}{4}.$$

Therefore, for every $\delta > \overline{\delta}$, either one of the two circumstances will occur:

1. There exists $M_\delta \in [0,\overline{M}]$ such that $[v^* - 3\eta/4, v^* - \eta/4] \cap V(M_\delta,\delta) \neq \{\varnothing\}$,
2. Or there exists $M_\delta \in [0,\overline{M}]$ such that $v > v^* - \eta/4$ for every $v \in V(M_\delta,\delta)$, and $v' < v^* - 3\eta/4$ for every $v' \in V(M_\delta+1,\delta)$.

I show that the second situation cannot occur for large enough δ. First, conditional on δ, $V(M,\delta)$ only depends on player 2's belief at h^M where h^M is the period M history that is consistent with type θ's equilibrium strategy. When all bad strategic types separate in period M, player 2's posterior belief is

$$\{\lambda(h^M)[\theta]/\alpha_1^*(a_1^j), 0, \lambda(h^M)[\alpha_1]\alpha_1(a_1^j)/\alpha_1^*(a_1^j)\}$$

His belief after period $M+1$ if type θ plays his equilibrium action a_1^* is:

$$\{\lambda(h^M)[\theta]/\alpha_1^*(a_1^j)\alpha_1^*(a_1^*), 0, \lambda(h^M)[\alpha_1]\alpha_1(a_1^j)\alpha_1(a_1^*)/\alpha_1^*(a_1^j)\alpha_1^*(a_1^*)\}$$

This is the same if type θ plays a_1^* in period M and a_1^j in period $M+1$ while the bad strategic types separate in period $M+1$. Therefore, the differences in continuation payoffs under these strategies cannot exceed $(1-\delta^2)v^1$, which contradicts the assumption that the payoff difference between these strategies is at least $\eta/2$.

B Extensions & Generalizations of Theorem 1

In this Appendix, I generalize Theorem 1 in Pei (2018) and extend its conclusions to situations in which the conditions in Theorem 1 do not apply. This includes, for example,

1. when $\mathrm{BR}_2(\alpha_1^*, \theta)$ is not a singleton,

2. when the probability of type α_1^* is too low or the probability $\phi_{\alpha_1^*}$ attaches to θ is too low such that $\lambda \notin \Lambda(a_1^*, \theta)$ when the commitment action is pure or $\lambda \notin \underline{\Lambda}(\alpha_1^*, \theta)$ when the commitment action is mixed.

The resulting payoff lower bound coincides with Theorem 1 when its requirements are met but is lower otherwise. Nevertheless, this new payoff lower bound is still informative and can rule out bad payoffs in many games, which is demonstrated by examples.

B.1 A Motivating Example

I start with an example in which despite the condition in Theorem 1 fails, reputation effects still lead to informative payoff lower bounds on the long-run player's payoff. Let $\Theta = \{\theta, \theta'\}$ and players' payoffs are given by:

θ	a_2^*	a_2^{**}	a_2'
a_1^*	$2, 3$	$1, 2$	$0, 0$
a_1'	$0, -1$	$0, -1$	$0, 0$

θ'	a_2^*	a_2^{**}	a_2'
a_1^*	$0, -1$	$0, 2$	$0, 3$
a_1'	$0, -1$	$0, -1$	$0, 0$

In the complete information game when the state is θ, there are two NEs in the stage game: (a_1^*, a_2^*) and (a_1', a_2'). One can apply the folk theorem result in Fudenberg et al.(1990) and show that $[0, 2]$ is the limiting equilibrium payoff set for type θ in a repeated complete information game when player 2 is short-lived.

Let $\Omega = \{a_1^*\}$ and $\phi_{a_1^*}(\theta) = 1$. Let us examine type θ's lowest equilibrium payoff, and in particular, how it is related to the value of $\lambda(\theta') \equiv \mu(\theta')/\mu(a_1^*)$:

- ▷ When $\lambda(\theta') < 1/3$, Theorem 1 implies that type θ can ensure payoff 2 in every NE.
- ▷ When $\lambda(\theta') \in [1/3, 2)$, the payoff lower bound in Theorem 1 no longer applies. The result in this subsection will generalize Theorem 1 which implies that type θ's NE payoff must be no less than $\min_{a_2 \in \{a_2^*, a_2^{**}\}} u_1(\theta, a_1^*, a_2)$, which equals to 1.

This payoff lower bound is tight, which is illustrated by the following NE:

- ▷ Type θ' always plays a_1^*. Type θ mixes between a_1^* and a_1' in period 0. If a_1^* is played in period 0, he plays a_1^* in every period. If a_1' is played in period 0, then play randomizes between (a_1^*, a_2^*) and (a_1', a_2') in all subsequent periods, depending on the realization of ξ_t.

▷ Player 2 plays a_2^{**} in every subsequent period if a_1^* was played in period 0. The probabilities with which type θ mixes in period 0 ensures that a_2^{**} is a best response. The probabilities with which (a_1^*, a_2^*) and (a_1', a_2') are prescribed by the public randomization device are designed such that type θ is indifferent between always playing a_1^* and always playing a_1'. Player 2 plays a_2' in every subsequent period after observing player 1's deviation.

B.2 Generalized Saturation Set & Strong Saturation Set

First, I generalize the definitions of saturation set and strong saturation set by allowing player 2 to have multiple best responses. Instead of defining them on $(\alpha_1^*, \theta) \in \Omega \times \Theta$, the new definitions will be based on $(\alpha_1^*, A_2^*) \in \Omega \times 2^{A_2}$, with $A_2^* \neq \{\varnothing\}$.

Fixing $\alpha_1^* \in \Omega$, for every $\tilde{\theta} \in \Theta$ and $\tilde{\mu} \in \Delta(\Omega \cup \Theta)$, let $\tilde{\lambda}(\tilde{\theta}) \equiv \tilde{\mu}(\tilde{\theta})/\tilde{\mu}(\alpha_1^*)$ and $\tilde{\lambda} = \left(\tilde{\lambda}(\tilde{\theta})\right)_{\tilde{\theta}\in\Theta} \in \mathbb{R}_+^k$ be the *likelihood ratio vector* with $k \equiv |\Theta|$. For every $A_2^* \subset A_2$ which is non-empty, let

$$\begin{aligned}\overline{\Lambda}(\alpha_1^*, A_2^*) \quad &\equiv \quad \Big\{\tilde{\lambda} \in \mathbb{R}_+^k \Big| \max_{a_2 \in A_2^*} \{u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2) + \sum_{\tilde{\theta}\in\Theta} \tilde{\lambda}(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^*, a_2)\} \\ &> \max_{a_2 \notin A_2^*} \{u_2(\phi_{\alpha_1^*}, \alpha_1^*, a_2) + \sum_{\tilde{\theta}\in\Theta} \tilde{\lambda}(\tilde{\theta}) u_2(\tilde{\theta}, \alpha_1^*, a_2)\}\Big\}.^6 \qquad \text{(B.1)}\end{aligned}$$

be the *generalized best response set*. The *generalized saturation set* of (α_1^*, A_2^*) is defined as:

$$\Lambda(\alpha_1^*, A_2^*) \equiv \Big\{\tilde{\lambda} \in \mathbb{R}_+^k \Big| \lambda' \in \overline{\Lambda}(\alpha_1^*, A_2^*) \text{ for every } \mathbf{0} \ll \lambda' \ll \tilde{\lambda}\Big\}, \qquad \text{(B.2)}$$

where "$\ll$" denotes weak dominance in product order. The *generalized strong saturation set* of (α_1^*, A_2^*) is defined as:

$$\underline{\Lambda}(\alpha_1^*, A_2^*) \equiv \mathbb{R}_+^k \Big\backslash \text{co}\Big(\mathbb{R}_+^k \backslash \Lambda(\alpha_1^*, A_2^*)\Big). \qquad \text{(B.3)}$$

The following Lemma provides necessary and sufficient conditions under which $\Lambda(\alpha_1^*, A_2^*)$ and $\underline{\Lambda}(\alpha_1^*, A_2^*)$ are non-empty:

Lemma B.1. *For any $\alpha_1^* \in \Omega$ and $A_2^* \subset A_2$, the following three statements are equivalent:*

1. $\Lambda(\alpha_1^*, A_2^*) \neq \{\varnothing\}$.
2. $\underline{\Lambda}(\alpha_1^*, A_2^*) \neq \{\varnothing\}$.
3. $BR_2(\alpha_1^*, \phi_{\alpha_1^*}) \subset A_2^*$.

PROOF OF LEMMA B.1: Suppose $\mathrm{BR}_2(\alpha_1^*, \phi_{\alpha_1^*})$ is not a subset of A_2^*, then according to (G.1), we have $\mathbf{0} \notin \overline{\Lambda}(\alpha_1^*, A_2^*)$, and therefore, $\Lambda(\alpha_1^*, A_2^*) = \underline{\Lambda}(\alpha_1^*, A_2^*) = \{\varnothing\}$.

Suppose $\mathrm{BR}_2(\alpha_1^*, \phi_{\alpha_1^*}) \subset A_2^*$, then $\mathbf{0} \in \overline{\Lambda}(\alpha_1^*, A_2^*)$, and by definition, $\mathbf{0} \in \Lambda(\alpha_1^*, A_2^*)$. Hence, there exists $\epsilon > 0$ such that $\lambda \in \Lambda(\alpha_1^*, A_2^*)$ for all $\lambda \in B(\mathbf{0}, \epsilon) \cap \overline{\Lambda}(\alpha_1^*, A_2^*) \neq \{\varnothing\}$. By definition,

$$\left\{\lambda \middle| \sum_{\tilde{\theta}} \lambda(\tilde{\theta})/\epsilon < 1\right\} \subset \underline{\Lambda}(\alpha_1^*, A_2^*) \neq \{\varnothing\}.$$

which concludes the proof. □

Let

$$v_\theta(\alpha_1^*, A_2^*) \equiv \min_{a_2 \in A_2^*} u_1(\theta, \alpha_1^*, a_2), \tag{B.4}$$

be player 1's '*generalized commitment payoff*'. This generalizes the definition in Pei (2018) by incorporating the possibility that player 2 can have multiple best responses to (α_1^*, θ). Intuitively, A_2^* is the set of actions that player 2 can play in the long run given that player 1 has successfully built up a reputation. Therefore, once we expand A_2^* in the set inclusion sense, player 1's '*worst case payoff*' decreases and the corresponding commitment payoff bound becomes weaker.

Remark: To understand why the new definitions in (B.2) and (B.3) generalize the ones in Pei (2018, section 3), note that when $A_2^* = \mathrm{BR}_2(\alpha_1^*, \theta) = \{a_2^*\}$, whether a likelihood ratio vector λ belongs to $\overline{\Lambda}(\alpha_1^*, \theta)$ or not only depends on its values on the following set of coordinates:

$$\Theta^b_{(\alpha_1^*, \theta)} \equiv \left\{\tilde{\theta} \in \Theta \middle| a_2^* \notin \mathrm{BR}_2(\alpha_1^*, \tilde{\theta})\right\}.$$

However, when $|A_2^*| \geq 2$, there is no clear bifurcation between good and bad strategic types. As a result, $\overline{\Lambda}(\alpha_1^*, A_2^*)$, $\Lambda(\alpha_1^*, A_2^*)$ and $\underline{\Lambda}(\alpha_1^*, A_2^*)$ may not be convex and $\underline{\Lambda}(\alpha_1^*, A_2^*)$ is not necessarily a '*triangle set*'. This is demonstrated by the following example:

▷ Let $\Theta = \{\theta, \theta', \theta''\}$, $A_1 = \{a_1^*, a_1'\}$ and $A_2 = \{a_2^*, a_2^{**}, a_2'\}$. Player 2's payoff is given by:

$u_2(\theta, \cdot, \cdot)$	a_2^*	a_2^{**}	a_2'
a_1^*	1	1	0
a_1'	1	1	0

$u_2(\theta', \cdot, \cdot)$	a_2^*	a_2^{**}	a_2'
a_1^*	1	-3	0
a_1'	1	-3	0

$u_2(\theta'', \cdot, \cdot)$	a_2^*	a_2^{**}	a_2'
a_1^*	-3	1	0
a_1'	-3	1	0

Let $\Omega = \{a_1^*\}$, $\phi_{a_1^*}(\theta) = 1$ and $A_2^* = \{a_2^*, a_2^{**}\}$. Notably, for every $\theta' \in \{\theta_1, \theta_2\}$, we have $\mathrm{BR}_2(a_1^*, \theta') \subset A_2^*$, but there exists $\pi \in \Delta\{\theta_1, \theta_2\}$ such that $\mathrm{BR}_2(a_1^*, \pi) = \{a_2'\}$. As a result, the sets $\overline{\Lambda}(a_1^*, A_2^*)$, $\Lambda(a_1^*, A_2^*)$

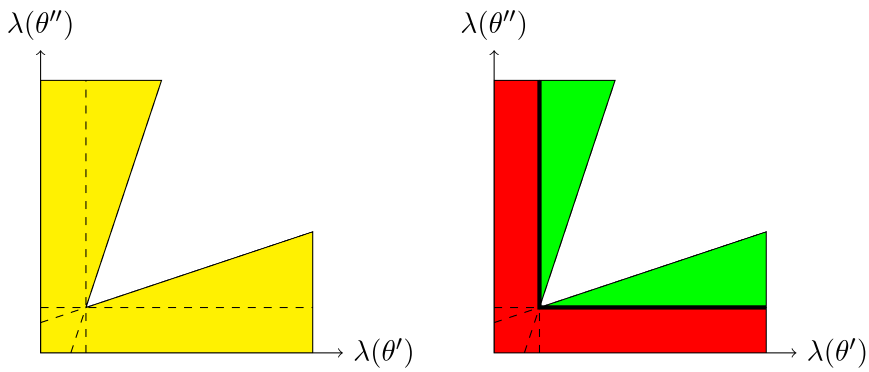

Figure 2: Left panel: $\overline{\Lambda}(a_1^*, A_2^*)$ in yellow. Right panel: $\Lambda(a_1^*, A_2^*) = \underline{\Lambda}(a_1^*, A_2^*)$ in red.

and $\underline{\Lambda}(a_1^*, A_2^*)$ are not convex. Moreover, $\underline{\Lambda}(a_1^*, A_2^*)$ is not a triangular set. These complications only arise when $|A_2^*| \geq 2$. I depict the three sets with $\lambda(\theta')$ and $\lambda(\theta'')$ being the two coordinates in Figure 2.

B.3 Statement of Result

Theorem 1' generalizes Theorem 1 along the following two dimensions. First, it allows for player 2 to have non-singleton best response sets. Second, it provides a meaningful lower bound on player 1's equilibrium payoff when the distribution condition in Theorem 1 fails. Let $\Pi(a_1^*, A_2^*)$ and $\underline{\Pi}(\alpha_1^*, A_2^*)$ be the exteriors of $\Lambda(a_1^*, A_2^*)$ and $\underline{\Lambda}(\alpha_1^*, A_2^*)$, respectively. Let

$$\mathcal{B}(\alpha_1^*, \theta) \equiv \left\{ A_2^* \middle| \mathrm{BR}_2(\theta, \alpha_1^*) \subset A_2^* \right\}.$$

The result is stated below:

Theorem 1'. *For every $\theta \in \Theta$, $\alpha_1^* \in \Omega$ with α_1^* being pure and $A_2^* \in 2^{A_2} \backslash \{\varnothing\}$:*

1. *If $\lambda \in \Lambda(\alpha_1^*, A_2^*)$, then $\liminf_{\delta \to 1} \underline{V}_\theta(\delta, \mu) \geq v_\theta(\alpha_1^*, A_2^*)$ for every u_1.*

2. *If $\lambda \in \Pi(\alpha_1^*, A_2^*)$ and $A_2^* \in \mathcal{B}(\alpha_1^*, \theta)$, then there exists u_1 such that*
$\limsup_{\delta \to 1} \underline{V}_\theta(\delta, \mu) < v_\theta(\alpha_1^*, A_2^*)$.

For every $\theta \in \Theta$, $\alpha_1^* \in \Omega$ with α_1^* being mixed and $A_2^* \in 2^{A_2} \backslash \{\varnothing\}$:

3. *If $\lambda \in \underline{\Lambda}(\alpha_1^*, A_2^*)$, then $\liminf_{\delta \to 1} \underline{V}_\theta(\delta, \mu) \geq v_\theta(\alpha_1^*, A_2^*)$ for every u_1.*

4. *If $\lambda \in \underline{\Pi}(\alpha_1^*, A_2^*)$, $A_2^* \in \mathcal{B}(\alpha_1^*, \theta)$ and $\alpha_1^* \notin co\Big(\Omega \Big\backslash \{\alpha_1^*\}\Big)$, then there exists u_1 such that*
$\limsup_{\delta \to 1} \underline{V}_\theta(\delta, \mu) < v_\theta(\alpha_1^*, A_2^*)$.

The proof is similar to that of Theorem 1 except for statement 3, which requires two additional steps and will be explained in the next subsection. Let me briefly comment on the role of $A_2^* \in \mathcal{B}(\alpha_1^*, \theta)$ in statements 2 and 4. In my construction, since there can be multiple commitment types, the bad strategic types are required to play mixed strategies. The condition that $A_2^* \in \mathcal{B}(\alpha_1^*, \theta)$ guarantees the existence of a convex combination of other strategic types (which do not involve type θ), under which player 2 has no incentive to play actions in A_2^*. The proofs of statements 2 and 4 in Theorem 1 directly carries over to this generalized environment, which I omit to avoid duplication.

Implication on Payoff Lower Bound: Theorem 1' further implies that we can apply the following algorithm to find a fully robust payoff lower bound for any given strategic type θ:

1. Let $A_2 \equiv \{a_2^1, ..., a_2^m\}$. For every $\alpha_1^* \in \Omega$, τ is an '*appropriate ranking*' of player 2's actions if $a_2^{1,\tau}, ..., a_2^{m,\tau}$ is a permutation of $a_2^1, ..., a_2^m$ with $u_1(\theta, \alpha_1^*, a_2^{i,\tau}) \geq u_1(\theta, \alpha_1^*, a_2^{j,\tau})$ for every $i > j$. Let $\mathcal{R}^{\alpha_1^*}$ be the set of appropriate ranking under α_1^*.

