# NOTES ON PANICS

GUIDO LORENZONI

## 1. A runnable entity

- A model inspired by He and Xiong
- Captures a simple entity with a balance sheet as follows:
  - a risky portfolio of uncertain value
  - short term liabilities, rolled over
- An entrepreneur needs to borrow 1 at date 0 to finance project with potential payoff $y_0$
- Payoff of the project evolves according to the geometric random walk

$$y_t = y_{t-1}\varepsilon_t$$

- With $E[\varepsilon_t] = 1$ and $\varepsilon_t \in [\underline{\varepsilon}, \overline{\varepsilon}]$
- Each period the project is completed with probability $\phi$, in which case it pays $y_t$, or it continues
- A project only pays off when completed. The expected present value of the project is

$$v_t = \beta E_t\left((1-\phi)\, v_{t+1} + \phi y_{t+1}\right)$$

  which yields

$$v_t = \frac{\phi}{1/\beta - 1 - \phi} y_t.$$

  We assume

$$v_0 > 1,$$

  so the project is profitable.
- Entrepreneur has no initial wealth
- E. finances the project selling debt contracts of random maturity (see below) to a large population of lenders
- Debt contracts have the following features
  - Each period $t \geq 1$, if the project is completed a lender gets

$$\min\left\{1, \frac{y_t}{d_{t-1}}\right\}$$

    where $d_{t-1}$ is the number of debt contracts outstanding from last period

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 - If the the project is not completed, a fraction $\delta$ of the $d_{t-1}$ contracts outstanding is drawn randomly and they are paid 1 each, at which point those debt contracts are fulfilled
 - If a contract is not drawn at date $t$, it remains outstanding
 - The borrower finances the payment $\delta d_{t-1}$ by issuing new debt contracts, so the budget constraint is

$$p_t \left[d_t - (1-\delta)\, d_{t-1}\right] \geq \delta d_{t-1} \tag{1}$$

- Timing in case of rollover is as follows:
 - the borrower announces it will issue $d_t$ debt contracts
 - the lenders observe $d_t$ and the price $p_t$ is determined (maybe on an auction)
 - if $p_t \left[d_t - (1-\delta)\, d_{t-1}\right] \geq \delta d_{t-1}$ the borrower continues
 - if $p_t \left[d_t - (1-\delta)\, d_{t-1}\right] < \delta d_{t-1}$ no new debt is issued, the borrower goes into default, the project is liquidated, and the existing debt holders receive

$$\min\left\{1, \frac{\lambda y_t}{d_{t-1}}\right\}$$

- The parameter $\lambda < 1$ captures the costs of early liquidation
- Assumption: borrower always wants to continue and always wants to have minimum debt, therefore he will always issue the minimum $d_t$ such that (1) is satisfied, if such a $d_t$ exists
- We could microfound last assumption by assuming borrower is a risk neutral agent who receives the equity value after realization of $y$ with no default
- Rational expectations require the price to satisfy

$$p_t = \beta E_t \left[(1-\phi)\, \rho_{t+1} \left\{\delta + (1-\delta)\, p_{t+1}\right\} + (1-\phi)\,(1-\rho_{t+1}) \min\left\{1, \frac{\lambda y_{t+1}}{d_t}\right\} + \phi \min\left\{1, \frac{y_{t+1}}{d_t}\right\}\right] \tag{2}$$

 where $\rho_{t+1}$ is an indicator equal to 1 in case of successful rollover and 0 otherwise
- Define the debt-to-capital ratio at the end of the period

$$x_t = \frac{d_t}{y_t},$$

- Markov equilibrium:
 - market price of debt is given by the decreasing continuous function $\mathcal{P}$

$$p_t = \mathcal{P}\left(\frac{d_t}{y_t}\right)$$

 - default occurs iff $y_t < \xi d_{t-1}$ for some scalar $\xi$

- rewrite the budget constraint (1) as
$$p_t\left(\frac{d_t}{y_t}-(1-\delta)\frac{d_{t-1}}{y_t}\right)\geq\delta\frac{d_{t-1}}{y_t}$$
or
$$\mathcal{P}\left(\frac{d_t}{y_t}\right)\left(\frac{d_t}{y_t}-(1-\delta)\frac{d_{t-1}}{y_{t-1}}\frac{y_{t-1}}{y_t}\right)\geq\delta\frac{d_{t-1}}{y_{t-1}}\frac{y_{t-1}}{y_t}$$
$$\mathcal{P}(x_t)\left(x_t-(1-\delta)\frac{x_{t-1}}{\epsilon_t}\right)\geq\delta\frac{x_{t-1}}{\epsilon_t}$$
- Each period the borrower takes the function $\mathcal{P}$ as given and chooses $x_t$ that satisfies last equation as an equality
- The following result is useful:

