# NOTES ON FINANCIAL ACCELERATOR

GUIDO LORENZONI

## 1. A baseline financial accellerator model

- Stochastic model with non state contingent debt, collateral constraints and aggregate investment
- Full global solution in the spirit of Brunnermeier-Sannikov
- A rich problem with a two dimensional state space
- Entrepreneurs risk neutral, with discount factor $\beta$
- Lenders risk neutral, with discount factor $q$
- Entrepreneurs only agents that can hold capital
- Adjustment cost function
- When entrepreneurs selling capital goods, these are turned back into consumption goods (at some cost)
- Entrepreneur's budget constraint

$$c_t + G\left(k_{t+1}, k_t\right) = A_t F\left(k_t, l_t\right) - w_t l_t - b_t + q b_{t+1}$$

- $G$ is a CRS investment cost function, which includes adjustment costs

$$G(k', k) \equiv k' - (1-\delta)k + \zeta(k' - (1-\delta)k)^2/k$$

- Collateral constraint

$$b_{t+1} \leq \theta p_{t+1} k_{t+1}$$

  for all realizations of $p_{t+1}$ that have positive probability at $t$
- More below on the price $p_t$ at which capital can be sold
- $A_t, w_t, p_t$ driven by Markov process $s_t$
- $G$ and $F$ are constant returns to scale
- Then the value function must satisfy

$$V\left(k_t, b_t, s_t\right) = v\left(b_t/k_t, s_t\right) k_t$$

  for some function $v$

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- Bellman equation

$$v\left(\tilde{b}_t, s_t\right) k_t = \max_{c_t, l_t, \tilde{b}_{t+1}, k_{t+1}} c_t + \beta E_t \left[ v\left(\tilde{b}_{t+1}, s_{t+1}\right)\right] k_{t+1}$$

subject to

$$c_t + G\left(k_{t+1}, k_t\right) = A_t F\left(k_t, l_t\right) - w_t l_t - \tilde{b}_t k_t + q \tilde{b}_{t+1} k_{t+1}$$

and

$$\tilde{b}_{t+1} \leq \theta \underline{p}_{t+1|t}$$

- Optimality for $k_{t+1}$ yields

$$\beta E_t \left[ v\left(\tilde{b}_{t+1}, s_{t+1}\right)\right] + q \lambda_t \tilde{b}_{t+1} = \lambda_t G_1\left(k_{t+1}, k_t\right)$$

- If it's optimal to consume $\lambda_t = 1$, in this case

$$G_1\left(k_{t+1}, k_t\right) = \beta E_t \left[ v\left(\tilde{b}_{t+1}, s_{t+1}\right)\right] + q \tilde{b}_{t+1}$$

- The LHS is marginal $Q$ the RHS is average $Q$ (Abel 1982 and Hayashi 1982)
- If the non-negativity of consumption is never binding this model yields standard Q theory predictions: asset price over capital stock is a sufficient statistic for the investment rate $k_{t+1}/k_t$
- In general we can have $\lambda_t > 1$ which implies marginal $Q$ smaller than average $Q$: firms have an incentive to issue more claims to finance investment, but entrepreneurs cannot buy these claims, since they are at $c_t = 0$
- If $\lambda_t > 1$ it means that either the collateral constraint is binding today or it will be binding in the future
- Optimality condition with respect to $\tilde{b}_{t+1}$ is

$$\lambda_t q k_{t+1} + \beta E_t \left[ \frac{\partial v\left(\tilde{b}_{t+1}, s_{t+1}\right)}{\partial \tilde{b}} \right] k_{t+1} - \mu_t = 0$$

and using envelope condition

$$q \lambda_t = \beta E_t \left[\lambda_{t+1}\right] + \mu_t / k_{t+1}$$

- Envelope condition for $k_t$ is

$$v\left(\tilde{b}_t, s_t\right) = \lambda_t \left[ A_t F_k\left(k_t, l_t\right) - G_2\left(k_{t+1}, k_t\right) - \tilde{b}_t \right]$$

- Combining with optimality for $k_{t+1}$

$$\lambda_t = \frac{\beta E_t \left[ v\left(\tilde{b}_{t+1}, s_{t+1}\right)\right]}{G_1\left(k_{t+1}, k_t\right) - q \tilde{b}_{t+1}} = \frac{\beta E_t \left[ \lambda_{t+1} \left[ A_{t+1} F_{k,t+1} - G_{2,t+1} - \tilde{b}_{t+1} \right]\right]}{G_{1,t} - q \tilde{b}_{t+1}} \quad (1)$$

- Suppose now entrepreneurs can trade used capital from other entrepreneurs, before employing the adjustment cost technology
- Then to reach capital $k_{t+1}$ they will choose to minimize total cost of achieving it

$$\min_{\hat{k}_t} G\left(k_{t+1}, \hat{k}_t\right) + p_t\left(\hat{k}_t - k_t\right)$$

