NOTES ON HYSTERESIS

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Based on "Aggregate Demand and the Dynamics of Unemployment" by Edouard Schaal and Mathieu Taschereau-Dumouchel

1. Model

- Workers produce a continuum of intermediate inputs
- All agents are risk neutral and discount future at rate β
- Continuum of workers $j \in [0,1]$
- If worker j is employed, produces good j

$$Y_j = Ae^z$$

  where z stochastic (see below)
- Unemployed workers enjoy leisure b
- To be employed workers need to be matched with firms (below on matching)
- Differentiated goods produced by each worker enter production function of final good

$$Y = \left(\int_{j \in E} Y_j^{\frac{\sigma-1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma-1}}$$

  where E set of employed workers
- Inverse demand function for good j

$$P_j = P \left(\frac{Y_j}{Y} \right)^{-\frac{1}{\sigma}}$$

- So revenue of firm employing worker j is

$$P_j Y_j = P \left(\frac{Y_j}{Y} \right)^{-\frac{1}{\sigma}} Y_j = P \left(Ae^z \right)^{1-\frac{1}{\sigma}} Y^{\frac{1}{\sigma}}$$

- With $1-u$ employed workers total output is

$$Y = Ae^z \left(1-u\right)^{\frac{\sigma}{\sigma-1}}$$

- So real revenues per firm are

$$\rho\left(u\right) = \frac{P_j}{P} Y_j = Ae^z \left(1-u\right)^{\frac{1}{\sigma-1}}$$

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- Now to the search and matching part
- Unemployed workers meet vacancies at rate

$$m(u, v)$$

where u is mass of unemployed and v mass of vacancies
- Assume m CRS
- Probability of finding a vacancy for unemployed worker

$$p\left(\frac{v}{u}\right) \equiv \frac{m(u,v)}{u} = m\left(1, \frac{v}{u}\right)$$

- Probability of finding a worker for a vacancy

$$q\left(\frac{v}{u}\right) \equiv \frac{m(u,v)}{v} = m\left(\frac{1}{v/u}, 1\right)$$

- Value functions
- Value of a job

$$J_t = \rho_t - w_t + \beta(1-\delta) E_t [J_{t+1}]$$

- Value of a employment

$$W_t = w_t + \beta E[(1-\delta) W_{t+1} + \delta U_{t+1}]$$

- Value of unemployment

$$U_t = b + \beta E[p_t W_{t+1} + (1-p_t) U_{t+1}]$$

- Nash bargaining, wages set so that γ fraction of surplus goes to workers

$$J_t = (1-\gamma) S_t$$

where S_t is the total surplus of a match

$$S_t \equiv J_t + W_t - U_t$$

- Total surplus dynamics

$$S_t = \rho_t - b + \beta E_t [(1-\delta)[J_{t+1} + W_{t+1}] + \delta U_{t+1} - p_t W_{t+1} - (1-p_t) U_{t+1}]$$
$$S_t = \rho_t - b + \beta E_t [(1-\delta)[J_{t+1} + W_{t+1} - U_{t+1}] - p_t (W_{t+1} - U_{t+1})]$$
$$S_t = \rho_t - b + \beta E[(1-\delta) S_{t+1} - p_t \gamma S_{t+1}]$$

- Given dynamics for ρ and θ we can recover dynamics of S from last equation
- Job creation
- Create vacancy if

$$\kappa \leq \beta q_t E_t J_{t+1} = \beta q_t (1-\gamma) E_t S_{t+1}$$

- Heterogeneity in κ
- Mass M of potential entrants with κ distributed with CDF $F(\kappa)$
- So
$$\theta_t = \frac{v_t}{u_t} = \frac{MF(\hat{\kappa}_t)}{u_t}$$
where
$$\hat{\kappa}_t = (1-\gamma)\beta q(\theta_t) E_t S_{t+1}$$
- Given $E_t S_{t+1}$ this equation gives us $\hat{\kappa}_t$
- Unemployment dynamics
$$u_{t+1} = (1-p(\theta_t)) u_t + \delta(1-u_t)$$

