Online Appendix for Trust and Betrayals

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A Proof of Lemma A.1

For every $t \in \mathbb{N}$, let $N_{L,t}$ and $N_{H,t}$ be the number of periods in which L and H are played from period 0 to $t-1$, respectively. The proof is done by induction on $N_{L,t}$.

When $N_{L,t} \leq 2(k-n)$, then the conclusion holds as $N_{H,t} \geq 2n + X$. According to (A.5) and (A.6) in Pei (2018b), we know that $\Delta(h^T)$ reaches $1-\eta^*$ before period T, after which play reaches the absorbing phase.

Suppose the conclusion holds for when $N_{L,t} \leq N$ with $N \geq 2(k-n)$, and suppose towards a contradiction that there exists h^T with $T \geq k + X$ and $N_{L,T} = N+1$, such that play remains in the active learning phase for every $h^t \preceq h^T$ but

$$(1-\delta)\sum_{t=0}^{T-1} \delta^t \mathbf{1}\{a_t = H\} - (1-\delta^X) > (1-\delta)\sum_{t=0}^{T-1} \delta^t \mathbf{1}\{a_t = L\} \cdot \frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}, \tag{A.1}$$

I obtain a contradiction in three steps.

Step 1: I show that for every $s < T$,

$$(1-\delta)\sum_{t=s}^{T-1} \delta^t \mathbf{1}\{a_t = H\} \geq (1-\delta)\sum_{t=s}^{T-1} \delta^t \mathbf{1}\{a_t = L\} \frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}. \tag{A.2}$$

Suppose towards a contradiction that the opposite of (A.2) holds, then (A.2) and (A.1) together imply that:

$$(1-\delta)\sum_{t=0}^{s-1} \delta^t \mathbf{1}\{a_t = H\} - (1-\delta^X) > (1-\delta)\sum_{t=0}^{s-1} \delta^t \mathbf{1}\{a_t = L\} \frac{\widetilde{\gamma}}{1-\widetilde{\gamma}} \tag{A.3}$$

and

$$(1-\delta)\sum_{t=s}^{T-1} \delta^t \mathbf{1}\{a_t = L\} > 0. \tag{A.4}$$

According to (A.4), $N_{L,s} < N_{L,T}$. Since $N_{L,T} = N+1$, we have $N_{L,s} \leq N$. Applying the induction hypothesis and (A.3), we know that play reaches the absorbing phase before h^s, leading to a contradiction.

Step 2: I show that for every k consecutive periods

$$\{a_r, ..., a_{r+k-1}\} \subset h^T,$$

the number of H in this sequence is at least $n+1$. According to (A.2) shown in the previous step and (**??**), H occurs at least $n+1$ times in the last k periods, i.e. $\{a_{T-k+1}, ..., a_T\}$.

Suppose towards a contradiction that there exists k consecutive periods in which H occurs no more than n times, then the conclusion above that H occurs at least $n+1$ times in the last k periods implies that there exists k consecutive periods $\{a_r, ..., a_{r+k-1}\}$ in which H occurs exactly n times and L occurs exactly $k-n$ times. According to (A.4) in Pei (2018b), we have

$$(1-\delta)\sum_{t=r}^{r+k-1} \delta^t \mathbf{1}\{a_t = H\} < (1-\delta)\sum_{t=r}^{r+k-1} \delta^t \mathbf{1}\{a_t = L\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}, \tag{A.5}$$

but according to (A.5) of Pei (2018b) and the definition of $\widehat{\gamma}$ in (A.3) of Pei (2018b), we also know that

$$\Delta(h^{r+k}) > \Delta(h^{r+1}). \tag{A.6}$$

Next, let us consider the following new sequence with length $T-k$:

$$\widetilde{h}^{T-k} \equiv \{\widetilde{a}_0, \widetilde{a}_1, ..., \widetilde{a}_{T-k-1}\} \equiv \{a_0, a_1, ..., a_{r-1}, a_{r+k}, ..., a_{T-1}\}$$