- Representative entrepreneur, so no trade and $\hat{k}_t = k_t$ in equilibrium
- First order condition

$$p_t = -G_2\left(k_{t+1}, k_t\right)$$

  gives us the price of capital that appears in the collateral constraint
- Then the optimality condition can be rewritten as

$$E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{A_{t+1}F_{k,t+1} + p_{t+1} - \tilde{b}_{t+1}}{G_{1,t} - q\tilde{b}_{t+1}}\right] = 1$$

- This is an asset pricing equation where

$$\beta\frac{\lambda_{t+1}}{\lambda_t}$$

  is the stochastic discount factor of the entrepreneurs and

$$\frac{A_{t+1}F_{k,t+1} + p_{t+1} - \tilde{b}_{t+1}}{G_{1,t} - q\tilde{b}_{t+1}}$$

  is the levered return on entrepreneurial capital
- We can also rewrite optimality for borrowing ratio as an asset pricing equation

$$1 = E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{q}\right] + \frac{\mu_t}{q\lambda_t}\frac{1}{k_{t+1}}$$

  which implies

$$E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{1}{q}\right] \leq 1$$

  here the expected return on bonds, discounted with the discount factor $\beta\lambda_{t+1}/\lambda_t$ can be $< 1$ if the collateral constraint is binding
- Rewrite (1) as

$$E_t\left[\beta\lambda_{t+1}\left[A_{t+1}F_k\left(k_{t+1}, l_{t+1}\right) + p_{t+1} - \tilde{b}_{t+1}\right]\right] = \lambda_t\left(G_{1,t} - q\tilde{b}_{t+1}\right)$$

  and then as

$$\begin{aligned} E_t\left[\beta\lambda_{t+1}\left[A_{t+1}F_k\left(k_{t+1}, l_{t+1}\right) + p_{t+1}\right]\right] &= \lambda_t G_{1,t} - \left(\lambda_t q - E_t\left[\beta\lambda_{t+1}\right]\right)\tilde{b}_{t+1} \\ &= \lambda_t G_{1,t} - \mu_t\tilde{b}_{t+1} \end{aligned}$$

so

$$E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}}{G_{1,t}}\right]\leq 1$$

- Agents are willing to accept a lower return on capital, since holding capital helps to relax the collateral constraint
- Using the same condition and $q\lambda_t \geq E_t\left[\beta\lambda_{t+1}\right]$ we also get that if

$$\lambda_{t+1} \text{ and } A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}-\tilde{b}_{t+1}$$

are negatively correlated we have

$$q\lambda_t E_t\left[A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}-\tilde{b}_{t+1}\right]\geq E_t\left[\beta\lambda_{t+1}\left[A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}-\tilde{b}_{t+1}\right]\right]=\lambda_t\left(G_{1,t}-q\tilde{b}_{t+1}\right)$$

which imply

$$E_t\left[\frac{A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}}{G_{1,t}}\right]\geq\frac{1}{q}$$

so the expected rate of return on capital is greater than the risk free interest rate
- New possibility: the collateral constraint can be slack even though the rate of return on entrepreneurial capital is greater than $1/q$
- Rewrite condition as

$$E_t\left[\beta\lambda_{t+1}\left[A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}-\tilde{b}_{t+1}\right]\right]=\lambda_t\left(G_{1,t}-q\tilde{b}_{t+1}\right)$$

- If constraint is slack $\mu_t=0$ and this becomes

$$E_t\left[\beta\lambda_{t+1}\left[A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}\right]\right]=\lambda_t G_{1,t}$$

or

$$E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}}{G_{1,t}}\right]=1$$

- If there are no shocks we have

$$\beta\frac{\lambda_{t+1}}{\lambda_t}=q$$

and

$$\frac{A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}}{G_{1,t}}=\frac{1}{q}$$

so collateral constraint can be slack only if investment is efficient at date $t$
- With risk, rate of return on entrepreneurial capital is correlated with $\lambda_{t+1}$
- Temporary productivity shocks generate negative correlation: high return on entrepreneurial wealth, high net worth, economy closer to efficient investment, lower return on entrepreneurial capital
- Then

$$1=E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}}{G_{1,t}}\right]<E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\right]E_t\left[\frac{A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}}{G_{1,t}}\right]$$

so

$$E_t\left[\frac{A_{t+1}F_k\left(k_{t+1},l_{t+1}\right)+p_{t+1}}{G_{1,t}}\right]>\frac{1}{q}$$

- This is a form of precautionary behavior: entrepreneurs are avoiding excess leverage because they anticipate states of the world in which the rate of return on their wealth will be higher than today (high $\lambda_{t+1}/\lambda_t$)
- Notice that entrepreneurs are risk neutral so "precautionary behavior" is really driven by general equilibrium forces