2. Dynamics

- Let's remove the z shock
- Markov equilibrium with unique state variable u
- Functional equation in S
$$S(u) = \rho(u) - b + \beta[(1-\delta) S(u') - p(\theta)\gamma S(u')]$$
$$u' = (1-p(\theta)) u + \delta(1-u)$$
$$\kappa = (1-\gamma) q(\theta) \beta S(u')$$
$$\theta u = MF(\kappa)$$
- Notice that if $\rho(u) = \rho$ and no heterogeneity in job creation, then we can show that there is a unique equilibrium in which $S(u)$ is flat (Mortensen and Nagypal)
- In this case there is a form of block recursivity: the value of a filled job can be solved without knowing the state u as solution to
$$S = \rho - b + \beta\left[(1-\delta) S - \frac{\gamma}{1-\gamma}\kappa\right]$$
then constant θ can be found from
$$\kappa = (1-\gamma) q(\theta) S$$
and unemployment dynamics from
$$u' = (1-p(\theta)) u + \delta(1-u)$$
- With general $\rho(u)$ this does not hold and we have
$$\theta = \Theta(u)$$
- Solving the functional equation

- Conjecture $S(.)$; for given u find θ that solves

$$\theta u = MF\left((1-\gamma)\, q(\theta)\, \beta S\left((1-p(\theta))\, u + \delta(1-u)\right)\right)$$

  (if there are multiple solutions... pick one)
- Let distribution κ have bounds $\underline{\kappa}, \overline{\kappa}$
- Then, if at $\theta = M/u$

$$(1-\gamma)\, q(\theta)\, \beta S\left((1-p(\theta))\, u + \delta(1-u)\right) > \overline{\kappa}$$

  then

$$\theta = M/u$$

  if

$$(1-\gamma)\, \beta S\left(u + \delta(1-u)\right) < \underline{\kappa}$$

  then

$$\theta = 0$$

- Use Cobb-Douglas matching function so

$$p(\theta) = \min\{\eta\theta^{\alpha}, \theta, 1\}$$

$$(1-p(\theta))\, u + \delta(1-u) < 1$$

$$q(\theta) = p(\theta)/\theta$$

- Update

$$S(u) = \rho(u) - b + \beta\left[(1-\delta)\, S(u') - p(\theta)\, \gamma S(u')\right]$$

- In matlab folder a code that follows the algorithm above
- Easy to add z shocks

3. A continuous time version

- The continuous time version of the model gives us two ODEs for u, S

$$\dot{u} = \delta(1-u) - p(\theta)\, u$$

$$rS = \rho(u) - b - (\delta + \gamma p(\theta))\, S + \dot{S}$$

- Where

$$p(\theta) = \eta\theta^{\alpha}$$

$$\rho(u) = A(1-u)^{\frac{1}{\sigma-1}}$$

- Consider the case with a fixed κ
- Then free entry condition is

$$\kappa = (1-\gamma)\, q(\theta)\, S$$

where

$$q(\theta) = \eta\theta^{\alpha-1}$$

gives a relation between S and θ

- Let's derive the latter relation explicitly and substitute in $p(\theta)$ to obtain the finding rate as a function of the surplus S

$$p = f(S) \equiv \xi S^{\frac{\alpha}{1-\alpha}}$$

where

$$\xi = \eta^{\frac{1}{1-\alpha}} \left(\frac{1-\gamma}{\kappa}\right)^{\frac{\alpha}{1-\alpha}}$$

- Then the ODEs become

$$\dot{u} = \delta(1-u) - f(S)u$$

$$\dot{S} = (r + \delta + \gamma f(S))S - \rho(u) + b$$

- Steady state conditions

$$\delta(1-u) = f(S)u$$

$$S = \frac{\rho(u) - b}{r + \delta + \gamma f(S)}$$

3.1. Numerical analysis.

- Choose all parameters except A, b
- Calibrate to 2 steady states u_L, u_H as follows
- Choose u_L, u_H
- Get values of S_L, S_H from