2. If α_1^* is pure, for every $\tau \in \mathcal{R}^{\alpha_1^*}$, find the smallest integer $k \in \{1, 2, ..., n\}$ such that

$$\lambda^{\alpha_1^*} \in \Lambda\Big(\alpha_1^*, \{a_2^{1,\tau}, ..., a_2^{k,\tau}\}\Big)$$

   where $\lambda^{\alpha_1^*}$ is the likelihood ratio vector between every strategic type and commitment type α_1^*. Let $V_\theta(\alpha_1^*) \equiv \max_{\tau \in \mathcal{R}^{\alpha_1^*}} u_1(\theta, \alpha_1^*, a_2^{k,\tau})$.

3. If α_1^* is mixed, for every $\tau \in \mathcal{R}^{\alpha_1^*}$, find the smallest integer $k \in \{1, 2, ..., n\}$ such that

$$\lambda^{\alpha_1^*} \in \underline{\Lambda}\Big(\alpha_1^*, \{a_2^{1,\tau}, ..., a_2^{k,\tau}\}\Big).$$

   Let $V_\theta(\alpha_1^*) \equiv \max_{\tau \in \mathcal{R}^{\alpha_1^*}} u_1(\theta, \alpha_1^*, a_2^{k,\tau})$.

Intuitively, $V_\theta(\alpha_1^*)$ is type θ's secured payoff by establishing reputation α_1^*. The following Corollary directly follows from Theorem 1', which establishes a lower bound on type θ's payoff when he can flexibly choose between various reputations in Ω:

Corollary B.1. *For every $\theta \in \Theta$ and every u_1,*

$$\liminf_{\delta \to 1} \underline{V}_\theta(\delta, \mu) \geq \max_{\alpha_1^* \in \Omega} V_\theta(\alpha_1^*). \tag{B.5}$$

B.4 Proofs of Statement 3

Statement 3 in Theorem 1' does not directly follow from the proof of statement 3 in Theorem 1 for two reasons. First, as shown in Figure 2, $\underline{\Lambda}(\alpha_1^*, A_2^*)$ is not necessarily a triangular set. Second, even if player 2 can be convinced that his best reply is in A_2^*, type θ player 1 can only guarantee his '*correlated commitment payoff*', given by the following linear program:

$$\min_{\alpha \in \Delta(A_1 \times A_2)} u_1(\theta, \alpha),$$

subject to:

$$\sum_{a_2 \in A_2^*} \alpha(a_1, a_2) = \alpha_1^*(a_1) \text{ for every } a_1 \in A_1.$$

The value of the above program can be strictly less than $v_\theta(\alpha_1^*, A_2^*)$ if α_1^* is mixed and $|A_2^*| \geq 2$ due to the correlations between a_1 and a_2, which can arise as the proof of statement 3 requires the use of non-stationary strategy, which is constructed by cherry-picking actions in the support of α_1^*.

In what follows, I show how to overcome these two challenges. In Part I, I show that for every $\lambda \in \underline{\Lambda}(\alpha_1^*, A_2^*)$, there exists $\psi = (\psi_i)_{i=1}^k$ with $\psi_i > 0$ such that

$$\lambda \in \{\tilde{\lambda} | \sum_{i=1}^k \tilde{\lambda}_i / \psi_i < 1\} \subset \underline{\Lambda}(\alpha_1^*, A_2^*). \tag{B.6}$$

Therefore, we adapt the proof of Theorem 1 when applying to set $\{\tilde{\lambda} | \sum_{i=1}^k \tilde{\lambda}_i / \psi_i < 1\}$. In Part II, I show that under the strategy constructed in the proof of Theorem 1, the expected discounted average distance between the constructed behavior strategy and α_1^* diminishes as $\delta \to 1$, which implies that the correlations between players' actions have negligible consequences in the limit.

Part I: In order to apply the Doob's Upcrossing Inequality as in the proof of statement 3 Theorem 1, we need to show the existence of $\psi = (\psi_i)_{i=1}^k$ for every $\lambda \in \underline{\Lambda}(\alpha_1^*, A_2^*)$ such that (B.6) holds. The key steps are summarized in the following two Lemmas:

Lemma B.2. *If $\underline{\Lambda}(\alpha_1^*, A_2^*)$ is non-empty, then*

$$\underline{\Lambda}(\alpha_1^*, A_2^*) = \Big\{\tilde{\lambda} \in \mathbb{R}_+^k \Big| \lambda' \notin co\Big(\mathbb{R}_+^k \backslash \overline{\Lambda}(\alpha_1^*, A_2^*)\Big) \textit{ for all } 0 \leq \lambda' \leq \tilde{\lambda}\Big\}. \tag{B.7}$$

PROOF OF LEMMA B.2: First, I show

$$\underline{\Lambda}(\alpha_1^*, A_2^*) \subset \Big\{\tilde{\lambda} \in \mathbb{R}_+^k \Big| \lambda' \notin \text{co}\Big(\mathbb{R}_+^k \backslash \overline{\Lambda}(\alpha_1^*, A_2^*)\Big) \text{ for all } 0 \leq \lambda' \leq \tilde{\lambda}\Big\}.$$

Suppose towards a contradiction that $\lambda \in \underline{\Lambda}(\alpha_1^*, A_2^*)$, but there exists $\tilde{\lambda} \le \lambda$ such that $\tilde{\lambda} = \alpha\lambda' + (1-\alpha)\lambda''$ for some $\alpha \in [0,1]$ and $\lambda', \lambda'' \notin \overline{\Lambda}(\alpha_1^*, A_2^*)$. Then, let $\tilde{\lambda}' \equiv (\lambda - \tilde{\lambda}) + \lambda'$ and $\tilde{\lambda}'' \equiv (\lambda - \tilde{\lambda}) + \lambda''$. By definition, $\tilde{\lambda}', \tilde{\lambda}'' \notin \Lambda(\alpha_1^*, A_2^*)$ and $\alpha\tilde{\lambda}' + (1-\alpha)\tilde{\lambda}'' = \lambda$, which contradicts the assumption that $\lambda \in \underline{\Lambda}^*(\alpha_1^*, A_2^*)$.

Next, I show

$$\left\{\tilde{\lambda} \in \mathbb{R}_+^k \middle| \lambda' \notin \text{co}\Big(\mathbb{R}_+^k \backslash \overline{\Lambda}(\alpha_1^*, A_2^*)\Big) \text{ for all } 0 \le \lambda' \le \tilde{\lambda}\right\} \subset \underline{\Lambda}(\alpha_1^*, A_2^*).$$

Suppose towards a contradiction that $\lambda \in \left\{\tilde{\lambda} \in \mathbb{R}_+^k \middle| \lambda' \notin \text{co}\Big(\mathbb{R}_+^k \backslash \overline{\Lambda}(\alpha_1^*, A_2^*)\Big) \text{ for all } 0 \le \lambda' \le \tilde{\lambda}\right\}$, but there exists $\lambda', \lambda'' \notin \Lambda(\alpha_1^*, A_2^*)$ and $\alpha \in [0,1]$ such that $\alpha\lambda' + (1-\alpha)\lambda'' = \lambda$. Then there exists $\tilde{\lambda}' \le \lambda'$ and $\tilde{\lambda}'' \le \lambda''$ such that $\tilde{\lambda}', \tilde{\lambda}'' \notin \overline{\Lambda}(\alpha_1^*, A_2^*)$. But then $\alpha\tilde{\lambda}' + (1-\alpha)\tilde{\lambda}'' \le \lambda$, which leads to a contradiction. □

Lemma B.3. *For every $\lambda \in \underline{\Lambda}(\alpha_1^*, A_2^*)$, there exists $\{\psi(\tilde{\theta})\}_{\tilde{\theta}\neq\theta}$ with $\psi(\tilde{\theta}) \in (0, +\infty]$ for all $\tilde{\theta} \neq \theta$, such that (B.6) applies.*

PROOF OF LEMMA B.3: According to Lemma B.2, if $\lambda \in \underline{\Lambda}(\alpha_1^*, A_2^*)$,

$$\left\{\lambda' \middle| 0 \le \lambda' \le \lambda\right\} \subset \underline{\Lambda}(\alpha_1^*, A_2^*).$$

Therefore, the following two closed convex sets: $\left\{\lambda' \middle| 0 \le \lambda' \le \lambda\right\}$ and $\text{co}\Big(\mathbb{R}_+^k \backslash \Lambda(\alpha_1^*, A_2^*)\Big)$, have empty intersection. According to the separating hyperplane theorem, there exists a k-dimensional vector, $\{\phi(\tilde{\theta})\}_{\tilde{\theta}\neq\theta}$ such that

$$\left\{\lambda' \middle| 0 \le \lambda' \le \lambda\right\} \subset \left\{\tilde{\lambda} \ge 0 \middle| \sum_{\tilde{\theta}\neq\theta} \tilde{\lambda}(\tilde{\theta})\phi(\tilde{\theta}) \le 1\right\},$$

and

$$\text{co}\Big(\mathbb{R}_+^k \backslash \Lambda(\alpha_1^*, A_2^*)\Big) \subset \left\{\tilde{\lambda} \ge 0 \middle| \sum_{\tilde{\theta}\neq\theta} \tilde{\lambda}(\tilde{\theta})\phi(\tilde{\theta}) > 1\right\}.$$

It is without loss of generality to adopt the following normalization:

$$\max_{0 \le \tilde{\lambda} \le \lambda} \sum_{\tilde{\theta}\neq\theta} \tilde{\lambda}(\tilde{\theta})\phi(\tilde{\theta}) = 1$$

Let λ^* be one of the points where the maximum is attained. If there exists $\hat{\theta}$ such that $\phi(\hat{\theta}) < 0$, then

▷ If $\lambda^*(\hat{\theta}) = 0$, then let $\hat{\phi}(\hat{\theta}) = 0$ and $\hat{\phi}(\tilde{\theta}) = \phi(\tilde{\theta})$ for $\tilde{\theta} \notin \{\hat{\theta}, \theta\}$, we have:

$$\text{co}\Big(\mathbb{R}_+^k \backslash \Lambda(\alpha_1^*, A_2^*)\Big) \subset \left\{\tilde{\lambda} \ge 0 \middle| \sum_{\tilde{\theta}\neq\theta} \tilde{\lambda}(\tilde{\theta})\hat{\phi}(\tilde{\theta}) > 1\right\}.$$

Moreover, for every $0 \le \hat{\lambda} \le \lambda$, let $\hat{\lambda}^*$ be defined such that:

$$\hat{\lambda}^*(\tilde{\theta}) \equiv \begin{cases} 0 & \text{if } \tilde{\theta} = \hat{\theta} \\ \hat{\lambda}(\tilde{\theta}) & \text{otherwise }. \end{cases}$$

Then, $0 \le \hat{\lambda}^* \le \lambda$. Since $\sum_{\tilde{\theta}\neq\theta} \hat{\lambda}^*(\tilde{\theta})\hat{\phi}(\tilde{\theta}) \le 1$, we have:

$$\left\{\lambda' \middle| 0 \le \lambda' \le \lambda\right\} \subset \left\{\tilde{\lambda} \ge 0 \middle| \sum_{\tilde{\theta}\neq\theta} \tilde{\lambda}(\tilde{\theta})\hat{\phi}(\tilde{\theta}) \le 1\right\}.$$

$\triangleright$ If $\lambda^*(\hat{\theta}) > 0$, then let

$$\hat{\lambda}^*(\tilde{\theta}) \equiv \begin{cases} 0 & \text{if } \tilde{\theta} = \hat{\theta} \\ \lambda^*(\tilde{\theta}) & \text{otherwise }. \end{cases}$$

We have $\sum_{\tilde{\theta}\neq\theta} \hat{\lambda}^*(\tilde{\theta})\phi(\tilde{\theta}) > 1$, which leads to a contradiction.

Therefore, there exists $\{\phi(\tilde{\theta})\}_{\tilde{\theta}\neq\theta}$ such that $\phi(\tilde{\theta}) \ge 0$ for every $\tilde{\theta} \neq \theta$. Let $\psi(\tilde{\theta}) \equiv 1/\phi(\tilde{\theta})$, we obtain the desirable conclusion. $\square$

Part II: In this part, I show that the payoff consequences of the correlations between players' actions have negligible impact on player 1's discounted average payoff when $\delta \to 1$. I consider a more general problem, in which player 1 only plays a subset of the infinite histories $\tilde{\mathcal{H}} \subset \mathcal{H}$ according to probability measure $\mathcal{P}$, which is defined as:

$$\mathcal{P}(\mathcal{H}') \equiv \mathcal{P}^*(\mathcal{H}' \cap \tilde{\mathcal{H}}) \Big/ \mathcal{P}^*(\tilde{\mathcal{H}})$$

for every $\mathcal{H}'$ where $\mathcal{P}^*$ is the probability measure over $\mathcal{H}$ induced by the commitment strategy α_1^*. In what follows, I will only discuss histories h^t that occur with positive probability under $\mathcal{P}$.

Suppose $\mathcal{P}^*(\tilde{\mathcal{H}}) = 1 - X$ for some $X \in (0, 1)$. At every history h^t, player 1 is playing a subset of the actions in A_1^* with positive probability, with $A_1^* \equiv \text{supp}(\alpha_1^*)$. Let $\{\alpha_1(h^t)\}_{h^t}$ be the behavior strategy induced by $\mathcal{P}$ with $\alpha_1(h^t) \in \Delta(A_1)$. Let $\epsilon(h^t)$ be the smallest $\epsilon \in [0, 1]$ such that there exists $\alpha_1 \in \Delta(A_1^*)$, satisfying:

$$\epsilon\alpha_1 + (1 - \epsilon)\alpha_1^* = \alpha_1(h^t) \tag{B.8}$$

The main result of this part is the following Lemma:

Lemma B.4. *For every $X \in (0,1)$, there exists $\eta_X : (0,1) \to \mathbb{R}_+$ with $\lim_{\delta\to 1} \eta(\delta) = 0$ such that:*

$$\sum_{t=0}^{\infty} \sum_{h^t \in \tilde{\mathcal{H}}} (1-\delta)\delta^t \epsilon(h^t)\mathcal{P}(h^t) \leq \eta_X(\delta). \tag{B.9}$$

To see the implications of Lemma B.4, notice that at history h^t, conditional on player 2 only plays actions in A_2^* with positive probability, type θ player 1's stage game payoff from playing $\alpha_1(h^t)$ is at least:

$$\epsilon \min_{\alpha \in \Delta(A_1 \times A_2)} u_1(\theta, \alpha) + (1-\epsilon) \underbrace{\min_{a_2 \in A_2^*} u_1(\theta, \alpha_1^*, a_2)}_{=v_\theta(\alpha_1^*, A_2^*)}.$$

If $\sum_{t=0}^{\infty} \sum_{h^t \in \tilde{\mathcal{H}}} (1-\delta)\delta^t \epsilon(h^t)\mathcal{P}(h^t)$ goes to 0 when δ goes to 1, i.e. the payoff consequences of the first term becomes negligible, then by adopting strategy $\{\alpha_1(h^t)\}_{h^t \in \mathcal{H}}$ and conditional on player 2 playing actions in A_2^* in all but a bounded number of periods (which we could obtain using the same argument as the proof of statement 3 in Theorem 1), type θ could guarantee himself an expected payoff arbitrarily close to $v_\theta(\alpha_1^*, A_2^*)$.

PROOF OF LEMMA B.4: I start with computing $\epsilon(h^t)$. Let

$$x(h^t) \equiv \sum_{a_1 \in A_1^* \setminus \text{supp}(\alpha_1(h^t))} \alpha_1^*(a_1), \tag{B.10}$$

which is the probability of actions that are being '*trimmed*' at h^t. Let

$$\beta \equiv \min_{a_1 \in A_1^*} \alpha_1^*(a_1), \tag{B.11}$$

which is strictly positive. We know that if $x(h^t) > 0$, then $x(h^t) \geq \beta$. Let

$$X(h^t) \equiv \sum_{s \geq t} \sum_{h^s \in \tilde{\mathcal{H}} \text{ with } h^s \succeq h^t} \mathcal{P}^*(h^s|h^t) x(h^s). \tag{B.12}$$

By definition, $X(h^0) = X$. Let $\hat{\mathcal{H}} \subset \tilde{\mathcal{H}}$ be the set of histories such that $x(h^t) > 0$. We have $\epsilon(h^t) = 1$ for every $h^t \in \hat{\mathcal{H}}$. According to (B.11),

$$\sum_t \sum_{h^t \in \hat{\mathcal{H}}} \mathcal{P}^*(h^t) x(h^t) \leq X$$

implies that

$$\sum_t \sum_{h^t \in \hat{\mathcal{H}}} \mathcal{P}^*(h^t) \leq X/\beta.$$

i.e. the expected number of such histories along any $h^t \in \tilde{\mathcal{H}}$ is bounded from above by $T \equiv \lceil X/\beta \rceil$ for any $t \in \mathbb{R}$. This gives rise to the following inequality:

$$\sum_{t=0}^{\infty} \sum_{h^t \in \hat{\mathcal{H}}} (1-\delta)\delta^t \epsilon(h^t)\mathcal{P}(h^t) \leq \frac{(1-\delta)\sum_t \sum_{h^t \in \hat{\mathcal{H}}} \mathcal{P}^*(h^t)}{1-X} \leq \frac{(1-\delta)X}{\beta(1-X)}. \tag{B.13}$$

where the first inequality uses the fact that $\mathcal{P}(\mathcal{H}') \leq \mathcal{P}^*(\mathcal{H}')/\mathcal{P}^*(\tilde{\mathcal{H}})$.