which is obtained by removing $\{a_r, ..., a_{r+k-1}\}$ from the original sequence and front-loading the subsequent play $\{a_{r+k}, ..., a_{T-1}\}$. The number of L in this new sequence is at most $N+1-(n-k)$, which is no more than N. According to the conclusion in Step 1:

$$(1-\delta)\sum_{t=r+k}^{T-1} \delta^t \mathbf{1}\{a_t = H\} > (1-\delta)\sum_{t=r+k}^{T-1} \delta^t \mathbf{1}\{a_t = L\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}. \tag{A.7}$$

This together with (A.5) and (A.1) imply that

$$(1-\delta)\sum_{t=0}^{T-k-1} \delta^t \mathbf{1}\{\widetilde{a}_t = H\} - (1-\delta^X) > (1-\delta)\sum_{t=0}^{T-k-1} \delta^t \mathbf{1}\{\widetilde{a}_t = L\}\frac{\widetilde{\gamma}}{1-\widetilde{\gamma}}.$$

According to the induction hypothesis, play will reach the absorbing phase before period $T-k$ if player 1 plays according to $\{\widetilde{a}_0, \widetilde{a}_1, ..., \widetilde{a}_{T-k-1}\}$.

1. Suppose $\widetilde{h}^{T-k}$ reaches the absorbing phase before period r, then play will also reach the absorbing phase

before period r according to the original sequence.

2. Suppose $\widetilde{h}^{T-k}$ reaches the absorbing phase in period s, with $s > t$, then according to (A.6), we have $\Delta(\widetilde{h}^s) \leq \Delta(h^{s+k})$, implying that play will reach the absorbing phase in period $s+k$ according to the original sequence.

This contradicts the hypothesis that play has never reached the absorbing phase before period T if play proceeds according to h^T.

Step 3: For every history $h^T \equiv \{a_0, a_1, ..., a_{T-1}\} \in \{H, L\}^T$ and $t \in \{1, ..., T-1\}$, define the operator $\Omega_t : \{H, L\}^T \to \{H, L\}^T$ as:

$$\Omega_t(h^T) = (a_0, ..., a_{t-2}, a_t, a_{t-1}, a_{t+1}, ..., a_{T-1}), \tag{A.8}$$

in another word, swapping the order between a_{t-1} and a_t. Recall the belief updating formula in (A.13) of Pei (2018b) and let

$$\mathcal{H}^{T,*} \equiv \Big\{h^T \Big| \Delta(h^t) < 1 - \eta^* \text{ for all } h^t \prec h^T\Big\}. \tag{A.9}$$

If $h^T \in \mathcal{H}^{T,*}$, then $\Omega_t(h^T) \in \mathcal{H}^{T,*}$ unless:

- $a_{t-1} = L$, $a_t = H$.
- *and*, $\Big(1 + \lambda(1-\gamma^*)\Big)\Delta(h^{t-1}) \geq 1 - \eta^*$.

Next, I show that the above situation cannot occur besides in the last k periods. Suppose towards a contradiction that there exists $t \leq T - k$ such that $h^T \in \mathcal{H}^{T,*}$ but $\Omega_t(h^T) \notin \mathcal{H}^{T,*}$. Then according to the conclusion in step 2, H occurs at least $n+1$ times in $\{a_t, ..., a_{t+k-1}\}$. Now, consider the sequence $\{a_{t-1}, ..., a_{t+k-1}\}$, in which H occurs at least $n+1$ times and L occurs at most $k-n$ times. This implies that:

