Notes on linear approximation for portfolio problems

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- These notes discuss how to approximate the solution to a portfolio problem
- They solve problem 2 in problem set 1, but they also introduce the general approach in Judd and Gu
- Problem, find θ that solves

  $$E\left[u'\left(\theta A+(1-\theta)A^*\right)(A-A^*)\right]=0$$

  for the two random variables A, A^* defined in the problem set
- Solve

  $$E\left[\frac{a^\rho\epsilon-\epsilon^*}{\theta a^\rho\epsilon+(1-\theta)\epsilon^*}\right]=0$$

  or

  $$E\left[\frac{(a^\rho-1)+a^\rho(\epsilon-1)-(\epsilon^*-1)}{\theta a^\rho+1-\theta+\theta a^\rho(\epsilon-1)+(1-\theta)(\epsilon^*-1)}\right]=0$$
- Let σ^2 be the variance of the shocks $\epsilon-1$
- Problem, when $\sigma^2=0$ problem is not well defined
- We solve a sequence of problems that converge as $\sigma\to0$
- Let $a^\rho=1+\pi\sigma^2$ and $a^\rho(\epsilon-1)=\left(1+\pi\sigma^2\right)\sigma\eta$ and $\epsilon^*=\sigma\eta^*$
- Consider the problem as $\sigma\to0$

  $$E\left[\frac{\pi\sigma^2+\left(1+\pi\sigma^2\right)\sigma\eta-\sigma\eta^*}{1+\theta\pi\sigma^2+\theta\left(1+\pi\sigma^2\right)\sigma\eta+(1-\theta)\sigma\eta^*}\right]=0$$
- Define

  $$H(\theta,\sigma)=E\left[\frac{\pi\sigma+\left(1+\pi\sigma^2\right)\eta-\eta^*}{1+\theta\pi\sigma^2+\theta\left(1+\pi\sigma^2\right)\sigma\eta+(1-\theta)\sigma\eta^*}\right]$$

- Problem
$$H(\theta, 0) = 0$$
for all θ!

- Judd-Guu approach, us Bifurcation Theorem, which amounts to applying implicit function theorem to
$$f(\theta, \sigma) = \begin{cases} \frac{H(\theta,\sigma)}{\sigma} & \text{if } \sigma \neq 0 \\ H_\sigma(\theta, 0) & \text{if } \sigma = 0 \end{cases}$$

- A simple example of the problem

- Apply implicit function theorem to
$$H(\theta, \sigma) = \sigma(\theta - \sigma)$$

- If you divide by σ you find the arm that converges

- Now
$$\lim_{\sigma \to 0} \frac{H(\theta, \sigma)}{\sigma} = H_\sigma(\theta, 0)$$
$$H_\theta(\theta(\sigma), \sigma)\,\theta'(\sigma) + H_\sigma(\theta(\sigma), \sigma) = 0$$

- At $\sigma = 0$ we have
$$H_\theta(\theta, 0) = 0$$
for all θ

- So if we want to have a well defined $\theta'(\sigma)$ at $\sigma = 0$ then we need
$$H_\sigma(\theta_0, 0) = 0$$

- This condition gives us the non-stochastic steady state!

- Better, it gives us the non-stochastic steady state that is a limit of the stochastic steady state

- Compute
$$E\left[\frac{\pi\sigma + \left(1 + \pi\sigma^2\right)\eta - \eta^*}{1 + \theta\pi\sigma^2 + \theta\left(1 + \pi\sigma^2\right)\sigma\eta + (1-\theta)\sigma\eta^*}\right]$$
$$H_\sigma(\theta, \sigma) = E\left[\frac{\pi + 2\pi\sigma\eta}{1 + \theta\pi\sigma^2 + \theta\left(1 + \pi\sigma^2\right)\sigma\eta + (1-\theta)\sigma\eta^*} - \frac{\pi\sigma + \left(1 + \pi\sigma^2\right)\eta - \eta^*}{\left(1 + \theta\pi\sigma^2 + \theta\left(1 + \pi\sigma^2\right)\sigma\eta + (1-\theta)\sigma\eta^*\right)^2}(\theta\eta\right.$$
$$E\left[\pi - (\eta - \eta^*)(\theta\eta + (1-\theta)\eta^*)\right] = 0$$
$$\pi - (\theta - (1-\theta)) = 0$$
$$1 + \pi - 2\theta = 0$$
$$\theta(0) = \frac{1}{2} + \frac{\pi}{2} = \frac{1}{2} + \frac{a^\rho - 1}{2\sigma^2}$$

- This is a “zero order” approximation we can then get a better approximation by computing

$$\theta'(\sigma)$$

- How can we do it?
- Differentiate

$$H_\theta(\theta(\sigma),\sigma)\,\theta'(\sigma)+H_\sigma(\theta(\sigma),\sigma)=0$$

- Gives

$$H_\theta(\theta(\sigma),\sigma)\,\theta''(\sigma)+H_{\theta\theta}(\theta(\sigma),\sigma)\,\theta'(\sigma)\,\theta'(\sigma)+2H_{\sigma\theta}(\theta(\sigma),\sigma)\,\theta'(\sigma)+H_{\sigma\sigma}(\theta(\sigma),\sigma)=0$$

- Compute it $\sigma=0$ the first two terms are 0 so we have

$$2H_{\sigma\theta}(\theta(0),0)\,\theta'(0)+H_{\sigma\sigma}(\theta(0),0)=0$$