Next, let us consider histories where $h^t \in \tilde{\mathcal{H}} \backslash \hat{\mathcal{H}} \equiv \breve{\mathcal{H}}$. For every history in this set, we have:

$$\alpha_1(h^t)[a_t] = \frac{\alpha_1^*(a_1)(1-X(h^t,a_1))}{\sum_{a_1' \in A_1^*} \alpha_1^*(a_1')(1-X(h^t,a_1'))}, \tag{B.14}$$

with $X(h^t,a_1) \in [0,1)$ for every $a_1 \in A_1$. By definition,

$$1-\epsilon(h^t) = \frac{1-\max_{a_1 \in A_1^*} X(h^t,a_1)}{\sum_{a_1' \in A_1^*} \alpha_1^*(a_1')(1-X(h^t,a_1'))}. \tag{B.15}$$

Moreover, since $x(h^t)=0$, we can use (B.12) and obtain:

$$X(h^t) = \sum_{a_1 \in A_1^*} \alpha_1^*(a_1) X(h^t,a_1). \tag{B.16}$$

i.e. $\{X(h^t)\}_{h^t \in \mathcal{H}}$ is a super-martingale with respect to $\mathcal{P}^*$ with equality holds for every $h^t \in \breve{\mathcal{H}}$. From (B.15) and (B.16), we have:

$$\epsilon(h^t) = \frac{\max_{a_1 \in A_1^*} X(h^t,a_1) - X(h^t)}{1-X(h^t)}.$$

and therefore,

$$\begin{aligned}\sum_{t=0}^{\infty} \sum_{h^t \in \breve{\mathcal{H}}} (1-\delta)\delta^t \mathcal{P}(h^t)\epsilon(h^t) &= \sum_{t=0}^{\infty} \sum_{h^t \in \breve{\mathcal{H}}} (1-\delta)\delta^t \frac{\big|\max_{a_1 \in A_1^*} X(h^t,a_1) - X(h^t)\big|\mathcal{P}^*(h^t)}{1-X} \\ &\leq \sum_{t=0}^{\infty} \sum_{h^t} (1-\delta)\delta^t \frac{K\big|X(h^t,a_1) - X(h^t)\big|\mathcal{P}^*(h^t)}{1-X}\end{aligned}$$

where $K \in \mathbb{R}_+$ is a constant that only depends on β. Since $\mathbb{E}[X(h^t,a_1)] = X(h^t)$ for every $h^t \in \breve{\mathcal{H}}$, according to a famous application of the Cauchy-Schwarz inequality (see Aumann and Mascheler 1995), we have:

$$\sum_{t=0}^{\infty}\sum_{h^t\in\check{\mathcal{H}}}(1-\delta)\delta^t\mathcal{P}(h^t)\epsilon(h^t)\leq K\sqrt{\frac{1-\delta}{1+\delta}}\sqrt{\frac{X}{1-X}}. \tag{B.17}$$

Summing up (B.13) and (B.17) and let

$$\eta_X(\delta)\equiv K\sqrt{\frac{1-\delta}{1+\delta}}\sqrt{\frac{X}{1-X}}+\frac{(1-\delta)X}{\beta(1-X)}. \tag{B.18}$$

The right hand side is converging to 0 for every $X\in(0,1)$ as $\delta\to 1$. $\square$

Final Step: Equipped with the results from the above two parts, we can directly apply the proof of the singleton A_2^* case. In particular, for every $\lambda\in\underline{\Lambda}(\alpha_1^*,A_2^*)$, there exists a hyperplane $\psi=(\psi_i)$ with all coefficients ψ_i being positive such that (B.6) holds. Then construct player 1's strategy as in Theorem 1. According to Part II, he can achieve a payoff arbitrarily close to $v_\theta(\alpha_1^*,A_2^*)$ as $\delta\to 1$.

C Limiting Equilibrium Payoffs without Commitment Types

In this Appendix, I characterize the limiting equilibrium payoff set in repeated Bayesian games between an informed long-run player and a sequence of short-run players when there are only rational types and values are interdependent. I focus on the product choice game example in Pei (2018), with stage game payoffs given by:[7]

$\theta=\theta_1$	C	S
H	$1,3$	$-1,2$
L	$2,0$	$0,1$

$\theta=\theta_0$	C	S
H	$1-\eta,0$	$-1-\eta,1$
L	$2,-2$	$0,0$

with $\eta\in(0,1)$.

Let $\mu\in(0,1)$ be the probability player 2's prior belief attaches to type θ_1 and $\delta\in(0,1)$ be player 1's discount factor. Let $v\equiv(v_{\theta_1},v_{\theta_0})\in\mathbb{R}^2$ be a generic payoff vector for player 1. Let $\text{NE}(\delta,\mu)\subset\Sigma$ be the set of Nash Equilibria (hereafter, NE for short) under parameter configuration (δ,μ) and let $\mathcal{V}^*(\delta,\mu)\in\mathbb{R}^2$ be the set of NE payoff vectors. The goal is to characterize $\mathcal{V}^*(\delta,\mu)$ when δ is arbitrarily close to 1, i.e. either computing $\lim_{\delta\to 1}\mathcal{V}^*(\delta,\mu)$ if it exists, or providing an upper bound for $\limsup_{\delta\to 1}\mathcal{V}^*(\delta,\mu)$ and a lower bound for $\liminf_{\delta\to 1}\mathcal{V}^*(\delta,\mu)$. The following two results, which characterize $\lim_{\delta\to 1}\mathcal{V}^*(\delta,\mu)$ when $\mu<1/2$ and $\mu=1/2$ respectively, directly follow from Proposition 4.2 in Pei (2018):

[7] The characterization result is generalized to the class of binary action (i.e. $|A_2|=2$) monotone supermodular repeated games in an on-going work (Pei 2016).

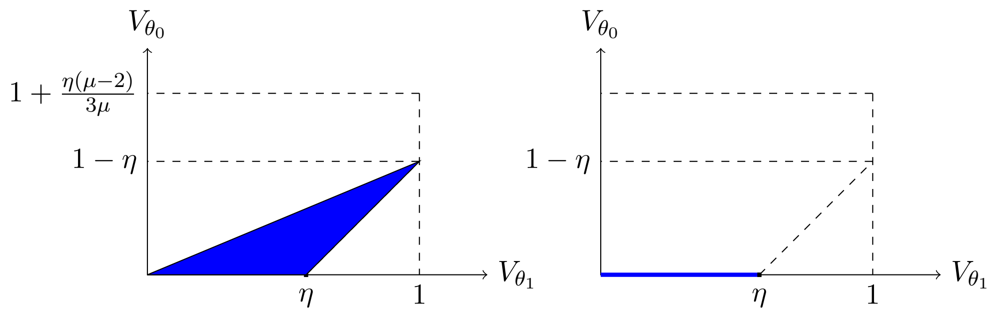

Figure 3: Limiting equilibrium payoff set in the incomplete information product choice game without commitment types (in blue). Left panel: $\mu = 1/2$. Right panel: $\mu < 1/2$.

Proposition C.1. *For every $\mu < 1/2$, $\lim_{\delta\to 1} \mathcal{V}^*(\delta, \mu) = \Big\{(v_{\theta_1}, 0)\Big| v_{\theta_1} \in [0, \eta]\Big\}$.*

Proposition C.2. *When $\mu = 1/2$, $\lim_{\delta\to 1} \mathcal{V}^*(\delta, \mu)$ is the triangle with vertices $(0, 0)$, $(\eta, 0)$ and $(1, 1 - \eta)$.*

The limiting equilibrium payoff set in these two cases are depicted in Figure I. When $\mu > 1/2$, let

$$\overline{v}(v_{\theta_1}, \mu) \equiv \Big(1 + \frac{\mu - 2}{3\mu}\eta\Big) v_{\theta_1}, \tag{C.1}$$

and

$$\underline{v}(v_{\theta_1}, \mu) \equiv \max\Big\{\Big(1 + \frac{\eta(1 - 2\eta)}{3} + 1\Big) v_{\theta_1} - \frac{2}{3}(2 - \mu)\eta, 0\Big\}. \tag{C.2}$$

Let

$$\overline{\mathcal{V}}(\mu) = \Big\{(v_{\theta_1}, v_{\theta_0})\Big| v_{\theta_1} \in [0, 1] \text{ and } v_{\theta_0} \in \Big[\underline{v}(v_{\theta_1}, \mu), \overline{v}(v_{\theta_1}, \mu)\Big]\Big\} \tag{C.3}$$

and

$$\underline{\mathcal{V}}(\mu) \equiv \Big\{(v_{\theta_1}, v_{\theta_0})\Big| v_{\theta_1} \in [0, 1] \text{ and } v_{\theta_0} \in \Big(\underline{v}(v_{\theta_1}, \mu), \overline{v}(v_{\theta_1}, \mu)\Big)\Big\} \tag{C.4}$$

By definition, $\text{clo}(\underline{\mathcal{V}}(\mu)) = \overline{\mathcal{V}}(\mu)$. The next result virtually characterizes player 1's limiting equilibrium payoff set when $\mu > 1/2$, which will be shown in the rest of this Appendix.

Proposition C.3. *When $\mu > 1/2$,*

$$\underline{\mathcal{V}}(\mu) \subset \liminf_{\delta\to 1} \mathcal{V}^*(\delta, \mu) \subset \limsup_{\delta\to 1} \mathcal{V}^*(\delta, \mu) \subset \overline{\mathcal{V}}(\mu). \tag{C.5}$$

The rest of this Appendix proceeds as follows. In subsection A.1, I establish $\underline{\mathcal{V}}(\mu) \subset \liminf_{\delta\to 1} \mathcal{V}^*(\delta, \mu)$ by constructing equilibria that arbitrarily approximate these payoffs when $\delta \to 1$. In subsection A.2, I establish $\limsup_{\delta\to 1} \mathcal{V}^*(\delta, \mu) \subset \overline{\mathcal{V}}(\mu)$ by showing that player 1 cannot obtain payoffs outside $\overline{\mathcal{V}}(\mu)$ when δ is close enough to 1. Due to the symmetric structure of the problem, I will be focusing on the '*maximizing*' direction, i.e. maximizing type θ_0's payoff given type θ_1's payoff. I will also comment on the '*minimizing*' direction.

C.1 Lower Bound of the Limiting Equilibrium Payoff Set

I show that for every $v \equiv (v_{\theta_1}, v_{\theta_0})$ with $v_{\theta_0} < \overline{v}(v_{\theta_1}, \mu)$, there exists $\overline{\delta} \in (0,1)$ such that for every $\delta > \overline{\delta}$, there exists an equilibrium in which player 1's payoff vector is v. Due to the existence of public randomization device, it is without loss of generality to focus on the attainability of $(1, v_{\theta_0})$ with $v_{\theta_0} < 1 + \frac{\mu-2}{3\mu}\eta$.

The constructed equilibrium has three phases: a '*normal phase*', a '*high phase*' and a '*low phase*'. Pick $\mu^* \in (1/2, \mu)$ close to μ, $\overline{\delta} \in (0,1)$ close to 1 and $\lambda \in \mathbb{R}_+$ close to 0.[8] Let v_t be player 1's continuation payoff in period t, which is computed via:

$$v_{t-1} = (1-\delta)u_t + \delta v_t$$

with $v_0 \equiv v$ and u_t being the stage game payoff vector for player 1 in period t. Let $v_t \equiv (v_{t,\theta_1}, v_{t,\theta_0}) \in \mathbb{R}^2$, which together with μ_t will be the state variables to keep track of.

Three Phase Construction: Play starts from the *normal phase*, in which player 2 always plays C and player 1's strategy is pinned down by the following belief updating process:

- If H is played in period t, then:

$$\mu_{t+1} = \min\Big\{1, \mu_t + \lambda \frac{2\mu^*-1}{2-\mu^*}(\mu_t - \mu^*)\Big\}. \tag{C.6}$$

- If L is played in period t, then:

$$\mu_{t+1} = \begin{cases} \mu_t - \lambda(\mu_t - \mu^*) & \text{if } v_{t+1,\theta_0} > (1-\eta)v_{t+1,\theta_1} \\ 1/2 & \text{if } v_{t+1,\theta_0} \leq (1-\eta)v_{t+1,\theta_1}. \end{cases} \tag{C.7}$$

Play transits from the *normal phase* to the *high phase* when μ_t reaches 1, after which players randomize between always playing (H, C) and always playing (L, S), depending on the realization of the public randomization device, with the probability of the former equals to v_{t,θ_1}.

Play transits from the *normal phase* to the *low phase* when μ_t reaches $1/2$, and by construction of the belief updating process, $v_{t,\theta_0} \leq (1-\eta)v_{t,\theta_1}$. The low phase delivers payoff v_t. I will later verify that it is in the interior of $\mathcal{V}(\delta, 1/2)$ when δ is large enough. Both the high phase and the low phase are absorbing.

[8] I will later specify the conditions on μ^*, $\overline{\delta}$ and λ.

Incentive Constraints: I verify players' incentive constraints in the above construction. First, I show that player 2 always has a strict incentive to play C in the normal phase. Let $x \equiv \frac{2\mu^*-1}{2-\mu^*}$. Player 2's expected payoff difference from playing C versus playing S in period t is:

$$\begin{aligned}\mathbb{E}[u_2(\theta, a_1, C) - u_2(\theta, a_1, S)|\mu_t] &= \frac{1}{1+x}\Big\{2[\mu^* + \lambda x(\mu_t - \mu^*)] - 1\Big\} + \frac{x}{1+x}\Big\{[\mu^* - \lambda(\mu_t - \mu^*)] - 2\Big\} \\ &\propto 2\mu^* + x\mu^* + \lambda x(\mu_t - \mu^*) - 1 - 2x \\ &> 2\mu^* + x\mu^* - 1 - 2x = 0. \end{aligned} \tag{C.8}$$

Next, I verify that when δ is large enough, the following two statements are true:

1. If play switches to the low phase in period t, then v_t is in the interior of $\mathcal{V}(\delta, 1/2)$,
2. In the normal phase as well as the first period of the high phase, v_t is always in the interior of the triangle consisting of vertices $(0,0)$, $(1, 1-\eta)$ and $(1, 1+\frac{\eta(\mu-2)}{3\mu})$.

For this purpose, we need to show that if player 1 plays H sufficiently frequently in the normal phase relative to L, play will transit to the high phase.

To establish this fact, I introduce some extra notation. First, $v = (1, v_{\theta_0})$ with $v_{\theta_0} \in \big(1-\eta, 1+\frac{\mu-2}{3\mu}\eta\big)$ can be written as the following convex combination:

$$v = p_{(H,C)} \cdot (1, 1-\eta) + p_{(L,C)} \cdot (2,2) + p_{(L,S)} \cdot (0,0) \tag{C.9}$$

with $p_{(H,C)}/p_{(L,C)} \geq (2-\mu)/(2\mu-1)$ and $p_{(H,C)}, p_{(L,C)}, p_{(L,S)} > 0$. Choose $\mu^* \in (1/2, \mu)$ close enough to μ such that:

$$\frac{2-\mu^*}{2\mu^*-1} < \frac{1}{2}\Big(\frac{2-\mu}{2\mu-1} + \frac{p_{(H,C)}}{p_{(L,C)}}\Big),$$

and let

$$\beta \equiv \frac{1}{2}\Big(\frac{2-\mu^*}{2\mu^*-1} + \frac{p_{(H,C)}}{p_{(L,C)}}\Big). \tag{C.10}$$

Moreover, there exists $\overline{\delta} \in (0,1)$ such that when $\delta > \overline{\delta}$, there exists $n, k \in \mathbb{N}$ such that:

$$\frac{\delta + \delta^2 + ... + \delta^n}{\delta^n(\delta + \delta^2 + ... + \delta^k)} < \beta < \frac{\delta^{k-1}(\delta + \delta^2 + ... + \delta^n)}{\delta + \delta^2 + ... + \delta^{k-1}}.$$

Let $\widehat{T} \in \mathbb{N}$ be the smallest integer T such that:

$$(1+\lambda x)^T > \frac{1-\mu^*}{\mu - \mu^*}(1-\lambda)^{-k}. \tag{C.11}$$

For an action sequence of length t denoted by $(a_{1,0},...,a_{1,t-1}) \in A_1^t$, let $H_t \equiv \sum_{k=0}^{t-1}(1-\delta)\delta^k \mathbf{1}\{a_{1,k}=H\}$ be the occupation measure of H and let $L_t \equiv \sum_{k=0}^{t-1}(1-\delta)\delta^k \mathbf{1}\{a_{1,k}=L\}$ be the occupation measure of L. The key step is the following Lemma:

Lemma C.1. *For every $\delta > \overline{\delta}$, if play remains in the normal phase in period $s \in \mathbb{N}$, then for every $t \leq s$,*

$$H_t \leq (1-\delta^{\widehat{T}}) + \beta L_t \tag{C.12}$$

PROOF OF LEMMA C.1: For every $\mu_t \in (\mu^*, 1)$, let

$$\overline{\mu}_{t+1}(\mu_t) \equiv \mu_t + \lambda x(\mu_t - \mu^*)$$

and

$$\underline{\mu}_{t+1}(\mu_t) \equiv \mu_t - \lambda(\mu_t - \mu^*),$$

which are the posterior beliefs in period $t+1$ after observing H and L when there is no phase transition. I start with the following useful observation, which says that player 2's posterior belief after 2 periods remains unchanged if we swap an L with an adjacent H.