$$S = \left(\frac{\delta}{\xi}\frac{1-u}{u}\right)^{\frac{1-\alpha}{\alpha}}$$

- Choose A and b so following the equation holds for both

$$(r + \delta + \gamma f(S))S = A(1-u)^{\frac{1}{\sigma-1}} - b$$

- That is, set

$$A = \frac{(r + \delta + \gamma f(S_H))S_H - (r + \delta + \gamma f(S_L))S_L}{(1-u_H)^{\frac{1}{\sigma-1}} - (1-u_L)^{\frac{1}{\sigma-1}}}$$

$$b = A(1-u_L)^{\frac{1}{\sigma-1}} - (r + \delta + \gamma f(S_L))S_L$$

- Then analyze dynamic properties of the ODEs, solving backward starting near SS
- The following is the phase diagram for an example
- In this example there are multiple equilibria, because for u in some interval two equilibrium paths are possible

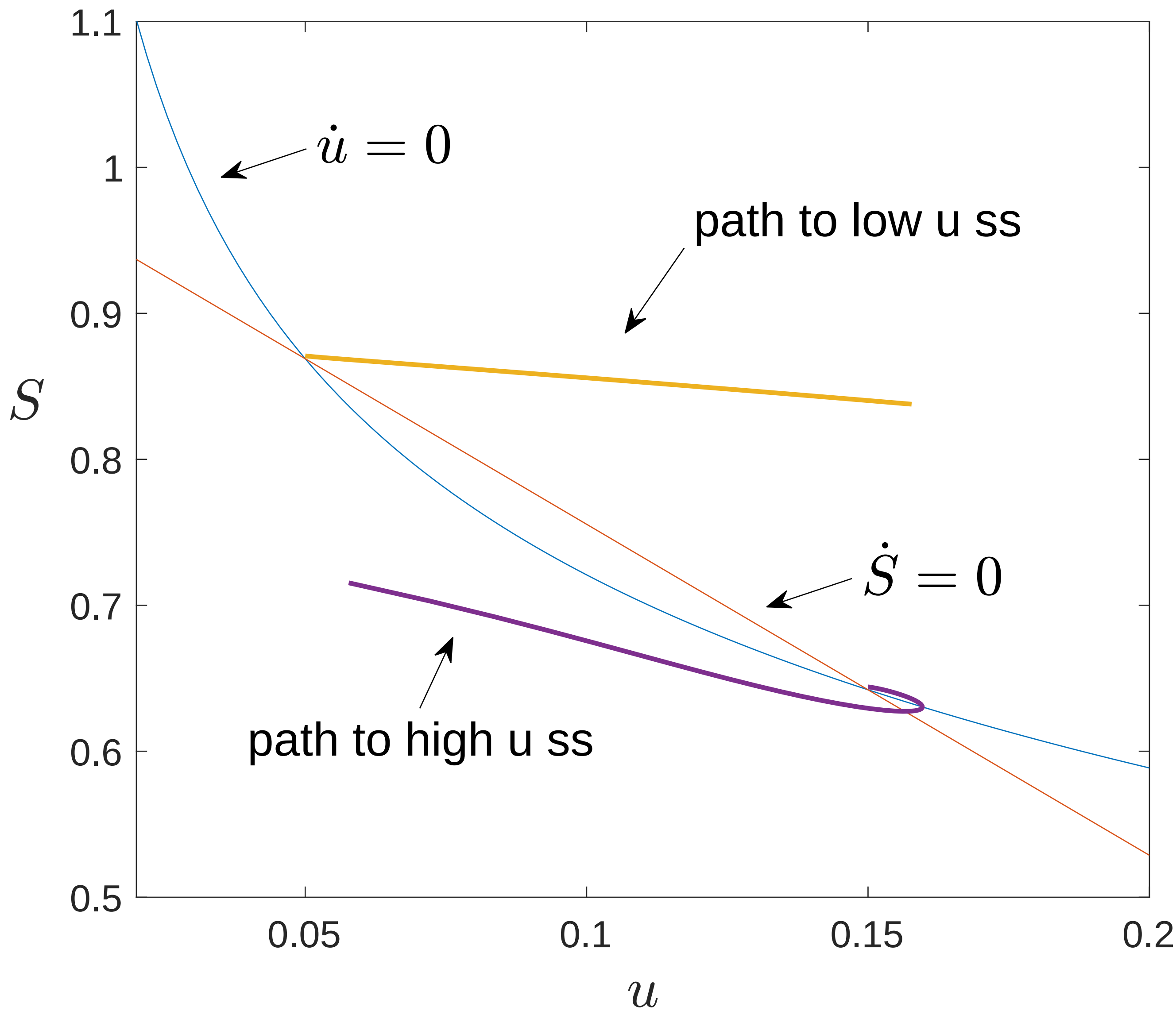

FIGURE 1. Two steady states and equilibrium dynamics

- In fact, the dynamics near the high u equilibrium are a sink-spiral, that is, the linearized system has two complex conjugate eigenvalues with a positive real part (see below a quick review of ODEs)
- This means that the model actually features indeterminacy as for u in some range there is a continuum of S where you can start the equilibrium dynamics
- The following exercise asks you to look for dynamics more like the ones emphasized in the paper: no multiple equilibria, but multiple steady states, so the initial condition for u matters for long run dynamics

- Maybe it can be done, maybe you need to add heterogeneity in κ as in the full blown model in the paper

Exercise 1. Can you find parameters such that the economy has two ss: one of them a saddle and the other one a source and not a spiral?

4. Quick review of dynamic systems

- If
$$\dot{x} = h(x)$$
and x is vector, let it have ss at 0 (normalization)
- Linearize near 0
$$\dot{x} = Dh \cdot x$$
- Local saddle if Dh has two real eigenvalues of opposite sign
- Let
$$Dh = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$
and notice that characteristic polynomial is
$$(h_{11} - \lambda)(h_{22} - \lambda) - h_{12}h_{21} = \lambda^2 - (h_{11} + h_{22})\lambda + h_{11}h_{22} - h_{21}h_{12}$$