$$\begin{aligned}
\Delta(h^{t+k}) &\geq \Delta(h^{t-1})\Big(1+\lambda(1-\gamma^*)\Big)^{n+1}\Big(1-\lambda\gamma^*\Big)^{k-n} \\
&= \Delta(h^{t-1})\underbrace{\Big(1+\lambda(1-\gamma^*)\Big)^{n}\Big(1-\lambda\gamma^*\Big)^{k-n}}_{\geq 1}\Big(1+\lambda(1-\gamma^*)\Big) \\
&\geq \Delta(h^{t-1})\Big(1+\lambda(1-\gamma^*)\Big) \\
&\geq 1-\eta^*,
\end{aligned} \tag{A.10}$$

where second inequality follows from $n/k > \widehat{\gamma}$ and (A.5) in Pei (2018b), and the 3rd inequality follows from the hypothesis that $\Omega_t(h^T) \notin \mathcal{H}^{T,*}$. Inequality (A.10) implies that play reaches the high phase before period $t + k \leq T$, contradicting the hypothesis that $h^T \in \mathcal{H}^{T,*}$.

To summarize, for every $t \leq T-k$, if $h^T \in \mathcal{H}^{T,*}$, then $\Omega_t(h^T) \in \mathcal{H}^{T,*}$. For every $t > T-k$, if $h^T \in \mathcal{H}^{T,*}$, then $\Omega_t(h^T) \in \mathcal{H}^{T,*}$ unless $a_{t-1} = L$ and $a_t = H$. Therefore, one can freely front-load the play of H from period 0 to $T-k-1$ and obtain the following revised sequence:

$$\{H, H, .., H, L, L, ..., L, a_{T-k}, ..., a_{T-1}\}, \tag{A.11}$$

which meets the following two requirements: (1) the revised sequence (A.11) still belongs to set $\mathcal{H}^{T,*}$; (2) sequence (A.11) satisfies (A.1).

According to the conclusion in Step 2: (1) the number of L from period 0 to $T-k-1$ cannot exceed $k-n-1$; (2) the number of L from period $T-k$ to $T-1$ cannot exceed $k-n-1$. This is because otherwise, there exists a sequence of length k that has at most n periods of H, contradicting the two conditions the revised sequence in (A.11) satisfies. Therefore, the total number of L in this sequence is at most $2(k-n-1)$, which contradicts the induction hypothesis that the number of L exceeds $2(k-n)$.

B Proof of Corollary 3

I show the result by induction on the number of types in the support of the belief. When $|\Theta| = 2$, recall the definitions of $\overline{a}_1(\cdot)$ as well as $\overline{\mathcal{H}}$ in Appendix B.1. Let $\overline{h}_1^t$ be the first history in $\overline{\mathcal{H}}$ such that type θ_2 has a strict incentive to play L but $\overline{a}_1(\overline{h}_1^t) = H$. This history exists since type θ_2's equilibrium payoff is strictly greater than $1-\theta_2$. Type θ_1 plays H with positive probability at $\overline{h}^t$, after which she fully reveals her private information.

One can similarly define $\overline{a}_1(\cdot)$ and $\overline{\mathcal{H}}$ in the continuation game starting from history $(\overline{h}_1^t, L)$. When δ is large enough, both types occur with positive probability at $(\overline{h}_1^t, L)$. Let $\overline{h}_2^t$ be the first history in $\overline{\mathcal{H}}$ starting from $(\overline{h}_1^t, L)$ such that type θ_2 has a strict incentive to play L but $\overline{a}_1(\overline{h}_1^t) = H$. The existence of this history also comes from the requirement that type θ_2's payoff is greater than $1-\theta_2$. Type θ_1 plays H with positive probability at $\overline{h}^t$, after which she fully reveals her private information. Similarly, one can define $(\overline{h}_3^t, \overline{h}_4^t, ...)$. As $\delta \to 1$, the length of this sequence goes to infinity, and at every such history, type θ_1 fully reveals her information after playing H.