Lemma C.2. *For every μ_t such that $\overline{\mu}_{t+1}(\mu_t) < 1$,*

$$\overline{\mu}_{t+2}\Big(\underline{\mu}_{t+1}(\mu_t)\Big) = \underline{\mu}_{t+2}\Big(\overline{\mu}_{t+1}(\mu_t)\Big). \tag{C.13}$$

PROOF OF LEMMA C.2: To see this,

$$\overline{\mu}_{t+2}\Big(\underline{\mu}_{t+1}(\mu_t)\Big) = \mu_t - \lambda(\mu_t - \mu^*) + (1-\lambda)\lambda x(\mu_t - \mu^*),$$

and

$$\underline{\mu}_{t+2}\Big(\overline{\mu}_{t+1}(\mu_t)\Big) = \mu_t + \lambda x(\mu_t - \mu^*) - \lambda\Big((\mu_t - \mu^*)(1+\lambda x)\Big) = \mu_t - \lambda(\mu_t - \mu^*) + (1-\lambda)\lambda x(\mu_t - \mu^*).$$

□

Back to the proof of Lemma C.1. Suppose towards a contradiction that there exists $s \in \mathbb{N}$ and an action sequence of length s: $(a_{1,0},...,a_{1,s-1})$, such that play remains in the *normal phase* in period s but there exists $t \leq s$ such that (C.12) fails. Let K be the number of L in $(a_{1,0},...,a_{1,t-1})$. The proof is done by induction on K. When $K \leq k$, the definition of $\widehat{T}$ implies that if the number of L is less than k and the number of H is more than $\widehat{T}$, then player 2's belief reaches 1 and play transits to the high phase. Suppose the conclusion holds for all $K \leq N$, when $K = N + 1$, I obtain a contradiction in three steps:

1. First, if play remains in the normal phase in period t, then for every $l < t$,

$$H_t - H_l > \beta(L_t - L_l)$$

   This is because otherwise, $H_t > (1 - \delta^{\widehat{T}}) + \beta L_t$ together with the above inequality imply that:

$$H_l > (1 - \delta^{\widehat{T}}) + \beta L_l$$

   Moreover, there exists at least one L in the sequence $(a_{1,l}, ..., a_{1,t-1})$ since $H_t - H_l > \beta(L_t - L_l)$. By the induction hypothesis, play will reach the high phase before period l, which leads to a contradiction.

2. Second, for every $n + k$ consecutive periods $(a_{1,r}, ..., a_{1,r+n+k-1})$,

$$\#\Big\{i \Big| i \in \{r, ..., r + n + k - 1\} \text{ and } a_{1,i} = H\Big\} \geq n + 1.$$

   This is because otherwise, the conclusion in step 1 implies that in the last $n + k$ periods, there are at least $n + 1$ periods where player 1 plays H. If the number of L in a $n + k$ sequence has is at most n, then there exists a $n + k$ sequence where H is played n times and L is played k times. This sequence satisfies:

   - $H_{r+n+k} - H_r \leq \beta(L_{r+n+k} - L_r)$ and $\mu_r < \mu_{r+n+k}$.

   Then, consider $(\tilde{a}_{1,0}, ..., \tilde{a}_{1,t-n-k-1}) \equiv (a_{1,0}, ..., a_{1,r-1}, a_{1,r+n+k}, ..., a_{1,t-1})$, which is an action sequence with length $t - n - k$. According to the conclusion in step 1,

$$H_t - H_{r+n+k} > \beta(L_t - L_{r+n+k}),$$

   but

$$H_{r+n+k} - H_r \leq \beta(L_{r+n+k} - L_r)$$

   so let $\tilde{H}_0, ..., \tilde{H}_{t-n-k}$ be the state variables computed via $(\tilde{a}_{1,0}, ..., \tilde{a}_{1,t-n-k-1})$, we have:

$$\tilde{H}_{t-n-k} > (1 - \delta^{\widehat{T}}) + \beta \tilde{L}_{t-n-k}$$

   and by induction hypothesis, belief will reach 1 and play reaches the *high phase* before period $t - n - k$ if player 1 plays according to $(\tilde{a}_{1,0}, ..., \tilde{a}_{1,t-n-k-1})$.

Back to the original action sequence, since $\mu_r < \mu_{r+n+k}$ and belief will reach 1 according to $(\tilde{a}_{1,0}, ..., \tilde{a}_{1,t-n-k-1})$ in period $\tau \leq t-n-k$, then by construction, belief will reach 1 according to $(a_{1,0}, ..., a_{1,t-1})$ in period no later than $\tau + n + k$, which leads to a contradiction.

3. Third, given that the number of H is at least $n+1$ in every sequence with length $n+k$, consider the following operation:

   - For two consecutive periods l and $l+1$ with $l+1 \leq t-n-k$, if $a_{1,l} = L$ and $a_{1,l+1} = H$, according to Lemma C.2, if $\overline{\mu}_{l+1}(\mu_l) < 1$, then

     $$\overline{\mu}_{l+2}\Big(\underline{\mu}_{l+1}(\mu_l)\Big) = \underline{\mu}_{l+2}\Big(\overline{\mu}_{l+1}(\mu_l)\Big).$$

     i.e. swapping $a_{1,l}$ and $a_{1,l+1}$ will not change the posterior beliefs in every period after $l+1$.

   First, if $\overline{\mu}_{l+1}(\mu_l) = 1$, then let us examine player 2's posterior belief after observing $(a_{1,l}, ..., a_{1,l+n+k-1})$. According to step 2, the number of H in this sequence is at least $n+1$. By construction, μ_s will reach 1 no later than period $l+n+k$.

   Second, if $\overline{\mu}_{l+1}(\mu_l) < 1$, then if play remains in the normal phase in period t under action sequence $(a_{1,0}, ..., a_{1,t-1})$, it will still remain in the normal phase in period t under action sequence

   $$(a_{1,0}, ..., a_{1,l-1}, a_{1,l+1}, a_{1,l}, a_{1,l+2}, ..., a_{1,t-1}).$$

   Conducting this operation for finitely many times, we could obtain a sequence with length $n+k$ such that the number of H is n and the number of L is k. Repeating the argument in step 2, we know that play will reach the *high phase* under this modified sequence. By construction, play will reach the high phase under the original sequence, which leads to a contradiction.

□

Back to verify players' incentive constraints. To show statement 1, note that Lemma C.1 implies the existence of $\xi < 1$, such that:

$$\beta\Big[p_{(L,C)} + (1-\delta)\Big] + (1-\delta^{\widehat{T}}) < \xi p_{(H,C)}.$$

Therefore, suppose play will transit to the low phase after playing L in period T, then $v_{T,\theta_0} \geq (1-\eta)v_{T,\theta_1}$ and

$$v_{T,\theta_1} \geq \frac{(1-\xi)p_{(H,C)}}{(1-\xi)p_{(H,C)} + p_{(L,C)}}$$

with the RHS uniformly bounded away from 0 for every action path remaining on the normal phase. Therefore,

$$v_{T+1} \equiv \delta^{-1}\Big(v_T - (1-\delta)(2,2)\Big) \in \mathcal{V}(\delta, 1/2)$$

when δ is close enough to 1.

To verify the second statement, we only need to show that if play remains in the normal phase in period T, then $v_{T,\theta_0} < \Big(1 + \frac{\mu-2}{3\mu}\eta\Big)v_{T,\theta_1}$, or equivalently,

$$p_{(H,C)} - H_T > \frac{1-\mu}{2\mu-1}(p_{(L,C)} - L_T) \tag{C.14}$$

Suppose towards a contradiction that (C.14) fails, then according to Lemma C.1,

$$H_T \leq (1 - \delta^{\widehat{T}}) + \beta L_T.$$

Adding up these two inequalities, we have:

$$\begin{aligned} p_{(H,C)} &\leq (1-\delta^{\widehat{T}}) + \beta L_T + \frac{1-\mu}{2\mu-1}(p_{(L,C)} - L_t) \\ &\leq (1-\delta^{\widehat{T}}) + \beta p_{(L,C)}. \end{aligned}$$

The 2nd inequality uses $\beta > \frac{1-\mu}{2\mu-1}$. This leads to a contradiction if $1 - \delta^{\widehat{T}} < \frac{1}{2}\Big(\frac{p_{(H,C)}}{p_{(L,C)}} - \beta\Big)p_{(L,C)}$, which will happen if δ is close enough to 1.

Remarks: One remarkable feature of the above equilibrium is that in the $\delta \to 1$ limit, type θ_0 can obtain payoffs strictly higher than $1-\eta$, which is the highest payoff he can get by imitating type θ_1 in every period. It also has the intuitive feature that player 2's belief increases once he observes H and decreases once he observes L. In periods where belief is close to 1, type θ_0 plays L for sure while type θ_1 mixes between H and L. This gives type θ_0 a stage game payoff strictly larger than $1-\eta$ while not revealing his type to player 2. The periods in which type θ_0 can steal extra payoffs occur with positive occupation measure in the $\delta \to 1$ limit.

The upper limit on type θ_0's equilibrium payoff for a fixed v_{θ_1} is determined by the following trade-off. On one hand, if the positive belief update after observing H increases relative to the negative belief update after observing

L, then player 2's belief will reach 1 faster and type θ_0 can '*steal*' extra payoffs more frequently. On the other hand, player 2's incentives to play C requires H to be played with high enough probability, or equivalently, the positive belief update following H must be small enough. The extent to which type θ_0 can steal extra payoff depends on the prior belief, as a more optimistic prior motivates player 2 to play C and allows player 1 to play L with higher probability. As a result, type θ_0's highest limiting equilibrium payoff is increasing in μ.

I relate this finding to two strands of literature on reputation effects. First, the constructed equilibrium features '*reputation cycles*', in which the low type boosts up his reputation by playing H while milking his reputation when it is sufficiently good. Instead of driven by changing types (Phelan 2006) or limited memory (Liu and Skrzypacz 2014), reputation cycles occur in my model because the low state is never fully revealed, so the building-milking cycle can last for unboundedly many rounds as $\delta \to 1$.[9] Second, my construction features slow learning (in particular, when μ_t approaches the lower bound μ^*), which also occurs in the war of attrition models of reputation, for example Abreu and Gul (2000), Atakan and Ekmekci (2013), Salomon and Forges (2015), etc. In these models, slow learning occurs as both players are patient and they are gradually screening their opponent's type. In my model, the uninformed players are myopic and slow learning is the result of interdependent values and player 2's incentive constraint: if belief deteriorates too fast following L, then it is impossible to motivate player 1 to rebuild his reputation after he plays L for a large number of periods.

Minimizing Direction: Constructing equilibria which achieves payoff $(v_{\theta_1}, v_{\theta_0})$ with $v_{\theta_0} > \underline{v}(v_{\theta_1}, \mu)$ is similar to the above construction, which I briefly summarize the differences to avoid duplication.

The constructed equilibrium still has three phases: a '*normal phase*', a '*high phase*' and a '*low phase*', which is characterized by $\mu^{**} \in (\mu, 1)$ and $\lambda \in (0, 1)$. Play starts from the *normal phase*, in which player 2 always plays S and player 1's strategy is pinned down by the following belief updating process:

- If L is played in period t, then:

$$\mu_{t+1} = \max\Big\{1/2, \mu_t - \lambda \frac{2-\mu^{**}}{2\mu^{**}-1}(\mu^{**} - \mu_t)\Big\}. \tag{C.15}$$

- If H is played in period t, then:

$$\mu_{t+1} = \begin{cases} \mu_t + \lambda(\mu^{**} - \mu_t) & \text{if } v_{t+1,\theta_0} \leq (1-\eta)v_{t+1,\theta_1} \\ \mu^{**} & \text{if } v_{t+1,\theta_0} > (1-\eta)v_{t+1,\theta_1}. \end{cases} \tag{C.16}$$

[9] If the state is fully revealed when player 2's prior belief is optimistic, then player 1's equilibrium payoff must be low. This is because type θ_0's continuation payoff is low after player 2 learns his type, and his incentive constraint to separate requires his continuation payoff by pooling with type θ_1 cannot be too high.

Play transits from the *normal phase* to the *low phase* in period T if $\mu_T = 1/2$, after which the continuation play delivers payoff vector v_T. Play transits from the *normal phase* to the *high phase* in period T if μ_T reaches μ^{**}, and by construction of the belief updating process, $v_{T,\theta_0} \in \Big[(1-\eta)v_{T,\theta_1}, (1-\eta)v_{T,\theta_1} + 2(1-\delta)\Big]$. The high phase delivers payoff v_T. Both the high phase and the low phase are absorbing. Similar to the maximizing direction, one can verify that:

1. Player 2 has a strict incentive to play S in the normal phase.
2. v_T is in the interior of $\mathcal{V}(\delta, 1/2)$ if play transits to the low phase in period T.
3. v_T is in the interior of the triangle consisting of $(0,0)$, $(1, 1-\eta)$ and $(1, \overline{v}(1, \mu^{**}))$ if play transits to the high phase in period T.

which completes the construction.

C.2 Upper Bound of the Limiting Equilibrium Payoff Set

In this subsection, I show that $\limsup_{\delta \to 1} \mathcal{V}^*(\delta, \mu) \subset \overline{\mathcal{V}}(\mu)$ for every $\mu \in (1/2, 1)$. First, it is obvious that every equilibrium payoff vector $v \gg (0,0)$. Moreover, Proposition 4.2 in Pei (2018) implies the following Lemma:

Lemma C.3. *For every μ and δ, if $(v_{\theta_1}, v_{\theta_0}) \in \mathcal{V}^*(\delta, \mu)$, then $v_{\theta_1} \leq 1$.*

In what follows, I show that $v_{\theta_0} \leq \overline{v}(v_{\theta_1}, \mu)$ for every $(v_{\theta_1}, v_{\theta_0}) \in \limsup_{\delta \to 1} \mathcal{V}^*(\delta, \mu)$. I start with introducing an auxiliary maximization problem.

Auxiliary Static Maximization Problem: For every $\widetilde{\mu} \in (1/2, 1)$ and $U_1 \in [-1, 2]$, let

$$U(U_1, \widetilde{\mu}) \equiv \max_{\{q^i, \alpha_1^i, \alpha_2^i\}_{i \in \mathbb{N}}} \sum_{i=1}^{\infty} q^i u_1(\theta_0, \alpha_1^i, \alpha_2^i),$$ [10]

subject to $\sum_{i=1}^{\infty} q^i = 1$, and for all $i \in \mathbb{N}$, we have $q^i \in [0,1]$,

$$\alpha_2^i \in \arg\max_{\alpha_2' \in \Delta(A_2)} \Big\{ \widetilde{\mu} u_2(\theta_1, \alpha_1^i, \alpha_2') + (1-\widetilde{\mu}) u_2(\theta_0, \alpha_1^i, \alpha_2') \Big\}$$

and

$$\sum_{i=1}^{\infty} q^i u_1(\theta_1, \alpha_1^i, \alpha_2^i) \leq U_1.$$

The function $U(U_1, \widetilde{\mu})$ is depicted in Figure 2. One can verify that $U(U_1, \widetilde{\mu}) \leq \overline{v}(U_1, \widetilde{\mu})$.

[10] As a convention, when the choice set is empty, let the maximum value be $-\infty$ and the minimum value be $+\infty$.

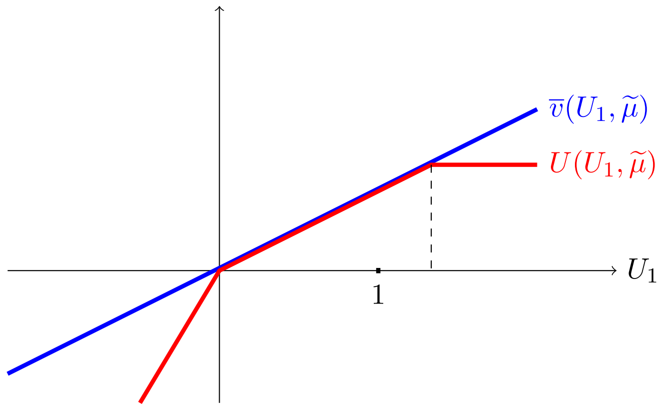

Figure 4: $U(U_1, \widetilde{\mu})$ (red line) and $\overline{v}(U_1, \widetilde{\mu})$ (blue line) as functions of U_1

For a Nash Equilibrium σ with player 1's payoff vector being $(v_{\theta_1}, v_{\theta_0})$, I establish an upper bound for:

$$D^{\sigma} \equiv v_{\theta_0} - U(v_{\theta_1}, \mu), \tag{C.17}$$

and show that the bound converges to 0 as $\delta \to 1$. I partition $\mathcal{H}$ into regular histories ($\mathcal{H}^R$) and irregular histories ($\mathcal{H}^I$), depending in the support of player 1's behavior strategy at that history. Let $h^t \in \mathcal{H}^I$ if and only if $\text{supp}\{\sigma_{\theta_1}(h^t)\} = \{H, L\}$ but $\text{supp}\{\sigma_{\theta_0}(h^t)\} \neq \{H, L\}$, otherwise, $h^t \in \mathcal{H}^R$.

Adjusted Strategy: I define an adjusted strategy $\widehat{\sigma}_1 : \mathcal{H} \to \Delta(A_1)$, which will be useful to establish the upper bound on (C.17). For every $h^t \in \mathcal{H}^R$:

- If $\sigma_{\theta_1}(h^t) = H$, then $\widehat{\sigma}_1(h^t) = H$.
- If $\sigma_{\theta_1}(h^t) = L$, then $\widehat{\sigma}_1(h^t) = L$.
- If $\text{supp}\{\sigma_{\theta_1}(h^t)\} = \text{supp}\{\sigma_{\theta_0}(h^t)\} = \{H, L\}$, then $\widehat{\sigma}_1(h^t) = \sigma_{\theta_1}(h^t)$ if $\sigma_{\theta_1}(h^t)$ assigns a larger probability to H compared to $\sigma_{\theta_0}(h^t)$, and $\widehat{\sigma}_1(h^t) = \mu(h^t)\sigma_{\theta_1}(h^t) + (1 - \mu(h^t))\sigma_{\theta_0}(h^t)$ otherwise.

For every $h^t \in \mathcal{H}^I$, let $\widehat{\sigma}_1(h^t) \equiv \sigma_{\theta_0}(h^t)$.

By construction, besides histories in which $\sigma_{\theta_1}(h^t) = H$ and $\sigma_{\theta_0}(h^t) = L$ or $\sigma_{\theta_1}(h^t) = L$ and $\sigma_{\theta_0}(h^t) = H$, $\widehat{\sigma}_1$ is always optimal for type θ_0. At regular histories, it is also optimal for type θ_1. Let $\mathcal{P}$ be the probability measure over $\mathcal{H}$ induced by $(\widehat{\sigma}_1, \sigma_2)$.