- This quadratic has two real roots of opposite sign iff
$$h_{11}h_{22} - h_{21}h_{12} < 0$$
- This restriction can be expressed as a restriction on the relative slope of the loci $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$
- The direction of the inequality depends on the signs of h_{12} and h_{22}
- In our case $h_{22} > 0$ and $h_{12} < 0$ so we get
$$\frac{h_{11}}{h_{12}} > \frac{h_{21}}{h_{22}}$$
or
$$-\frac{h_{11}}{h_{12}} < -\frac{h_{21}}{h_{22}}$$
so the locus $\dot{x}_1 = 0$ must have smaller derivative than the locus $\dot{x}_2 = 0$
- If a steady state is not a saddle it can be:
  - a source (all eigenvalues have positive real part, if $h_{11} + h_{22} > 0$)
  - or a sink (all eigenvalues have negative real part, if $h_{11} + h_{22} < 0$)
- In both cases it can be a spiral if eigenvalues have imaginary parts, that is, if
$$(h_{11} + h_{22})^2 - 4(h_{11}h_{22} - h_{21}h_{12}) < 0$$
- A saddle cannot be a spiral

5. Back to the (continuous time) model

- Notice that if multiple steady states are possible, then the loci have to cross multiple times and they cannot cross always from the same side
- Result: if there are multiple ss, if one is a saddle, the one immediately next to it cannot be a saddle
- Result: if a ss is a spiral, then multiple equilibria exist (sufficient condition, not necessary)
- In our example above, the high u steady state was a sink/spiral
- Compute Dh for the model

$$h_1(u,S) = \delta(1-u) - f(S)\,u$$

$$h_2(u,S) = (r+\delta+\gamma f(S))\,S - \rho(u) + b$$

$$h_{11} = -\delta - f(S)$$

$$h_{12} = -f'(S)\,u$$

$$h_{21} = -\rho'(u)$$

$$h_{22} = r + \delta + \gamma f(S) + \gamma f'(S)\,S$$

- Determinant

$$h_{11}h_{22} - h_{12}h_{21} = -(\delta + f(S))(r+\delta+\gamma f(S) + \gamma f'(S)\,S) - f'(S)\,u\rho'(u)$$

- Notice that if

$$\rho' = 0$$

  (as in baseline DMP) then determinant is always < 0 and we can only have a saddle (so ss must be unique)
- But if $\rho' < 0$ and large enough in absolute value, we can have other configurations