## 2. Linear technology

- Suppose $F\left(k,l\right)=k$ and $A_t=a_t$ that is an i.i.d. shock
- Model can be analyzed with single state variable

$$s_t\equiv a_t-\tilde{b}_t$$

- Recursive equilibrium is given by

$$\lambda\left(s\right),x\left(s\right),b\left(s\right)$$

where

$$x_t=\frac{k_{t+1}}{k_t},$$

the three functions must satisfy three sets of conditions for all $s>\underline{s}$, where $\underline{s}$ is a lower bound to be determined
- Recursive condition for $\lambda$

$$\lambda\left(s\right)=\beta\frac{E\left[\lambda\left(a'-b\left(s\right)\right)\left[a'-b\left(s\right)-G_2\left(x\left(a'-b\left(s\right)\right),1\right)\right]\right]}{G_1\left(x\left(s\right),1\right)-qb\left(s\right)},$$

- Condition for $x\left(s\right)$ that

$$s+qb\left(s\right)x\left(s\right)\geq G\left(x\left(s\right),1\right)$$

with strict equality if $\lambda\left(s\right)>1$
- Condition for the borrowing ratio $b\left(s\right)$

$$q\lambda\left(s\right)\geq\beta E\left[\lambda\left(a'-b\left(s\right)\right)\right]$$

and

$$b\left(s\right)\leq-\theta\min_{a'}G_2\left(x\left(a'-b\left(s\right)\right),1\right)$$

with complementary slackness
- Equilibrium can be computed recursively
- As initial condition think of finite horizon problem, set $\lambda=1$ in the final period and $G_2$ to some fixed value
- Code stoch_KM.m computes equilibrium using following algorithm

- Iteration, endogenous gridpoint method, find $\hat{b}$ that satisfies

$$b = -\theta \min_{a'} G_2\left(x\left(a' - b\right), 1\right),$$

- Choose candidate pairs $(b, \lambda)$ as follows
- Set $b = \hat{b}$ and let

$$\hat{\lambda} = \max\{\frac{\beta}{q} E\left[\lambda\left(a' - b\right)\right], 1\},$$

  then choose any $\lambda$ in $[\hat{\lambda}, \infty)$
- Set $b < \hat{b}$ and if

$$\frac{\beta}{q} E\left[\lambda\left(a' - b\right)\right] < 1$$

  discard, otherwise set

$$\lambda = \frac{\beta}{q} E\left[\lambda\left(a' - b\right)\right]$$

- For each pair $(b, \lambda)$ find $x$ that solves

$$\lambda\left[G_1\left(x, 1\right) - qb\right] = \beta E\left[\lambda\left(a' - b\right)\left[a' - b - G_2\left(x\left(a' - b\right), 1\right)\right]\right],$$

  or

$$\lambda\left[G_1\left(0, 1\right) - qb\right] \geq \beta E\left[\lambda\left(a' - b\right)\left[a' - b - G_2\left(x\left(a' - b\right), 1\right)\right]\right],$$

  if $\lambda = 1$ this is the optimal solution for all $s$ that satisfy

$$s \geq G\left(x, 1\right) - qbx,$$

  if $\lambda > 1$ this is the optimal solution for

$$s = G\left(x, 1\right) - qbx$$

- The lower bound for $s$ is

$$\underline{s} = \min_{x \geq 0} G\left(x, 1\right) - q\hat{b}x$$

  (which arises when $\lambda \to \infty$)
- Functional form used for $G$ is

$$G\left(k', k\right) = k' - k + \frac{\xi}{2} \frac{\left(k' - k\right)^2}{k}$$

  or

$$G\left(x, 1\right) = x - 1 + \frac{\xi}{2}\left(x - 1\right)^2$$

  so derivatives are

$$G_1 = 1 + \xi\left(x - 1\right)$$

  and

$$G_2 = -1 - \xi\left(x - 1\right) - \frac{\xi}{2}\left(x - 1\right)^2$$

- Then this equation

$$\lambda\left[G_1(x,1)-qb\right]=\beta E\left[\lambda\left(a'-b\right)\left[a'-b-G_2\left(x\left(a'-b\right),1\right)\right]\right],$$

  becomes

$$x=1+\frac{1}{\xi}\left\{\frac{\beta E\left[\lambda\left(a'-b\right)\left[a'-b-G_2\left(x\left(a'-b\right),1\right)\right]\right]}{\lambda}+qb-1\right\}$$

- Frictionless benchmark

$$G_1(x,1)=qE\left[a-G_2(x,1)\right]$$

  investment constant with $x$ solving

$$1+\xi(x-1)=q\left[Ea+1+\xi(x-1)+\frac{\xi}{2}(x-1)^2\right]$$

- Assume that

$$r<Ea<r+\frac{\xi}{2}r^2$$

  where $r=1/q-1$ to ensure that a solution to the frictionless problem exists and is bounded

**Choose:** solution with $x<1+r$ to satisfy transversality condition