Decomposing D^{σ}: Let $\overline{\mu}(h^t), \underline{\mu}(h^t)$ be the posterior beliefs of player 2 after observing the action played at h^t with $\overline{\mu}(h^t) \geq \mu(h^t) \geq \underline{\mu}(h^t)$. Let $q(h^t)$ be the probability that the posterior belief is $\overline{\mu}(h^t)$ according to $\widehat{\sigma}_1(h^t)$.

Let $\overline{V}(h^t)$ and $\underline{V}(h^t)$ be the corresponding continuation values of type θ_1 at belief $\overline{\mu}(h^t)$ and $\underline{\mu}(h^t)$, respectively. Let $\widetilde{u}(h^t)$ be type θ_0's stage game payoff at h^t. For every $h^t \in \mathcal{H}^R$, let

$$D^\sigma(h^t) \equiv (1-\delta)\widetilde{u}(h^t) + \delta\Big\{q(h^t)U\big(\overline{V}(h^t), \overline{\mu}(h^t)\big) + (1-q(h^t))U\big(\underline{V}(h^t), \underline{\mu}(h^t)\big)\Big\} - \overline{v}\big(V(h^t), \mu(h^t)\big)$$

and for every $h^t \in \mathcal{H}^I$, let

$$D^\sigma(h^t) \equiv (1-\delta)\widetilde{u}(h^t) + \delta\overline{v}\big(\underline{V}(h^t), \underline{\mu}(h^t)\big) - \overline{v}\big(V(h^t), \mu(h^t)\big).$$

The Abel's summation formula implies the following decomposition of D^σ:

$$D^\sigma = \sum_{t=0}^{\infty}\sum_{h^t} \delta^t \mathcal{P}(h^t) D^\sigma(h^t). \tag{C.18}$$

Regular Histories: Let $M \equiv \max_{(\theta,a)\in\Theta\times A} u_1(\theta, a) - \min_{(\theta,a)\in\Theta\times A} u_1(\theta, a)$. If $\sigma_{\theta_1}(h^t) = H$ and $\sigma_{\theta_0}(h^t) = L$ or $\sigma_{\theta_1}(h^t) = L$ and $\sigma_{\theta_0}(h^t) = H$, then

$$\sum_{h^s \succsim h^t} \delta^{s-t}\mathcal{P}(h^s)D^\sigma(h^s) \leq (1-\delta)M,$$

which converges to 0 as $\delta \to 1$. Therefore, from now and onwards, I ignore such histories. Moreover, if $\mu(h^t) < 1/2$, then

$$\sum_{h^s \succsim h^t} \delta^{s-t}\mathcal{P}(h^s)D^\sigma(h^s) \leq 0,$$

so therefore, I assume that $\mu(h^t) \geq 1/2$.

Since $\widehat{\sigma}_1$ is also optimal for type θ_1, we have:

$$V(h^t) = (1-\delta)u(h^t) + \delta\Big(q(h^t)\overline{V}(h^t) + (1-q(h^t))\underline{V}(h^t)\Big), \tag{C.19}$$

where $u(h^t)$ is the stage game payoff of type θ_1 at h^t when he plays $\widehat{\sigma}_1(h^t)$. By construction, we have:

$$\tilde{u}(h^t) \leq U(u(h^t), \mu(h^t)) \leq \overline{v}(u(h^t), \mu(h^t)), \tag{C.20}$$

we know that:

$$D^\sigma(h^t) \leq \frac{2\delta\eta}{3}\Big\{\frac{q(h^t)\overline{V}(h^t)(\overline{\mu}(h^t) - \mu(h^t))}{\overline{\mu}(h^t)\mu(h^t)} + \frac{(1-q(h^t))\underline{V}(h^t)(\underline{\mu}(h^t) - \mu(h^t))}{\underline{\mu}(h^t)\mu(h^t)}\Big\}. \tag{C.21}$$

Moreover, if $\overline{q}(h^t) \notin \{0,1\}$, type θ_1's incentive constraint requires that:

$$\left|\overline{V}(h^t) - \underline{V}(h^t)\right| < M(1-\delta). \tag{C.22}$$

According to the construction of $\widehat{\sigma}_1$, we know that either

$$q(h^t) = \frac{\overline{\mu}(h^t)(\mu(h^t) - \underline{\mu}(h^t))}{\mu(h^t)(\overline{\mu}(h^t) - \underline{\mu}(h^t))} \text{ and } 1 - q(h^t) = \frac{\underline{\mu}(h^t)(\overline{\mu}(h^t) - \mu(h^t))}{\mu(h^t)(\overline{\mu}(h^t) - \underline{\mu}(h^t))}$$

or

$$q(h^t) = \frac{\mu(h^t) - \underline{\mu}(h^t)}{\overline{\mu}(h^t) - \underline{\mu}(h^t)} \text{ and } 1 - q(h^t) = \frac{\overline{\mu}(h^t) - \mu(h^t)}{\overline{\mu}(h^t) - \underline{\mu}(h^t)}.$$

Moreover,

$$0 \leq \frac{(\overline{\mu}(h^t) - \mu(h^t))(\mu(h^t) - \underline{\mu}(h^t))}{\overline{\mu}(h^t) - \underline{\mu}(h^t)} \leq \frac{1}{\sqrt{2}}\mathbb{E}^{\mathcal{P}}\left[||\overrightarrow{\mu}(h^{t+1}) - \overrightarrow{\mu}(h^t)||\Big|h^t\right] \tag{C.23}$$

where $\overrightarrow{\mu}(h^t)$ denotes the entire belief vector and $||\cdot||$ is the $\mathcal{L}^2$-norm. Plugging the expressions of $q(h^t)$ and $1-q(h^t)$ into (C.21), using the facts that $\mu(h^t) \geq 1/2$, (C.22) and (C.23), we obtain the existence of a constant $K \in \mathbb{R}_+$ such that:

$$D^\sigma(h^t) \leq (1-\delta)K\mathbb{E}^{\mathcal{P}}\left[||\overrightarrow{\mu}(h^{t+1}) - \overrightarrow{\mu}(h^t)||\Big|h^t\right]. \tag{C.24}$$

Irregular Histories: First, consider the case when $\text{supp}\Big(\sigma_{\theta_0}(h^t)\Big) = \{H\}$ and $\text{supp}\Big(\sigma_{\theta_1}(h^t)\Big) = \{H, L\}$. Type θ_1's incentive constraint gives:

$$V(h^t) = (1-\delta)u_1(\theta_1, L, \sigma_2(h^t)) + \delta\overline{V}(h^t) = (1-\delta)u_1(\theta_1, H, \sigma_2(h^t)) + \delta\underline{V}(h^t),$$

which implies that:

$$\begin{aligned} D^\sigma(h^t) &= (1-\delta)\Big(u_1(\theta_0, H, \sigma_2(h^t)) - \Big(\frac{5}{3} - \frac{2\eta}{3\mu(h^t)}\Big)u_1(\theta_1, H, \sigma_2(h^t))\Big) \\ &\quad +\delta\underbrace{\Big(\Big(\frac{5}{3} - \frac{2\eta}{3\underline{\mu}(h^t)}\Big)\underline{V}(h^t) - \Big(\frac{5}{3} - \frac{2\eta}{3\mu(h^t)}\Big)\underline{V}(h^t)\Big)}_{<0 \text{ since } \underline{\mu}(h^t)<\mu(h^t)} \\ &\leq \eta(1-\delta)\underbrace{(u_1(\theta_1, H, \sigma_2(h^t)) - 1)}_{\leq 0} \leq 0. \end{aligned} \tag{C.25}$$

Next, consider the case when $\text{supp}\Big(\sigma_{\theta_0}(h^t)\Big) = \{L\}$ and $\text{supp}\Big(\sigma_{\theta_1}(h^t)\Big) = \{H, L\}$, then similar to the previous case, we have:

$$\begin{aligned} D^\sigma(h^t) &= (1-\delta)\Big(u_1(\theta_0, L, \sigma_2(h^t)) - \Big(\frac{5}{3} - \frac{2\eta}{3\mu(h^t)}\Big)u_1(\theta_1, L, \sigma_2(h^t))\Big) \\ &\quad +\delta\Big(\underbrace{\Big(\frac{5}{3} - \frac{2\eta}{3\underline{\mu}(h^t)}\Big)\underline{V}(h^t) - \Big(\frac{5}{3} - \frac{2\eta}{3\mu(h^t)}\Big)\underline{V}(h^t)}_{<0 \text{ since } \underline{\mu}(h^t)<\mu(h^t)}\Big) \\ &\leq 2\eta(1-\delta) \quad \underbrace{u_1(\theta_1, L, \sigma_2(h^t))}_{\geq 0 \text{ only when } \sigma_2(h^t)[C]>0.} \end{aligned} \tag{C.26}$$

which is strictly positive only when $\sigma_2(h^t)[C] > 0$. In this case, $D^\sigma(h^t) \leq 2(1-\delta)\eta$.

Upper Bound on the Expected Number of Irregular Histories: Recall that $\mathcal{P}$ is the probability measure over $\mathcal{H}$ induced by $(\widehat{\sigma}_1, \sigma_2)$. I put an upper bound on:

$$Q \equiv \mathbb{E}^{\widetilde{\mathcal{P}}}\Big[\mathbf{1}\Big\{h^t \in \mathcal{H}^I, \sigma_2(h^t)[C] > 0 \text{ and } \sigma_{\theta_0}[H] = 0\Big\}\Big] \tag{C.27}$$

This bound is interesting since it can be applied to the previous part and obtain:

$$\mathbb{E}^{\widetilde{\mathcal{P}}}\Big[\sum_{t=0}^{\infty}\sum_{h^t \in \mathcal{H}^I} \delta^t D^\sigma(h^t)\Big] \leq 2(1-\delta)\eta Q. \tag{C.28}$$

which provides an upper bound on type θ_0's gain in continuation value at irregular histories.

For every $h^t \in \mathcal{H}$ with $\widetilde{\mathcal{P}}(h^t) > 0$, let

$$\lambda(h^t) \equiv (1 - \mu(h^t))/\mu(h^t).$$

Under the probability measure induced by $(\widehat{\sigma}_1, \sigma_2)$, if $h^t \in \mathcal{H}^R$,

$$\mathbb{E}[\lambda(h^{t+1})|h^t] = \lambda(h^t).$$

If $h^t \in \mathcal{H}^I$,

$$\mathbb{E}[\lambda(h^{t+1})|h^t] > \lambda(h^t).$$

while under the probability measure induced by $(\sigma_{\theta_1}, \sigma_2)$,

$$\mathbb{E}[\lambda(h^{t+1})|h^t] = \lambda(h^t).$$

even if $h^t \in \mathcal{H}^I$. For every $h^t \in \mathcal{H}^I$ with $\mathcal{P}(h^t) > 0$, if $\sigma_2(h^t)[C] > 0$, according to player 2's incentive constraint, $\lambda(h^{t+1}) = 0$ occurs with probability at least $\frac{1+\lambda(h^t)}{2}$. Therefore, if h^t occurs with probability $\mathcal{P}(h^t)$, we have:

$$\mathbb{E}[\lambda(h^{t+1})] \geq \frac{\mathbb{E}[\lambda(h^t)]}{1 - \frac{1}{2}\mathcal{P}(h^t)}$$

Since $\mathbb{E}[\lambda(h^t)] \leq 1$ for every $t \in \mathbb{N}$, we have:

$$Q \leq -2 \log \lambda(h^0).$$

Upper Bound on D^σ: In the final step, I establish an upper bound on D^σ for every $\sigma \in \text{NE}(\delta, \mu)$. To begin with:

$$\begin{aligned}
D^\sigma &\leq \mathbb{E}\Big[\sum_{t=0}^{\infty} \sum_{h^t} \delta^t D^\sigma(h^t)\Big] \\
&= \mathbb{E}\Big[\sum_{t=0}^{\infty} \sum_{h^t \in \mathcal{H}^R} \delta^t D^\sigma(h^t)\Big] + \mathbb{E}^{\widetilde{\mathcal{P}}}\Big[\sum_{t=0}^{\infty} \sum_{h^t \in \mathcal{H}^I} \delta^t D^\sigma(h^t)\Big] \\
&\leq \mathbb{E}\Big[K \sum_{t=0}^{\infty} \sum_{h^t \in \mathcal{H}^R} (1-\delta)\delta^t \mathbb{E}^{\widetilde{\mathcal{P}}}[||\overrightarrow{\mu}(h^{t+1}) - \overrightarrow{\mu}(h^t)||]\Big] + 2(1-\delta)\eta Q \\
&\leq \mathbb{E}\Big[K \sum_{t=0}^{\infty} \sum_{h^t} (1-\delta)\delta^t \mathbb{E}[||\overrightarrow{\mu}(h^{t+1}) - \overrightarrow{\mu}(h^t)||]\Big] + 2(1-\delta)\eta Q. \qquad \text{(C.29)}
\end{aligned}$$

A well-known application of the Cauchy-Schwarz inequality implies:[11]

$$\mathbb{E}\Big[\sum_{t=0}^{\infty} \sum_{h^t} (1-\delta)\delta^t \mathbb{E}[||\overrightarrow{\mu}(h^{t+1}) - \overrightarrow{\mu}(h^t)||]\Big] \leq 2\sqrt{\frac{1-\delta}{1+\delta}}\sqrt{\mu(1-\mu)}. \qquad \text{(C.30)}$$

Replacing $\mu(h^0)$ with μ and let

$$D^{\delta,\mu} \equiv \sup_{\sigma \in \text{NE}(\delta,\mu)} D^\sigma.$$

and let

$$A(\delta) \equiv 2K\sqrt{\frac{1-\delta}{1+\delta}}\sqrt{\mu(1-\mu)} + 2(1-\delta)\eta Q,$$

[11] The undiscounted version of this inequality was present in Aumann and Maschler (1995), which is used to bound the average variation in beliefs over a long finite horizon.

inequalities (C.28), (C.29) and (C.30) together imply that $D^{\delta,\mu} \leq A(\delta)$, with the RHS converging to 0 as $\delta \to 1$.

Minimum Payoff: To show that $v_{\theta_0} \geq \underline{v}(v_{\theta_1}, \mu)$ for every $(v_{\theta_1}, v_{\theta_0}) \in \limsup_{\delta\to 1} \mathcal{V}^*(\delta, \mu)$, I introduce the following auxiliary minimization problem:

- For every $\widetilde{\mu} \in (1/2, 1)$ and $U_1 \in [-1, 2]$, let

$$W(U_1, \widetilde{\mu}) \equiv \min_{\{q^i,\alpha_1^i,\alpha_2^i\}_{i\in\mathbb{N}}} \sum_{i=1}^{\infty} q^i u_1(\theta_0, \alpha_1^i, \alpha_2^i)$$

  subject to $\sum_{i=1}^{\infty} q^i = 1$, and for all $i \in \mathbb{N}$, $q^i \in [0, 1]$ and

$$\alpha_2^i \in \arg\max_{\alpha_2' \in \Delta(A_2)} \Big\{ \widetilde{\mu} u_2(\theta_1, \alpha_1^i, \alpha_2') + (1 - \widetilde{\mu}) u_2(\theta_0, \alpha_1^i, \alpha_2') \Big\},$$

  and

$$\sum_{i=1}^{\infty} q^i u_1(\theta_1, \alpha_1^i, \alpha_2^i) \geq U_1.$$

Solving this linear program, we have $W(U_1, \widetilde{\mu}) \geq \underline{v}(U_1, \widetilde{\mu})$. The rest of the steps are symmetric to the proof of $v_{\theta_0} \leq \overline{v}(v_{\theta_1}, \mu)$, which I omit to avoid repetition.

D Monotone Supermodular Games with $|A_2| \geq 3$

In this Appendix, I extend the results in monotone supermodular games (Theorems 2 and 3 in Pei 2018) to the case where player 2 has three or more actions. A counterexample in which these conditions are violated is presented in Appendix G.3 of Pei (2018). To begin with, let us recall the definition of monotone-supermodular payoffs:

Condition 1 (M-SM)**.** *The stage game payoff is monotone-supermodular if:*

1. $u_1(\theta, a_1, a_2)$ *is strictly decreasing in* a_1 *and strictly increasing in* a_2*.*

2. $u_1(\theta, a_1, a_2)$ *has strictly increasing differences (or SID) in* (a_1, θ) *and weakly increasing differences (or WID) in* (a_2, θ)*.* $u_2(\theta, a_1, a_2)$ *has SID in* (a_1, a_2) *and* (θ, a_2)*.*

In monotone supermodular games, I establish the commitment payoff bound when the high states are more likely and establish the uniqueness of player 1's equilibrium behavior and payoff when the low states are more likely. The arguments in Appendices C and D in Pei (2018) carry over besides two extra conditions on u_1 for the conclusions in Lemma C.1 and Lemma C.3. I introduce and explain these conditions in the next two subsections.

For notation simplicity, I focus on the case where public history only consists of player 1's action. This is without loss of generality as shown in Appendix C of Pei (2018).

D.1 Semi-Separability in Monotone Supermodular Games

In this subsection, I introduce the following semi-separability condition, which is required to extend the conclusion of Lemma C.1 to the case with $|A_2| \geq 3$.

Condition 2 (Semi-S)**.** *Player* 1*'s payoff is semi-separable if there exist* $f : A_1 \times A_2 \to \mathbb{R}$*,* $v : \Theta \times A_1 \to \mathbb{R}$ *and* $c : \Theta \times A_1 \to \mathbb{R}$ *with* $v(\theta, a_1)$ *non-decreasing in* θ *and* $\min_{\theta \in \Theta} v(\theta, a_1) \geq 0$ *for every* $a_1 \in A_1$ *such that:*

$$u_1(\theta, a_1, a_2) = f(a_1, a_2)v(\theta, a_1) + c(\theta, a_1) \tag{D.1}$$

This semi-separability condition accommodates the usual notion of separable payoffs, where $u_1(\theta, a_1, a_2) = f(a_1, a_2)+c(\theta, a_1)$. It allows for a larger class of payoff structures, and in particular, player 1's cardinal preferences over player 2's action could depend on the state. Non-separability in players' payoffs occurs, for example, in the Spence education signalling model where the employer (player 2) assigns job positions to the worker (player 1) after observing his years of education, or in the warranty provision example where the consumer (player 2) decides how much to purchase after observing the warranty provided by the firm (player 1).