*Claim* 1. Given a non-negative, decreasing continuous function $\mathcal{P}(.)\geq 0$, there is a $\xi$ (which could be $\infty$) such that if $x\leq\xi\epsilon$ there is a solution $x'\geq(1-\delta)x/\epsilon$ to the equation
$$\mathcal{P}(x')x'=(\delta+(1-\delta)\mathcal{P}(x'))\frac{x}{\epsilon},$$
and if $x>\xi\epsilon$ there is no solution

- Graphical argument focusing on first looking at the "debt Laffer curve"
$$\mathcal{P}(x')\left(x'-(1-\delta)\frac{x}{\epsilon}\right)$$
and then translating it in terms of the solutions to
$$\frac{\mathcal{P}(x')x'}{\delta+(1-\delta)\mathcal{P}(x')}=\frac{x}{\epsilon},$$
where function on LHS is independent of $x/\epsilon$
- Whenever $x\leq\xi\epsilon$ denote the smallest solution as $x'=f\left(\frac{x}{\epsilon}\right)$
- Notice that $f$ is defined only on domain $x/\epsilon\in[0,\xi]$
- Notice that the inverse of $f$ can be derived in closed form
$$f^{-1}(x')=\frac{\mathcal{P}(x')x'}{\delta+(1-\delta)\mathcal{P}(x')}$$
in some range $x'\in[(1-\delta)x/\epsilon,f^{-1}(\xi)]$
- So if $\epsilon\geq\xi x$ the borrower offers $x'=f\left(\frac{x}{\epsilon}\right)$ new debt contracts and successfully rolls over
- The rational expectations condition (2) is

$$\begin{aligned}(3)\quad \mathcal{P}(x)=\beta E[\phi\min\left\{1,\frac{\varepsilon}{x}\right\}+&\\ +(1-\phi)\left(\iota\left(\frac{x}{\varepsilon}\leq\xi\right)\left(\delta+(1-\delta)\mathcal{P}\left(f\left(\frac{x}{\varepsilon}\right)\right)\right)+\iota\left(\frac{x}{\varepsilon}>\xi\right)\min\left\{1,\lambda\frac{\varepsilon}{x}\right\}\right)]\end{aligned}$$

**Definition 2.** A Markov equilibrium is given by a scalar $\xi > 0$, a function $f$ and a function $\mathcal{P}$, such that $\xi$ and $f$ are constructed as in Claim 1 and $\mathcal{P}$ satisfies (3).

- There can be an interval $[0, \bar{x}]$ where $x$ is small enough and we are sure no liquidation or default will occur next period, then the price is given by
$$\mathcal{P}(x) = \beta E\left[(1-\phi)(\delta + (1-\delta)\mathcal{P}(f(x/\varepsilon))) + \phi \min\{1, \varepsilon/x\}\right]$$
However, eventually if $\varepsilon < 1$ is realized for many periods, we escape to the region where default possible. Therefore $\mathcal{P}(x) < 1$ for all $x > 0$
- At date 0, we need to check that
$$\mathcal{P}(x_0)\, x_0 \geq 1$$
for some $x_0$, so the project can be financed

## 2. Algorithm

- We compute a finite horizon model where the project matures with probability 1 at date $T$
- At date $T$ lenders get
$$\min\left\{1, \frac{\varepsilon_T}{x_T}\right\}$$
- Calculate
$$\mathcal{P}_{T-1}(x_T) = \beta E\left[\min\left\{1, \frac{\varepsilon_T}{x_T}\right\}\right]$$
- Find $\xi_{T-1}$
$$\xi_{T-1} = \max_x \frac{\mathcal{P}_{T-1}(x)\, x}{\delta + (1-\delta)\mathcal{P}_{T-1}(x)} = \frac{\beta E[\epsilon]}{\delta}$$
and $f_{T-1}$ from inverting
$$\frac{\mathcal{P}_{T-1}(x)\, x}{\delta + (1-\delta)\mathcal{P}_{T-1}(x)}$$
(which is monotone everywere in this first round)
- Iterate on $P$

$$\mathcal{P}_{T-2}(x) = \beta E[\phi \min\left\{1, \frac{\varepsilon}{x}\right\} +$$
$$+ (1-\phi)\left(\iota\left(\frac{x}{\varepsilon} \leq \xi_{T-1}\right)\left(\delta + (1-\delta)\mathcal{P}\left(f_{T-1}\left(\frac{x}{\varepsilon}\right)\right)\right) + \iota\left(\frac{x}{\varepsilon} > \xi_{T-1}\right)\min\left\{1, \lambda\frac{\varepsilon}{x}\right\}\right)]$$