Next, suppose the conclusion holds for all posterior beliefs that have at most k elements in the support. When there are $k+1$ types, consider the incentives of type θ_2. After reaching history $\overline{h}_n^t$ for a given $n \in \mathbb{N}$, type θ_1 needs to play H with positive probability at $\overline{h}_n^t$. This is because otherwise, there exists type θ_j with $j > 2$ that cannot extract information rent in the future, leading to a contradiction. After type θ_1 plays H at $\overline{h}_n^t$, there are at most k types in the support of the posterior belief. The conclusion is then implied by the induction hypothesis.

C Proof of Theorem 2

Notation: For every $\sigma_\theta : \mathcal{H} \rightarrow \Delta(A_1)$ and $\sigma_2 : \mathcal{H} \rightarrow \Delta(A_2)$, let $\mathcal{H}(\sigma_\theta, \sigma_2)$ be the set of histories that occur with positive probability under the measure induced by $(\sigma_\theta, \sigma_2)$. Let $\overline{\sigma}_\theta : \mathcal{H} \rightarrow \Delta(A_1)$ be such that $\overline{\sigma}_\theta(h^t) = H$ for every $h^t \in \mathcal{H}$. Let $\underline{\sigma}_\theta : \mathcal{H} \rightarrow \Delta(A_1)$ be such that $\underline{\sigma}_\theta(h^t) = L$ for every $h^t \in \mathcal{H}$.

Completely Mixed Strategies: Suppose towards a contradiction that there exists a Nash equilibrium $\sigma = \Big((\sigma_\theta)_{\theta \in \Theta}, \sigma_2\Big)$ that attains payoff within ϵ of v^*, and there exists a type $\widehat{\theta}$ that has a completely mixed best reply to σ_2. Then both $\overline{\sigma}_\theta$ and $\underline{\sigma}_\theta$ are type $\widehat{\theta}$'s best replies to σ_2. Since the stage game payoff is monotone-supermodular according to the orders $T \succ N$, $H \succ L$ and $\theta_1 \succ \theta_2 \succ ... \succ \theta_m$, Lemma C.1 in Pei (2018a) implies that:

1. For every $\theta_j \succ \widehat{\theta}$, type θ_j plays H with probability 1 at every $h^t \in \mathcal{H}(\overline{\sigma}_\theta, \sigma_2)$.
2. For every $\theta_k \prec \widehat{\theta}$, type θ_k plays L with probability 1 at every $h^t \in \mathcal{H}(\underline{\sigma}_\theta, \sigma_2)$.

I consider the above two cases separately. First, suppose $\widehat{\theta} \neq \theta_m$, then type θ_m will play L with probability 1 at every $h^t \in \mathcal{H}(\underline{\sigma}_\theta, \sigma_2)$, from which she is supposed to receive payoff no less than $v_m^* - \epsilon$. On the other hand, the argument in Fudenberg and Levine (1992) implies that in every Nash equilibrium, there are at most

$$T_{\theta_m} \equiv \log \pi_0(\theta_m) \Big/ \log(1 - \gamma^*) \tag{C.1}$$

periods in which player 2 plays T. That is to say, for every $\epsilon > 0$, there exists $\overline{\delta} \in (0, 1)$ such that when $\delta > \overline{\delta}$, type θ_m's payoff is less than ϵ in every Nash equilibrium. Pick ϵ to be small enough such that $\epsilon < v_m^*/2$, we obtain a contradiction.

Second, suppose $\widehat{\theta} = \theta_m$, then types $\theta_1 \sim \theta_{m-1}$ will play H with probability 1 at every $h^t \in \mathcal{H}(\overline{\sigma}_\theta, \sigma_2)$. Therefore, after playing L for the first time, type θ_m will reveal her type so her continuation payoff is at most $1 - \theta_m$. Hence, her discounted average payoff in the repeated game cannot exceed $(1 - \delta) + \delta(1 - \theta_m)$. Let ϵ be small enough such that $(1 - \delta) + \delta(1 - \theta_m) < v_m^* - \epsilon$, we have a contradiction.