Let $h^t \equiv (a_{1,0}, ..., a_{1,t-1})$ be a generic period t public history, with $\mathcal{H}^t$ be the set of period t public histories and $\mathcal{H} \equiv \cup_{t=0}^{\infty} \mathcal{H}^t$ be the set of public histories. Let $\Sigma_1 : \mathcal{H} \to \Delta(A_1)$ be an interim strategy for player 1. Let $\overline{\Sigma}_1 \subset \Sigma_1$ be the subset of player 1's strategies such that for every $\overline{\sigma}_1 \in \overline{\Sigma}_1$:

$$\overline{\sigma}_1(h^t) = \overline{a}_1 \text{ for every } h^t = (\overline{a}_1, ..., \overline{a}_1). \tag{D.2}$$

Let $\underline{\Sigma}_1 \subset \Sigma_1$ be the subset of player 1's strategies such that for every $\underline{\sigma}_1 \in \underline{\Sigma}_1$:

$$\underline{\sigma}_1(h^t) = \underline{a}_1 \text{ for every } h^t = (\underline{a}_1, ..., \underline{a}_1). \tag{D.3}$$

Intuitively, $\overline{\Sigma}_1$ is the set of player 1's interim strategies that always plays the highest action on the equilibrium path, $\underline{\Sigma}_1$ is the set of player 1's interim strategies that always plays the lowest action on the equilibrium path. The following result generalizes Lemma C.1 in Pei (2018):

Lemma D.1. *If player* 1*'s payoff is monotone supermodular and semi-separable, then for every pair of states* $\theta \succ \theta^*$ *and in every Nash Equilibrium* $\sigma = (\sigma_\omega, \sigma_2)_{\omega \in \Omega}$*:*

1. *If there exists $\overline{\sigma}_1 \in \overline{\Sigma}_1$ such that $\overline{\sigma}_1$ is type θ^*'s best reply to σ_2, then $\sigma_\theta \in \overline{\Sigma}_1$.*

2. *If there exists $\underline{\sigma}_1 \in \underline{\Sigma}_1$ such that $\underline{\sigma}_1$ is type θ's best reply to σ_2, then $\sigma_{\theta^*} \in \underline{\Sigma}_1$.*

Intuitively, one needs to show that when always playing the highest action is a best response for player 1 in a low state, then the equilibrium strategy in a high state must be playing the highest action for sure at every on-path history. Since player 2 can learn the state only via the history of player 1's actions, every *action path* of player 1 induces a distribution over player 2's action paths, which is independent of θ.[12] Viewing the game in its normal form, player 1 is choosing a distribution of a_1 (the occupation measure) to induce a distribution of a_2. Therefore, the structure of the problem is similar to that of one-shot signalling games. The conclusion in Lemma D.1 requires player 1's equilibrium strategy to be *monotone* with respect to the state.

As shown in Liu and Pei (2017), supermodularity of players' payoffs cannot guarantee the monotonicity of *all* equilibria in one-shot signalling games. When $|A_2| = 2$ (or '*binary action games*'), player 1's equilibrium strategy must be monotone if the game is '*monotone supermodular*'. The proof relies on the fact that every pair of elements in $\Delta(A_2)$ can be ranked via first order stochastic dominance (FOSD), so that every type of player 1 is choosing between a set of equilibrium (mixed) action profiles that can be completely ranked.

When $|A_2| \geq 3$, however, not every pair of elements in $\Delta(A_2)$ can be ranked via FOSD. The semi-separability condition is equivalent to the '*increasing absolute differences over distributions*' property introduced in Liu and Pei (2017), which provides an endogenous *complete ranking* over the distributions on A_2 conditional on every $a_1 \in A_1$.[13] This finding is reported as Proposition 3 in Liu and Pei (2017), which I restate as Lemma D.2:

Lemma D.2. *u_1 is semi-separable if and only if for every $a_1 \in A_1$ and $\alpha_2, \alpha_2' \in \Delta(A_2)$, $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ is either non-negative and non-decreasing for all $\theta \in \Theta$, or non-positive and non-increasing for all $\theta \in \Theta$.*

PROOF OF LEMMA D.2: See the proof of Proposition 3 in Liu and Pei (2017). $\square$

PROOF OF LEMMA D.1: I will only show statement 1 since the proof of statement 2 is symmetric. Suppose towards a contradiction that $\overline{\sigma}_1$ is type θ^*'s best reply but $\sigma_\theta \notin \overline{\Sigma}_1$. Let $\overline{\alpha}_t \in \Delta(A_1 \times A_2)$ be the distribution over action profiles in period t under $(\overline{\sigma}_1, \sigma_2)$ and let $\alpha_t \in \Delta(A_1 \times A_2)$ be the distribution over action profiles in period

[12] This is no longer true when player 2 can also observe exogenous signals about θ beyond player 1's action. As a result, there exist equilibria where player 1 uses non-monotone strategies, as shown in Feltovich et al.(2002) in an education signalling context.

[13] The other finding in Liu and Pei (2017), that every equilibrium in a one-shot signalling game is monotone when player 2's payoff is strictly quasi-concave in a_2 cannot be applied to this repeated signalling game setting. This is because player 2's belief about $\Theta \times A_1$ can change over time, so the occupation measure on A_2 may attach positive probability to more than two elements despite player 2 plays at most two actions with positive probability in every period.

t under $(\sigma_\theta, \sigma_2)$. Let

$$\overline{\alpha}^\delta \equiv \sum_{t=0}^{\infty}(1-\delta)\delta^t \overline{\alpha}_t \quad \text{and} \quad \alpha^\delta \equiv \sum_{t=0}^{\infty}(1-\delta)\delta^t \alpha_t$$

be the occupation measures over action profiles induced by $(\overline{\sigma}_1, \sigma_2)$ and $(\sigma_\theta, \sigma_2)$. Let $\overline{\alpha}_{2,t}$ and $\alpha_{2,t}$ be the marginal distributions of $\overline{\alpha}_t$ and α_t on A_2, respectively. Let $\hat{\alpha}_t$ be defined as:

$$\hat{\alpha}_t(a_1, a_2) \equiv \begin{cases} \alpha_{2,t}(a_2) & \text{if } a_1 = \overline{a}_1 \\ 0 & \text{if } a_1 \neq \overline{a}_1 \end{cases} \tag{D.4}$$

Let $\hat{\alpha}^\delta \equiv \sum_{t=0}^{\infty}(1-\delta)\delta^t \hat{\alpha}_t$. Let $\overline{\alpha}_2^\delta$, α_2^δ and $\hat{\alpha}_2^\delta$ be the marginal distributions of $\overline{\alpha}^\delta$, α^δ and $\hat{\alpha}^\delta$ on A_2, respectively. By construction, $\alpha_2^\delta = \hat{\alpha}_2^\delta$.

Type θ^*'s incentive constraint requires that $u_1(\theta^*, \overline{\alpha}^\delta) \geq u_1(\theta^*, \alpha^\delta)$. Since u_1 is strictly decreasing in a_1 and $\sigma_\theta \notin \overline{\Sigma}$, we have

$$u_1(\theta^*, \overline{\alpha}^\delta) \geq u_1(\theta^*, \alpha^\delta) > u_1(\theta^*, \hat{\alpha}^\delta). \tag{D.5}$$

Using '$\succsim_{a_1}$' (or '$\succ_{a_1}$') to denote the order over $\Delta(A_2)$ based on a_1. According to Lemma D.2, we have $\overline{\alpha}_2^\delta \succ_{\overline{a}_1} \hat{\alpha}_2^\delta$. On the other hand, type θ's and type θ^*'s incentive constraints together imply:

$$u_1(\theta^*, \overline{\alpha}^\delta) - u_1(\theta^*, \alpha^\delta) \geq 0 \geq u_1(\theta, \overline{\alpha}^\delta) - u_1(\theta, \alpha^\delta). \tag{D.6}$$

Consider $u_1(\tilde{\theta}, \overline{\alpha}^\delta) - u_1(\tilde{\theta}, \alpha^\delta)$ as a function of $\tilde{\theta}$ and decompose it into:

$$u_1(\tilde{\theta}, \overline{\alpha}^\delta) - u_1(\tilde{\theta}, \alpha^\delta) = u_1(\tilde{\theta}, \overline{\alpha}^\delta) - u_1(\tilde{\theta}, \hat{\alpha}^\delta) + u_1(\tilde{\theta}, \hat{\alpha}^\delta) - u_1(\tilde{\theta}, \alpha^\delta).$$

Since $\overline{\alpha}_2^\delta \succ_{\overline{a}_1} \hat{\alpha}_2^\delta$ and u_1 exhibits WID in $(\tilde{\theta}, a_2)$, we have $u_1(\tilde{\theta}, \overline{\alpha}^\delta) - u_1(\tilde{\theta}, \hat{\alpha}^\delta)$ being non-decreasing in $\tilde{\theta}$. Since $\sigma_\theta \notin \Sigma_1$ and u_1 exhibits SID in $(\tilde{\theta}, a_1)$, we have $u_1(\tilde{\theta}, \hat{\alpha}^\delta) - u_1(\tilde{\theta}, \alpha^\delta)$ is strictly increasing in $\tilde{\theta}$. Therefore, $\theta \succ \theta^*$ implies that:

$$u_1(\theta^*, \overline{\alpha}^\delta) - u_1(\theta^*, \alpha^\delta) < u_1(\theta, \overline{\alpha}^\delta) - u_1(\theta, \alpha^\delta). \tag{D.7}$$

Inequality (D.7) contradicts inequality (D.6), which establishes statement 1. □

D.2 No Intermediate Reward Condition

Let $\overline{\theta} \equiv \max \Theta$, $\underline{\theta} \equiv \min \Theta$, $\overline{a}_1 \equiv \max A_1$, $\underline{a}_1 \equiv \min A_1$ and $\overline{a}_2 \equiv \max A_2$. To avoid unnecessary complications, I make the generic assumption that $\mathrm{BR}_2(\theta, \overline{a}_1)$ is a singleton for every $\theta \in \Theta$. The *no intermediate reward* condition is stated below:

Condition 3 (N-IR). *u_1 satisfies no intermediate reward if for every $\theta \in \Theta$, at least one of the following inequalities is true:*

- $u_1(\theta, \overline{a}_1, \overline{a}_2) \leq u_1(\theta, \underline{a}_1, \underline{a}_2)$.
- $u_1(\theta, \overline{a}_1, \overline{a}_2) > \max_{a_2' \prec \overline{a}_2} u_1(\theta, \underline{a}_1, a_2')$.

As one can see, this condition is trivially satisfied when $n = 2$. This enables us to obtain a partition on the set of states similar to the binary action case. Let

$$\Theta_g \equiv \left\{ \theta \in \Theta \middle| u_1(\theta, \overline{a}_1, \overline{a}_2) > u_1(\theta, \underline{a}_1, \underline{a}_2) \text{ and } \mathrm{BR}_2(\theta, \overline{a}_1) = \{\overline{a}_2\} \right\}, \tag{D.8}$$

be the set of good states, let

$$\Theta_n \equiv \left\{ \theta \in \Theta \middle| u_1(\theta, \overline{a}_1, \overline{a}_2) \leq u_1(\theta, \underline{a}_1, \underline{a}_2) \right\}, \tag{D.9}$$

be the set of negative states and let

$$\Theta_p \equiv \left\{ \theta \in \Theta \middle| u_1(\theta, \overline{a}_1, \overline{a}_2) > \max_{a_2' \prec \overline{a}_2} u_1(\theta, \underline{a}_1, a_2') \text{ and } \mathrm{BR}_2(\theta, \overline{a}_1) \prec \overline{a}_2 \right\}, \tag{D.10}$$

be the set of positive states. As in the binary action case, one or two of the sets within $\{\Theta_g, \Theta_p, \Theta_n\}$ can be empty. My analysis starts from the following Lemma, which resembles Lemma 4.1 in Pei (2018):

Lemma D.3. *If the game's payoff is monotone supermodular and u_1 satisfies no intermediate reward, then*

1. *$\{\Theta_g, \Theta_p, \Theta_n\}$ is a partition of Θ.*
2. *For every $\theta_g \in \Theta_g$, $\theta_p \in \Theta_p$ and $\theta_n \in \Theta_n$, we have $\theta_g \succ \theta_p \succ \theta_n$.*
3. *If $\Theta_p, \Theta_n \neq \{\varnothing\}$, then for every $\theta_n \in \Theta_n$, we have $BR_2(\theta_n, \overline{a}_1) \prec a_2$.*

PROOF OF LEMMA D.3: The 1st statement follows directly from the N-IR condition, as there exists no θ such that:

$$\max_{a_2' \prec \overline{a}_2} u_1(\theta, \underline{a}_1, a_2') \geq u_1(\theta, \overline{a}_1, \overline{a}_2) > u_1(\theta, \underline{a}_1, \underline{a}_2).$$

Next, I show the 2nd statement. For every $\theta_g \in \Theta_g$ and $\theta_p \in \Theta_p$, since $\mathrm{BR}_2(\theta_g, \overline{a}_1) = \{\overline{a}_2\}$ while $\mathrm{BR}_2(\theta_p, \overline{a}_1) \prec \overline{a}_2$, supermodularity of u_2 with respect to (θ, a_2) implies that $\theta_g \succ \theta_p$. For every $\theta_p \in \Theta_p$ and $\theta_n \in \Theta_n$, $u_1(\theta_p, \overline{a}_1, \overline{a}_2) > u_1(\theta_p, \underline{a}_1, \underline{a}_2)$ while $u_1(\theta_n, \overline{a}_1, \overline{a}_2) \leq u_1(\theta_n, \underline{a}_1, \underline{a}_2)$. Given that $(\overline{a}_1, \overline{a}_2) \succ (\underline{a}_1, \underline{a}_2)$ and u_1 having SID between θ and (a_1, a_2), we know that $\theta_p \succ \theta_n$. For every $\theta_g \in \Theta_g$ and $\theta_n \in \Theta_n$, $u_1(\theta_g, \overline{a}_1, \overline{a}_2) >$

$u_1(\theta_g, \underline{a}_1, \underline{a}_2)$ while $u_1(\theta_n, \overline{a}_1, \overline{a}_2) \leq u_1(\theta_n, \underline{a}_1, \underline{a}_2)$. Given that $(\overline{a}_1, \overline{a}_2) \succ (\underline{a}_1, \underline{a}_2)$ and u_1 having SID between θ and (a_1, a_2), we know that $\theta_g \succ \theta_n$.

I now proceed to the 3rd statement. Given that $\Theta_p, \Theta_n \neq \{\varnothing\}$, the 2nd statement implies that $\theta_p \succ \theta_n$. Since $\mathrm{BR}_2(\theta_p, \overline{a}_1) \prec \overline{a}_2$ and u_2 exhibits SID in (θ, a_2), we have $\mathrm{BR}_2(\theta_n, \overline{a}_1) \precsim \mathrm{BR}_2(\theta_p, \overline{a}_1) \prec \overline{a}_2$. $\square$

Assume that $\overline{a}_1 \in \Omega$ and $\mathrm{BR}_2(\phi_{\overline{a}_1}, \overline{a}_1) = \{\overline{a}_2\}$. That is to say, there exists a commitment type that always plays $\overline{a}_1$ and player 2's best response to that commitment type is $\overline{a}_2$.

Next, I introduce the optimistic and pessimistic prior belief conditions, which are similar to those in section 4 of Pei (2018). For every $\mu \in \Delta(\Omega)$, μ is *optimistic* if:

$$\begin{aligned} &\mu(\overline{a}_1) u_2(\phi_{\overline{a}_1}, \overline{a}_1, \overline{a}_2) + \sum_{\theta \in \Theta_g \cup \Theta_p} \mu(\theta) u_2(\theta, \overline{a}_1, \overline{a}_2) \\ > \; &\max_{a_2' \prec \overline{a}_2} \Big\{ \mu(\overline{a}_1) u_2(\phi_{\overline{a}_1}, \overline{a}_1, a_2') + \sum_{\theta \in \Theta_g \cup \Theta_p} \mu(\theta) u_2(\theta, \overline{a}_1, a_2') \Big\}, \end{aligned} \tag{D.11}$$

and is *pessimistic* if

$$\begin{aligned} &\mu(\overline{a}_1) u_2(\phi_{\overline{a}_1}, \overline{a}_1, \overline{a}_2) + \sum_{\theta \in \Theta_g \cup \Theta_p} \mu(\theta) u_2(\theta, \overline{a}_1, \overline{a}_2) \\ \leq \; &\max_{a_2' \prec \overline{a}_2} \Big\{ \mu(\overline{a}_1) u_2(\phi_{\overline{a}_1}, \overline{a}_1, a_2') + \sum_{\theta \in \Theta_g \cup \Theta_p} \mu(\theta) u_2(\theta, \overline{a}_1, a_2') \Big\}, \end{aligned} \tag{D.12}$$

The following two results are generalizations of Theorems 2 and 3 in Pei (2018). Both results study games in which the stage game payoffs are monotone-supermodular, with u_1 being semi-separable and has no intermediate rewards. I start with a theorem that establishes the commitment payoff bound when player 2's prior is optimistic.

Theorem 2'. *If μ satisfies (D.11), then for every $\theta \in \Theta$:*

$$\liminf_{\delta \to 1} \underline{V}_\theta(\delta, \mu) \geq u_1(\theta, \overline{a}_1, \overline{a}_2).$$

Then, I proceed to the result that shows the uniqueness of the good strategic types' on-path behavior when player 2's prior is pessimistic.