Stationary Strategies: The above argument rules out completely mixed strategies. To rule out stationary strategies, one only needs to show that no type plays stationary pure strategies. First, suppose towards a contradiction that type $\widehat{\theta}$ plays L in every period, then in every Nash equilibrium, there are at most

$$T_{\widehat{\theta}} \equiv \log \pi_0(\widehat{\theta}) \Big/ \log(1 - \gamma^*) \tag{C.2}$$

periods in which player 2 plays T. Therefore, her equilibrium payoff vanishes to 0 as δ approaches 1, contradicting the fact that $v_{\widehat{\theta}} \geq 1 - \widehat{\theta} > 0$.

Second, suppose towards a contradiction that type $\widehat{\theta}$ plays H in every period. If $\widehat{\theta} \neq \theta_1$, then her equilibrium payoff is at most $1 - \widehat{\theta}$, which is strictly less than $v^*_{\widehat{\theta}}$, leading to a contradiction. If $\widehat{\theta} = \theta_1$, then type θ_2 is separated from type θ_1 the first time she plays L, after which her continuation payoff is no more than $1 - \theta_2$. Therefore, type θ_2's equilibrium payoff is at most $(1-\delta) + \delta(1-\theta_2)$, which is strictly less than v^*_2 when δ is close enough to 1. This leads to a contradiction.

D Extensions

Product Choice Game Consider the following game introduced by Mailath and Samuelson (2001), Ekmekci (2011), as well as many other papers. Player 1 is a seller choosing between high quality and low quality. Player 2 is a buyer choosing between buy and not buy. Players' stage game payoffs are given by:

-	Buy	Not Buy
High Effort/Quality	$1-\theta, b$	$-d(\theta), 0$
Low Effort/Quality	$1, -c$	$0, 0$

As in the baseline model, θ and $d(\theta)$ are the costs of providing high quality when the buyer buys and not buy, respectively. My model incorporates the *separable payoff case* in Ekmekci (2011) where $\theta = d(\theta)$.

Entry Deterrence/Limit Pricing Game Player 1 is an incumbent choosing between a low price (or *fight*) and a normal price (or *accommodate*). Player 2 is the entrant deciding between entering the market and staying out. Players' payoffs are given by:

-	Out	Enter
Low Price	$1-\theta, 0$	$-d(\theta), -b$
Normal Price	$1, 0$	$0, c$

where θ and $d(\theta)$ are the incumbent's costs from limit pricing (if the entrant stays out) and predation (if the entrant enters), respectively. As in Milgrom and Roberts (1982), θ depends on the efficiency of her production technology, which tends to be her private information. When studying the incumbent's payoff and equilibrium behavior, this is equivalent to the previous payoff matrix as the myopic players' incentives depend only on the their gains from staying out conditional on θ and the incumbent's action.

Monetary Policy: Player 1 is a central bank that is facing a continuum of households (player 2s), each has negligible mass. In every period, the central bank chooses the inflation level while at the same time, households form their expectations about inflation. To simplify matters, I assume that both the actual inflation and the expected inflation are binary variables. In line with the classic work of Barro (1986), players' stage game payoffs are given by:

-	Low Expectation	High Expectation
Low Inflation	$1-\theta, x_1$	$-d(\theta), -y_1$
High Inflation	$1, -y_2$	$0, x_2$

where $x_1, x_2, y_1, y_2 > 0$ are parameters, $\theta \in \Theta \subset (0,1)$ is the central bank's private information. To make sense of this payoff matrix, households want to match their expectations with the actual inflation. The central bank's payoff decreases with the actual inflation and increases with the amount of surprised inflation (defined as actual inflation minus expected inflation). As argued in Barro (1986), the central bank can strictly benefit from surprised inflation as it can increase real economic activities, decrease unemployment rate and increase governmental revenue. How the central bank trades-off these benefits with the costs of inflation is captured by θ, which depends on the central banker's ideology and tends to be her persistent private information. Economically, the parametric assumption that $\theta < 1$ implies that inflation is costly for the central bank if it is perfectly anticipated by the households.

E Miscellaneous

E.1 Counterexample under Complete Information

I provide a counterexample to Theorem 2 under complete information. I show that when $|\Theta| = 1$, there exists a sequential equilibrium in which player 1 plays a stationary mixed strategy and attains payoff v^*, equal to $1-\theta$.

The long-run player plays H with probability γ^* at every history. The myopic players play T in period 0. Their actions in period $t(\geq 1)$ depends on the game's outcome in period $t-1$ (denoted by a_{t-1}):

$$\text{Player 2's action in period } t = \begin{cases} N & \text{if } a_{t-1} = N \\ T & \text{if } a_{t-1} = H \\ p^*T + (1-p^*)N & \text{if } a_{t-1} = L \end{cases} \tag{E.1}$$

where

$$p^* \equiv \frac{1-\theta/\delta}{1-\theta}, \tag{E.2}$$

which is strictly between 0 and 1 when δ is close enough to 1.

E.2 ϵ-Stackelberg Strategies

First, I construct a Nash equilibrium in which some type of the long-run player plays a stationary ϵ-Stackelberg strategy but the long-run player's equilibrium payoff is 0. Consider the following strategy profile:

- The myopic players play N at every history.
- Type θ_1 plays H with probability $\gamma^* + \epsilon$ at every history.
- Types other than θ_1 plays L at every history.

Let ϵ be small enough such that:

$$\epsilon < \frac{c}{b+c} \frac{1-\pi_0(\theta_1)}{\pi_0(\theta_1)}, \tag{E.3}$$

the above strategy profile is a Nash equilibrium for every $\delta \in (0,1)$.

Next, I show that no type will play any stationary ϵ-Stackelberg strategy in any sequential equilibrium when ϵ is small enough so that every stationary ϵ-Stackelberg strategy is completely mixed.

Proposition E.1. *For every ϵ small enough, there exists no sequential equilibrium in which some type of the long-run player plays a stationary ϵ-Stackelberg strategy.*

Proof. Let ϵ be small enough so that every stationary ϵ-Stackelberg strategy is completely mixed. Suppose towards a contradiction that there exists a sequential equilibrium (σ, π) with $\sigma = \Big((\sigma_\theta)_{\theta \in \Theta}, \sigma_2 \Big)$ and $\pi : \mathcal{H} \rightarrow \Delta(\Theta)$ such that $\sigma_{\widehat{\theta}}$ is a stationary ϵ-Stackelberg strategy. Consider the subgame after player 2 plays T in period $t \in \mathbb{N}$. Both $\overline{\sigma}_\theta$ and $\underline{\sigma}_\theta$ are type $\widehat{\theta}$'s best reply to σ_2 in that subgame. Lemma C.1 in Pei (2018) implies that:

1. For every $\theta < \widehat{\theta}$, type θ will play H with probability 1 in period t.
2. For every $\theta > \widehat{\theta}$, type θ will play L with probability 1 in period t.

Therefore, after observing H in period 0, player 2's posterior attaches probability 1 to the event that $\theta \leq \widehat{\theta}$. For every $\theta \leq \widehat{\theta}$, we have shown before that she will play H with probability strictly greater than γ^* at every history where T is played with positive probability. Hence, T is the myopic players' strict best reply after they observe H in period 0 and regardless of player 1's action choices after period 0. As a result, in the subgame following the myopic players play T in period 0, type $\widehat{\theta}$ can obtain continuation payoff $(1-\delta)(1-\widehat{\theta}) + \delta$ by playing H in period 0 and playing L in all subsequent periods, which is strictly more than her payoff by playing H in every period, which is $1 - \widehat{\theta}$. This contradicts the previous claim that $\overline{\sigma}_\theta$ is her best reply. $\square$

References

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