Theorem 3'. *If μ satisfies (D.12), then there exists $\epsilon > 0$ such that when $\mu(\Omega) < \epsilon$, type θ_g plays $\overline{a}_1$ with probability 1 at every on-path history in every Nash Equilibrium for every $\theta_g \in \Theta_g$.*

The proof follows along the same line as that of Theorems 2 and 3 in Pei (2018). The role of the no intermediate reward condition is embodied in the proof of Lemma D.4, which plays the same role as Lemma C.3 in Pei (2018).

For every $\hat{\mu} \in \Delta(\Theta)$, $\hat{\mu} \in \mathcal{B}$ if and only if:

$$\overline{a}_2 \notin \arg\max_{a_2 \in A_2} \sum_{\theta \in \Theta} \hat{\mu}(\theta) u_2(\theta, \overline{a}_1, a_2). \tag{D.13}$$

Abusing notation, let $\underline{\theta} \equiv \min_\theta \operatorname{supp}(\hat{\mu})$, we have the following result:

Lemma D.4. *For every on-path history h^t, if player 2's belief about the state is $\hat{\mu}$ with $\hat{\mu} \in \mathcal{B}$ and strategic type $\underline{\theta}$ occurs with strictly positive probability at h^t, then*

$$V_{\underline{\theta}}(h^t) \leq \max_{a_2 \neq \overline{a}_2} u_1(\underline{\theta}, \underline{a}_1, a_2). \tag{D.14}$$

PROOF OF LEMMA D.4: Let

$$\Theta^* \equiv \Big\{\tilde{\theta} \in \Theta_p \cup \Theta_n \Big| \mu(h^t)(\tilde{\theta}) > 0\Big\}.$$

According to (D.13) and the fact that $\hat{\mu} \in \mathcal{B}$, we know that $\Theta^* \neq \{\varnothing\}$. The rest of the proof is done via induction on $|\Theta^*|$. When $|\Theta^*| = 1$, there exists a pure strategy $\sigma^*_{\underline{\theta}} : \mathcal{H} \to A_1$ in the support of $\sigma_{\underline{\theta}}$ such that (D.13) holds for all h^s satisfying $h^s \in \mathcal{H}^{(\sigma^*_{\underline{\theta}}, \sigma_2)}$ and $h^s \succeq h^t$. At every such h^s, player 2 has a strict incentive not to play $\overline{a}_2$. When playing $\sigma^*_{\underline{\theta}}$, type $\underline{\theta}$'s stage game payoff is no more than $\max_{a_2 \neq \overline{a}_2} u_1(\underline{\theta}, \underline{a}_1, a_2)$ in every period.

Suppose towards a contradiction that the conclusion holds when $|\Theta^*| \leq k-1$ but fails when $|\Theta^*| = k$, then there exists $h^s \in \mathcal{H}^\sigma(\underline{\theta})$ with $h^s \succeq h^t$ such that

- $\mu(h^\tau) \in \mathcal{B}$ for all $h^s \succeq h^\tau \succeq h^t$.
- $V_{\underline{\theta}}(h^s) > \max_{a_2 \neq \overline{a}_2} u_1(\underline{\theta}, \underline{a}_1, a_2)$.
- For all a_1 such that $\mu(h^s, a_1) \in \mathcal{B}$, $\sigma_{\underline{\theta}}(h^s)(a_1) = 0$.

According to the martingale property of beliefs, there exists a_1 such that $(h^s, a_1) \in \mathcal{H}^\sigma$ and $\mu(h^s, a_1)$ satisfies (D.13). Since $\mu(h^s, a_1)(\underline{\theta}) = 0$, there exists $\tilde{\theta} \in \Theta^* \backslash \{\underline{\theta}\}$ such that $(h^s, a_1) \in \mathcal{H}^\sigma(\tilde{\theta})$. Our induction hypothesis suggests that:

$$V_{\tilde{\theta}}(h^s) = \max_{a_2 \neq \overline{a}_2} u_1(\tilde{\theta}, \underline{a}_1, a_2).$$

The incentive constraints of type $\underline{\theta}$ and type $\tilde{\theta}$ at h^s require the existence of $(\alpha_{1,\tau}, \alpha_{2,\tau})_{\tau=0}^\infty$ with $\alpha_{i,\tau} \in \Delta(A_i)$ such that:

$$\mathbb{E}\Big[\sum_{\tau=0}^\infty (1-\delta)\delta^\tau \Big(u_1(\underline{\theta}, \alpha_{1,\tau}, \alpha_{2,\tau}) - u_1(\underline{\theta}, \underline{a}_1, a_2)\Big)\Big] > 0 \geq \mathbb{E}\Big[\sum_{\tau=0}^\infty (1-\delta)\delta^\tau \Big(u_1(\tilde{\theta}, \alpha_{1,\tau}, \alpha_{2,\tau}) - u_1(\tilde{\theta}, \underline{a}_1, a_2)\Big)\Big],$$

where $\mathbb{E}[\cdot]$ is taken over probability measure $\mathcal{P}^{\sigma}$. However, the supermodularity condition implies that,

$$u_1(\underline{\theta}, \alpha_{1,\tau}, \alpha_{2,\tau}) - u_1(\underline{\theta}, \underline{a}_1, a_2) \leq u_1(\tilde{\theta}, \alpha_{1,\tau}, \alpha_{2,\tau}) - u_1(\tilde{\theta}, \underline{a}_1, a_2),$$

leading to a contradiction.

□

Replace Lemma 4.1, C.1 and C.3 in Pei (2018) with Lemma D.1, D.3 and D.4 in the current document, the proofs of Theorems 2' and 3' follow along the same line as the ones for Theorems 2 and 3. I omit the details to avoid repetition.

Remark: Given the no-intermediate reward condition, one can also generalize it by defining no intermediate reward relative to $a_2 \neq \min A_2$. The main problem with $a_2 \neq \overline{a}_2$ is described as follows. Suppose player 2's prior belief $\hat{\mu}$ is such that his unique best response is strictly below a_2 even if all types of player 1 play $\overline{a}_1$. Then there exists $\mu_1, \mu_2, ..., \mu_n \in \Delta(\Theta)$ and $\alpha_1, ..., \alpha_n \in [0,1]$ such that $\sum_{i=1}^{n} \alpha_i \mu_i = \hat{\mu}$ while player 2's best response when all types of player 1 playing $\overline{a}_1$ is no less than a_2 under μ_i for all $i \in \{1, 2, .., n\}$. This invalidates Lemma D.4.

E Multiple Equilibrium Behaviors in Private Value Reputation Games

In this Appendix, I show that player 1 has multiple equilibrium behaviors in private value reputation games with incomplete information. This substantiates my claim in subsection 4.5 that interdependent values is the driving force behind behavioral uniqueness. To make the comparison clear, I consider stage games with monotone-supermodular payoffs. The only difference is that player 2's payoff does not depend on θ.

Formally, players' stage game payoffs are given by $u_1(\theta, a_1, a_2)$ and $u_2(a_1, a_2)$. This private value reputation game has *monotone-supermodular* payoffs if $(\Theta, \succ)$, $(A_1, \succ)$ and $(A_2, \succ)$ be finite ordered sets and u_1, u_2 satisfy:

Assumption 1 (Monotonicity)**.** *$u_1(\theta, a_1, a_2)$ is strictly decreasing in a_1 and strictly increasing in a_2.*

Assumption 2 (Supermodularity)**.** *$u_1(\theta, a_1, a_2)$ has strictly increasing differences (or SID) in (a_1, θ) and weakly increasing differences (or WID) in (a_2, θ). $u_2(a_1, a_2)$ has SID in (a_1, a_2).*

Furthermore, I assume that $|A_2| = 2$, i.e. $A_2 = \{\overline{a}_2, \underline{a}_2\}$. Let $\text{BR}_2 : A_1 \to 2^{A_2}$ be player 2's best response correspondence. I focus on the interesting case where $\text{BR}_2(\overline{a}_1) = \{\overline{a}_2\}$ and $\text{BR}_2(\underline{a}_1) = \{\underline{a}_2\}$. Partition the set of states into good and bad states: $\Theta \equiv \Theta_g \cup \Theta_b$, with:

$$\Theta_g \equiv \{\theta | u_1(\theta, \overline{a}_1, \overline{a}_2) > u_1(\theta, \underline{a}_1, \underline{a}_2)\} \tag{E.1}$$

and let Θ_b be the complement of Θ_g. Let $\Omega \equiv \{\overline{a}_1, \underline{a}_1\}$. My construction can accommodate the presence of other pure commitment types. Consider the following class of equilibria, indexed by $T \in \mathbb{N}$.

- All types in Θ_b always plays $\underline{a}_1$. All types in Θ_g plays $\overline{a}_1$ in all periods besides period T.
- In every period besides periods 0 and T, player 2 plays $\overline{a}_2$ if player 1's past play coincides with the equilibrium play of the types in Θ_g, and plays $\underline{a}_2$ otherwise. In period T, if the history is off-path, player 2 plays $\underline{a}_2$, if the history is on-path, player 2's equilibrium play is computed via her best response problem.

The integer index T is required to be large enough such that (i) $T \geq 1$ and (ii) for every $\alpha_2, \alpha_2' \in \Delta(A_2)$ and every $\theta \in \Theta_b$:

$$u_1(\theta, \overline{a}_1, \alpha_2) + \delta^T u_1(\theta, \underline{a}_1, \alpha_2') < u_1(\theta, \underline{a}_1, \alpha_2) + \delta^T u_1(\theta, \underline{a}_1, \underline{a}_2) \tag{E.2}$$

For every $\delta \in (0,1)$, the above inequality is satisfied for every large enough T since $u_1(\theta, \underline{a}_1, \alpha_2) - u_1(\theta, \overline{a}_1, \alpha_2) > 0$ and is uniformly bounded from below for all α_2 and $\theta \in \Theta_g$ according to the monotonicity assumption. This implies that no matter how large δ is, player 1 always has multiple possible behaviors on the equilibrium path. Intuitively, no bad strategic type is willing to pool with the commitment type for more than T periods, and the continuation game becomes similar to Fudenberg and Levine (1989) in which player 1 can have multiple continuation payoffs after deviating from his commitment action.

F Tightness of Payoff Lower Bound

In this Appendix, I discuss the tightness of the commitment payoff bound in Theorem 2 in (Pei 2018, hereafter, Theorem 2). To introduce my notion of tightness, let $\lambda(\theta)$ be the likelihood ratio between strategic type θ and strategic type $\overline{\theta}$. Let $\lambda \equiv \{\lambda(\theta)\}_{\theta \in \Theta}$ be the likelihood ratio vector. I will treat λ as a primitive and examine the case in which:

$$\sum_{\theta \in \Theta_p \cup \Theta_g} \lambda(\theta) \mathcal{D}(\theta, \overline{a}_1) \geq 0. \tag{F.1}$$

The above inequality, which is essentially (4.12), translates (4.2) into an environment where there are only strategic types. Fixing a commitment strategy α_1 and ϕ_{α_1}, a reputation game is an '*elaboration*' of λ that contains $(\alpha_1, \phi_{\alpha_1})$ if $\mu(\theta)/\mu(\overline{\theta}) = \lambda(\theta)$ for every $\theta \in \Theta$, $\alpha_1 \in \Omega$ and ϕ_{α_1} be the distribution over θ conditional on player 1 being commitment type α_1. The results below consider situations where (F.1) applies and $\Theta_p \neq \{\varnothing\}$.

Proposition F.1. *For every λ satisfying (F.1), $a_1 \in A_1$, $\phi_{a_1} \in \Delta(\Theta)$ and $\eta > 0$, there exists an elaboration of λ that contains (a_1, ϕ_{a_1}) and $\overline{\delta} \in (0,1)$, such that for every $\delta > \overline{\delta}$, there exists an equilibrium σ of this elaboration* *such that for every* $\theta \in \Theta$*:*

$$\left| V_\theta^\sigma(\delta) - \max\{u_1(\theta, \overline{a}_1, \overline{a}_2), u_1(\theta, \underline{a}_1, \underline{a}_2)\} \right| < \eta. \tag{F.2}$$

Proposition F.2. *If* $\Theta_n = \{\varnothing\}$*, then for every* λ *satisfying (F.1),* $\alpha_1 \in \Delta(A_1)$*,* $\phi_{\alpha_1} \in \Delta(\Theta)$ *and* $\eta > 0$*, there exists an elaboration of* λ *that contains* $(\alpha_1, \phi_{\alpha_1})$ *and* $\overline{\delta} \in (0,1)$*, such that for every* $\delta > \overline{\delta}$*, there exists an equilibrium* σ *of this elaboration such that for every* $\theta \in \Theta$*:*

$$\left| V_\theta^\sigma(\delta) - \max\{u_1(\theta, \overline{a}_1, \overline{a}_2), u_1(\theta, \underline{a}_1, \underline{a}_2)\} \right| < \eta. \tag{F.3}$$

These tightness results are similar in spirit as Theorem 2 of Cripps et al.(1996), which implies that player 1 cannot secure a strictly higher equilibrium payoff by building up other reputations. The condition that $\Theta_p \neq \{\varnothing\}$ is needed, which is demonstrated by the following counterexample:

$\theta = \theta_1$	C	S
H	$1, 2$	$-1, 0$
L	$2, 1$	$0, 0$

$\theta = \theta_0$	C	S
H	$-1, 1$	$-2, 0$
L	$2, 1/2$	$0, 0$

In this game, C is player 2's dominant strategy and type θ_0 can guarantee payoff 2 by always playing L. This is strictly higher than the upper bound identified in Theorem 2, which is 0. The condition that $\Theta_p \neq \{\varnothing\}$ is sufficient to circumvent the above problem by ensuring that $\mathcal{D}(\theta, \overline{a}_1) < 0$ for every $\theta \in \Theta_p \cup \Theta_n$. This guarantees the existence of bad equilibria in which playing $a_1 \neq \overline{a}_1$ signals that $\theta \in \Theta_n \cup \Theta_p$. This motivates player 2 to play $\underline{a}_2$ and gives player 1 a low payoff.

It is also worth noticing that having another commitment type $a_1 \neq \overline{a}_1$ such that $\text{BR}_2(a_1, \phi_{a_1}) = \{\overline{a}_2\}$ cannot guarantee payoff $u_1(\theta, \overline{a}_1, \overline{a}_2)$. To see this, consider the following $3 \times 2 \times 2$ game:

$\theta = \theta_1$	h	l
H	$1, 3$	$-2, 0$
M	$2, 1$	$-1, 0$
L	$3, -1$	$0, 0$

$\theta = \theta_0$	h	l
H	$1 - 2\eta, -1$	$-2 - 2\eta, 0$
M	$2 - \eta, -3$	$-1 - \eta, 0$
L	$3, -5$	$0, 0$

Suppose $\Omega = \{M\}$, ϕ_M is the Dirac measure on θ_1, $\eta \in (0, 1/2)$, with prior belief given by:

$$\mu(M) = \epsilon, \quad \mu(\theta_1) = (1-\epsilon)p \text{ and } \mu(\theta_0) = (1-\epsilon)(1-p)$$

where $\epsilon > 0$ is small enough and

$$\epsilon + (1-\epsilon)p - 3(1-\epsilon)(1-p) > 0.$$

Consider the following equilibrium, in which type θ_1's payoff is $1/2$ and type θ_0's payoff is $\frac{1}{2}-\eta$.

- With probability $\frac{\epsilon}{3(1-\epsilon)(1-p)}$, type θ_0 always plays M. With complementary probability, he plays the same strategy as type θ_1.
- Type θ_1 plays H in period 0, and plays either H or L in the follow-up periods, depending on the realizations of the public randomization device.
- Player 2 plays h in period 0. Starting from period 1, if player 1 has always played M in the past, she plays h with probability $1-\frac{1}{2\delta}$. If player 1 played H in period 0 and has always followed the recommendations of the public randomization device, she plays h if the public randomization device recommends player 1 to play H and l otherwise. If player 1 played L in period 0 or has deviated from the recommended actions, then she plays l in every subsequent period.
- The probability with which the public randomization device recommends H is $1-\frac{1}{2\delta}$.

I show Propositions F.1 and F.2 in the next two subsections. First, I argue that when $\mathcal{D}(\alpha_1,\phi_{\alpha_1})\leq 0$, the results are trivially true. This is because there exists an equilibrium in which all types in $\Theta_g\cup\Theta_p$ plays $\overline{a}_1$ all the time and all types in Θ_n plays $\underline{a}_1$ all the time. Player 2 plays $\overline{a}_2$ if and only if player 1 has always been playing $\overline{a}_1$ in the past. Therefore, I will be focusing on cases in which $\mathcal{D}(\alpha_1,\phi_{\alpha_1})>0$. My construction can be generalized by allowing for transparent pure strategy commitment types (Cripps et al.1996). The generalization to the case where α_1 is mixed, $\mathcal{D}(\phi_{\alpha_1},\alpha_1)>0$ and neither Θ_p nor Θ_n is empty is difficult, since constructing equilibrium is hard in presence of a mixed strategy commitment type and multiple strategic types that have non-trivial incentives (unlike statements 2 and 4 in Theorem 1).

F.1 Proof of Proposition F.1

Let $\overline{\theta}_n\equiv\max\Theta_n$ and $\underline{\theta}_p\equiv\min\Theta_p$. When $a_1=\overline{a}_1$, then consider the equilibrium in which all types in $\Theta_g\cup\Theta_p$ plays $\overline{a}_1$ all the time and all types in Θ_n plays $\underline{a}_1$ all the time. Player 2 plays $\overline{a}_2$ if and only if player 1 has always been playing $\overline{a}_1$ in the past.

When $a_1=\underline{a}_1$, if $\Theta_n\neq\{\varnothing\}$, then the above strategy profile remains to be an equilibrium when the probability of type $\underline{a}_1$ satisfies:

$$\mu(\underline{a}_1)\mathcal{D}(\phi_{\underline{a}_1},\underline{a}_1)+\sum_{\theta_n\in\Theta_n}\mu(\theta_n)\mathcal{D}(\theta_n,\underline{a}_1)<0. \tag{F.4}$$

If $\Theta_n=\{\varnothing\}$, then consider the equilibrium in which types in $\Theta\backslash\{\underline{\theta}_p\}$ plays $\overline{a}_1$ all the time. Type $\underline{\theta}_p$ mixes between always playing $\underline{a}_1$ and always playing $\overline{a}_1$. The probability with which he plays $\underline{a}_1$ in period 0 is such that after observing $\underline{a}_1$ in period 0, player 2's belief in period 1, $\mu_1(\cdot)$, satisfies:

$$\mu_1(\underline{a}_1)\mathcal{D}(\phi_{\underline{a}_1}, \underline{a}_1) + \mu_1(\underline{\theta}_p)\mathcal{D}(\underline{\theta}_p, \underline{a}_1) = 0. \tag{F.5}$$

This probability exists when $\mu(\underline{a}_1)$ is small enough. Starting from period 1, player 2 plays $\overline{a}_2$ for sure if all the past actions were $\overline{a}_1$. He plays $a\overline{a}_2 + (1-s)\underline{a}_2$ if all the past actions were $\underline{a}_1$, with $s \in (0,1)$ chosen to make type $\underline{\theta}_p$ indifferent in period 0. He plays $\underline{a}_2$ for sure at other histories.

When $a_1 \notin \{\underline{a}_1, \overline{a}_1\}$, I focus on the case where $\Theta_p, \Theta_n \neq \{\varnothing\}$. I will explain other cases later. Let $q, r \in \mathbb{R}_+$ be defined as:

$$u_1(\underline{\theta}_p, a_1, q\overline{a}_2 + (1-q)\underline{a}_2) = u_1(\underline{\theta}_p, \overline{a}_1, \overline{a}_2), \tag{F.6}$$

and

$$u_1(\overline{\theta}_n, a_1, r\overline{a}_2 + (1-r)\underline{a}_2) = u_1(\overline{\theta}_n, \underline{a}_1, \underline{a}_2) \tag{F.7}$$

respectively. By Assumption 1 (monotonicity), we know that $q \in (0,1)$ and by the definition of Θ_n, we know that $r > 0$. I consider two cases separately.

1. If $r \geq q$, all strategic types in $\Theta_g \cup \Theta_p \backslash \{\underline{\theta}_p\}$ plays $\overline{a}_1$ all the time and all types in Θ_n plays $\underline{a}_1$ all the time. Type $\underline{\theta}_p$ mixes between playing $\overline{a}_1$ all the time and playing a_1 all the time. Starting from period 1, Player 2 plays $\overline{a}_2$ for sure if player 1 has always been playing $\overline{a}_1$ in the past, plays $\overline{a}_2$ with probability q if player 1 has always been playing a_1. Plays $\underline{a}_2$ for sure otherwise.

2. If $r < q$, all strategic types in $\Theta_g \cup \Theta_p$ plays $\overline{a}_1$ all the time and all types in $\Theta_n \backslash \{\overline{\theta}_n\}$ plays $\underline{a}_1$ all the time. Type $\overline{\theta}_n$ mixes between playing $\underline{a}_1$ all the time and playing a_1 all the time. Starting from period 1, Player 2 plays $\overline{a}_2$ for sure if player 1 has always been playing $\overline{a}_1$ in the past, plays $\overline{a}_2$ with probability r if player 1 has always been playing a_1. Plays $\underline{a}_2$ for sure otherwise.

Remark: If $\Theta_n = \{\varnothing\}$, then an equilibrium can be constructed as in the case with $r \geq q$. The result can be extended to the case where $\Theta_p = \{\varnothing\}$ but $\mathcal{D}(\overline{\theta}_n, \overline{a}_1) < 0$. Just modify the definition of q via:

$$u_1(\underline{\theta}_g, a_1, q\overline{a}_2 + (1-q)\underline{a}_2) = u_1(\underline{\theta}_g, \overline{a}_1, \overline{a}_2), \tag{F.8}$$

where $\underline{\theta}_g \equiv \min \Theta_g$.

F.2 Proof of Proposition F.2

I construct an equilibrium when α_1 is mixed and $\overline{a}_1 \in \text{supp}(\alpha_1)$. Cases in which $\overline{a}_1 \notin \text{supp}(\alpha_1)$ can be dealt with similarly. The idea is: type $\underline{\theta}_p$'s payoff is $u_1(\underline{\theta}_p, \overline{a}_1, \overline{a}_2)$. In period 0, he randomizes between always playing actions in $\text{supp}(\alpha_1)$ and always playing $\overline{a}_1$.

- ▷ Every type above $\underline{\theta}_p$ always plays $\overline{a}_1$. Player 2 plays $\overline{a}_2$ for sure given that player 1 has always played $\overline{a}_1$ in the past.

- ▷ For every $t \in \mathbb{N}$, type $\underline{\theta}_p$ only plays actions in $\text{supp}(\alpha_1)$ with positive probability in period t. In particular, at history $h^t = (\overline{a}_1, ..., \overline{a}_1)$, the probability in which he plays $a_1 \neq \overline{a}_1$ is chosen such that player 2's posterior belief satisfies: $\mu(\underline{\theta}_p)/\mu(\alpha) = \xi(a_1)$ for some $\xi(a_1) > 0$. His continuation value after playing a_1, denoted by $V(a_1)$ satisfies:
$$(1-\delta)u_1(\underline{\theta}_p, a_1, \overline{a}_2) + \delta V(a_1) = u_1(\underline{\theta}_p, \overline{a}_1, \overline{a}_2).$$

When δ is large enough, $V(a_1) \in [u_1(\underline{\theta}_p, \underline{a}_1, \underline{a}_2), u_1(\underline{\theta}_p, \overline{a}_1, \overline{a}_2)]$. When $\xi|\mathcal{D}(\underline{\theta}_p, \overline{a}_1)| > \mathcal{D}(\phi_{\alpha_1}, \alpha_1)$, then the continuation value of type $\underline{\theta}_p$ is $u_1(\underline{\theta}_p, \underline{a}_1, \underline{a}_2)$ in every equilibrium. When $2\xi|\mathcal{D}(\underline{\theta}_p, \alpha_1)| < \mathcal{D}(\phi_{\alpha_1}, \alpha_1)$, then according to Theorem 1 of Pei (2018), for every $\eta > 0$, there exists $\overline{\delta} \in (0,1)$ such that for every $\delta > \overline{\delta}$, type $\underline{\theta}_p$'s continuation value is at least $u_1(\underline{\theta}_p, \alpha_1, \overline{a}_2) - \eta$, which is above $u_1(\underline{\theta}_p, \underline{a}_1, \underline{a}_2)$ when η is chosen to be small enough. By the upper-hemi-continuity of Nash Equilibrium payoffs, there exists $\xi(a_1) \in \left(\frac{\mathcal{D}(\phi_{\alpha_1}, \alpha_1)}{2\mathcal{D}(\underline{\theta}_p, \alpha_1)}, \frac{\mathcal{D}(\phi_{\alpha_1}, \alpha_1)}{\mathcal{D}(\underline{\theta}_p, \overline{a}_1)}\right)$ for every a_1 such that $V(a_1)$ is a continuation value for type $\underline{\theta}_p$.

G Weaker Version of the Supermodularity Assumption

In this Appendix, I characterize Assumption 2' (in section 5 of Pei 2018) using the primitives of the model. I also discuss how this extension broadens the scope of application of my result, by focusing on the example of repeated prisoner's dilemma with reciprocal altruism.

G.1 Characterizing Assumption 2'

Recall that Assumption 2' requires that there exists $f : \Theta \to \mathbb{R}_+$ such that $\tilde{u}_1(\theta, a_1, a_2) \equiv f(\theta)u_1(\theta, a_1, a_2)$ has SID in (a_1, θ) and WID in (a_2, θ). To characterize this assumption using the primitives, let $\psi_j : \Theta \times A_{-j} \times A_j \times A_j \to \mathbb{R}$ be defined as:
$$\psi_j(\theta, a_{-j}|a_j, a_j') \equiv u_1(\theta, a_j, a_{-j}) - u_1(\theta, a_j', a_{-j}). \tag{G.1}$$

for $j \in \{1,2\}$. Let $\Theta \equiv \{\theta_0, ..., \theta_n\}$ with $\theta_0 \prec \theta_1 \prec ... \prec \theta_n$. For every $i \in \{1,2,...,n\}$, let

$$\underline{g}_i \equiv \max_{a_1 \in A_1, a_2 \succ a_2'} \left\{ \psi_2(\theta_{i-1}, a_1|a_2, a_2') \Big/ \psi_2(\theta_i, a_1|a_2, a_2') \right\} \tag{G.2}$$

and let

$$\overline{g}_i \equiv \min_{a_2 \in A_2, a_1 \succ a_1'} \left\{ \psi_1(\theta_{i-1}, a_2|a_1, a_1') \Big/ \psi_1(\theta_i, a_2|a_1, a_1') \right\}. \tag{G.3}$$

The characterization result is stated as the following Proposition:

Proposition G.1. *There exists $f : \Theta \to \mathbb{R}_+$ such that $\tilde{u}_1(\theta, a_1, a_2) \equiv f(\theta)u_1(\theta, a_1, a_2)$ has SID in (a_1, θ) and WID in (a_2, θ) if and only if $\underline{g}_i < \overline{g}_i$ for every $i \in \{1, 2, ..., n\}$.*

PROOF OF PROPOSITION G.1: I start with the '*only if*' direction. Suppose towards a contradiction that $\underline{g}_i \geq \overline{g}_i$ for some i, then consider any $f(\theta_i), f(\theta_{i-1}) > 0$.

▷ If $f(\theta_i)/f(\theta_{i-1}) \geq \underline{g}_i$, then $f(\theta_i)/f(\theta_{i-1}) \geq \overline{g}_i$, implying that there exists $(a_1, a_1', a_2) \in A_1 \times A_1 \times A_2$ with $a_1 \succ a_1'$ such that

$$f(\theta_i)\psi_1(\theta_i, a_2|a_1, a_1') \Big/ f(\theta_{i-1})\psi_1(\theta_{i-1}, a_2|a_1, a_1') \geq 1.$$

Since $\psi_1(\theta, a_2|a_1, a_1') < 0$, we have

$$f(\theta_i)\psi_1(\theta_i, a_2|a_1, a_1') \leq f(\theta_{i-1})\psi_1(\theta_{i-1}, a_2|a_1, a_1')$$

contradicting the SID of u_1 with respect to (θ, a_1).

▷ If $f(\theta_i)/f(\theta_{i-1}) < \underline{g}_i$, then similarly, there exists $(a_2, a_2', a_1) \in A_2 \times A_2 \times A_1$ with $a_2 \succ a_2'$ such that

$$f(\theta_i)\psi_2(\theta_i, a_1|a_2, a_2') \Big/ f(\theta_{i-1})\psi_2(\theta_{i-1}, a_1|a_2, a_2') < 1.$$

Since $\psi_2(\theta, a_1|a_2, a_2') > 0$, we have

$$f(\theta_i)\psi_2(\theta_i, a_1|a_2, a_2') < f(\theta_{i-1})\psi_2(\theta_{i-1}, a_1|a_2, a_2')$$

contradicting the WID of u_1 with respect to (θ, a_2).

For the '*if*' direction, suppose $\underline{g}_i < \overline{g}_i$ for all i, then there exists $(f(\theta_i))_{i=0}^n$ such that $f(\theta_i)/f(\theta_{i-1}) \in [\underline{g}_i, \overline{g}_i)$ for every $i \in \{1, ..., n\}$. For any pair of states (θ_j, θ_k) with $j > k$, we have:

$$f(\theta_j)/f(\theta_k) \in \Big[\prod_{i=k+1}^{j} \underline{g}_i, \prod_{i=k+1}^{j} \overline{g}_i\Big).$$

This implies the following two inequalities:

$$\begin{aligned} \frac{f(\theta_j)}{f(\theta_k)} &\geq \prod_{i=k+1}^{j} \underline{g}_i = \prod_{i=k+1}^{j} \max_{a_1 \in A_1, a_2 \succ a_2'} \Big\{\psi_2(\theta_{i-1}, a_1|a_2, a_2')\Big/\psi_2(\theta_i, a_1|a_2, a_2')\Big\} \\ &\geq \max_{a_1 \in A_1, a_2 \succ a_2'} \Big\{\psi_2(\theta_k, a_1|a_2, a_2')\Big/\psi_2(\theta_j, a_1|a_2, a_2')\Big\}, \end{aligned}$$

and

$$\begin{aligned} \frac{f(\theta_j)}{f(\theta_k)} &< \prod_{i=k+1}^{j} \overline{g}_i = \prod_{i=k+1}^{j} \min_{a_2 \in A_2, a_1 \succ a_1'} \Big\{\psi_1(\theta_{i-1}, a_2|a_1, a_1')\Big/\psi_1(\theta_i, a_2|a_1, a_1')\Big\} \\ &\leq \min_{a_2 \in A_2, a_1 \succ a_1'} \Big\{\psi_1(\theta_k, a_2|a_1, a_1')\Big/\psi_1(\theta_j, a_2|a_1, a_1')\Big\}, \end{aligned}$$

which together imply that

$$f(\theta_j)\psi_1(\theta_j, a_2|a_1, a_1') > f(\theta_k)\psi_1(\theta_k, a_2|a_1, a_1') \text{ and } f(\theta_j)\psi_2(\theta_j, a_1|a_2, a_2') \geq f(\theta_{i-1})\psi_2(\theta_k, a_1|a_2, a_2').$$

$\square$

G.2 Repeated Prisoner's Dilemma with Reciprocal Altruism

I apply Proposition G.1 to study repeated prisoner's dilemma with reciprocal altruism. A patient long-run player is interacting with a sequence of short-run players, one in each period. Players' *material payoffs* are given by:

-	C	D
C	$1, 1$	$-c_1, 1 + b_2$
D	$1 + b_1, -c_2$	$0, 0$

where $b_1, b_2, c_1, c_2 > 0$. Player 1 has private information about his level of altruism $\theta \in \{\theta_0, \theta_1, ..., \theta_n\} \equiv \Theta$, i.e. the fraction of player 2's material payoff he internalizes. Player 2 internalizes fraction $\beta(\theta)$ of player 1's material payoff, with $\beta(\cdot)$ a strictly increasing function. Every player's *stage game payoff* is a linear combination of his/her material payoff and an altruism term depending on his/her opponent's material payoff, with the relative weight between the two depending on θ:[14]

θ	C	D
C	$1+\theta, 1+\beta(\theta)$	$-c_1+\theta(1+b_2), (1+b_2)-\beta(\theta)c_1$
D	$(1+b_1)-\theta c_2, -c_2+\beta(\theta)(1+b_1)$	$0, 0$

This model captures situations in which a foreign entity (for example, firm, NGO, missionary, army, etc.) comes to a remote village in a developing country and needs to collaborate with different members of the local community in different periods. The local people are suspicious about the foreigner's intention (θ) and are learning about it over time via their previous interactions. Importantly, the local people's willingness to cooperate is higher when they believe the foreigner is more altruistic or the foreigner is cooperating with higher probability.

Assumption 2 fails in this game, as player 1's returns from player 2's cooperation *decreases* with the θ. This is because a more altruistic player internalizes a larger fraction of his opponent's cost from cooperation. Nevertheless, Assumption 2' can be satisfied under some conditions on the parameters, which are characterized in Proposition G.2. These conditions are meaningful as one can apply my reputation results on monotone-supermodular after doing an affine transformation on player 1's payoff.

Proposition G.2. *The game is supermodular if and only if the following three inequalities hold:*

$$\max\Theta < \min\{\frac{b_1}{1+c_2}, \frac{c_1}{1+b_2}\}, \tag{G.4}$$

$$(1+b_1)(1+b_2) \geq c_1c_2, \tag{G.5}$$

and

$$(1+c_1)(1+c_2) \geq b_1b_2. \tag{G.6}$$

PROOF OF PROPOSITION G.2: Inequality (G.4) guarantees Assumption 1. In what follows, I check Assumption 2'. For any $\theta_{i-1} \leq \theta_i$, we have:

$$\underline{g}_i = \max\Big\{\frac{1+c_1-\theta_{i-1}b_2}{1+c_1-\theta_i b_2}, \frac{1+b_1-\theta_{i-1}c_2}{1+b_1-\theta_i c_2}\Big\},$$

$$\overline{g}_i = \min\Big\{\frac{c_1-\theta_{i-1}(1+b_2)}{c_1-\theta_i(1+b_2)}, \frac{b_1-\theta_{i-1}(1+c_2)}{b_1-\theta_i(1+c_2)}\Big\}.$$

[14]Reciprocal altruism can help explain various evidence in the lab, including public good games, ultimatum bargaining games and centipede games. For details, see Levine (1998). Sethi and Somanathan (2001) show that reciprocal preferences can survive in a larger class of evolutionary environments compared to purely altruistic or purely spiteful preferences. My linear formulation of altruism was adopted in the aforementioned papers.

According to Proposition G.1, Assumption 2' is satisfied if and only if $\underline{g}_i < \overline{g}_i$ for every i. Notice that

$$\frac{c_1 - \theta_{i-1}(1+b_2)}{c_1 - \theta_i(1+b_2)} > \frac{1 + c_1 - \theta_{i-1}b_2}{1 + c_1 - \theta_i b_2}$$

and

$$\frac{b_1 - \theta_{i-1}(1+c_2)}{b_1 - \theta_i(1+c_2)} > \frac{1 + b_1 - \theta_{i-1}c_2}{1 + b_1 - \theta_i c_2}$$

are implied by $\theta_{i-1} < \theta_i$. Therefore, one only needs to check:

$$\frac{c_1 - \theta_{i-1}(1+b_2)}{c_1 - \theta_i(1+b_2)} > \frac{1 + b_1 - \theta_{i-1}c_2}{1 + b_1 - \theta_i c_2}$$

and

$$\frac{b_1 - \theta_{i-1}(1+c_2)}{b_1 - \theta_i(1+c_2)} > \frac{1 + c_1 - \theta_{i-1}b_2}{1 + c_1 - \theta_i b_2}.$$

The first inequality is implied by (G.5) and the second one is implied by (G.6), which concludes the proof.

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